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NOVEL CONCEPTS OF SOME OPEN SETS IN BITOPOLOGICAL SPACES

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Abstract. In this paper, new classes of some open sets are $(i,j)^{\star}$-$\mathcal{S}$-open set,$(i, j)^{\star}$-$\mathcal{S}_{\mathcal{T}}$-set, $(i, j)^{\star}$-$\mathcal{H}$ closed set $(i, j)^{\star}$-$\alpha^{\star}$-sets and $(i, j)^{\star}$-$\mathcal{B}_{\alpha}{}^{\star}$-sets. We introduced and investigated on the line of research.

1. Introduction

In 1963, J.C.Kelly [2] expressed the geometrical existence of bitopological space that is a non empty set X together with two arbitrary topologies defined on X and it plays an important role to study the shapes of objects. General topologist have introduced and investigated of open sets in bitopological spaces. In 1967, C. W. Patty [6] introduced some properties of bitopological spaces. In 1991, M. L. Thivagar *et al*., [3] established the properties of a new type of biological open sets which are entirely different from Kellys pairwise open sets called $\tau_{1,2}$-open set and $\tau_1\tau_2$-open set. In 2006, M. L. Thivagar *et al*., [5] also discussed the weak forms of some an open sets with their continuity and defined various types of bitopological generalized closed sets and so on. Many researcher [7, 8] [resp. $(1,2)^*$-regular and $(1,2)^*$-semi-preopen set] studied new sets in bitopological spaces. Sheik John *et al*., [10] was various notions of topology by allowing for bitopological spaces instead of topological spaces.

In this paper, new classes of some open sets are $(i, j)^{\star}$-$\mathcal{S}$-open set, $(i, j)^{\star}$-$\mathcal{S}_{\tau}$ -set, $(i, j)^{\star}$-$\mathcal{H}$ closed set $(i, j)^{\star}$-$\alpha^{\star}$-sets and $(i, j)^{\star}$-$\mathcal{B}_{\alpha}{}^{\star}$-sets. We introduced and investigated on the line of research.

2. Preliminaries

All over this paper, (X,τ_1,τ_2) (momentarily, X) represent bitopological spaces.

Definition 2.1.

Let I be a subset of X. Then I is called $\tau_{1,2}$-open [9] if Y = O ∪Q where O $\in\tau_1$ and Q $\in\tau_2$.

The complement of $\tau_{1,2}$-open set is said to be $\tau_{1,2}$-closed.

So as to $\tau_{1,2}$-open sets but not direction of form a topology.

Proposition 2.2. *[1]* In a bitopological space (X,τ_1,τ_2), each $(1, 2)^{\star}$-semi-open set is $(1, 2)^{\star}$-β-open.

In the rest of the paper, we denote a bitopological space by (X,τ_1,τ_2), where $(X,\tau_1,\tau_2) = (X,\tau_i,\tau_j)$. The operators int and cl of a subset H of X are represent by $\tau_{i,,j}$-$int(H)$ and $\tau_{i,j}$-$cl(H)$.

In future a bitopological space (X,τ_i,τ_j) will be simply called as a space.

3. $(i, j)^\star$-$\mathcal{S}$-open sets and $(i, j)^\star$-$\mathcal{T}$-sets

Definition 3.1.

A subset H of a space (X,τ_i,τ_j) is said to be$(i, j)^\star$-$\mathcal{S}$-open if $\tau_{i,j}$-int$(\tau_{i,j}$cl(H)) $\subseteq$ $\tau_{i,j}$-cl$(\tau_{i,j}$-int(H)).

$H^c = X - H$ is $(i, j)^\star$-$\mathcal{S}$-closed.

Definition 3.2 .

A subset H of a space (X,τ_i,τ_j) is called a $(i, j)^\star$- $\mathcal{T}$–set if $\tau_{i,j}$-int(H) = $\tau_{i,j}$int$(\tau_{i,j}$-cl(H)).

Theorem 3.3. For a subset of a space (X,τ_i,τ_j), the following results are receive.

(i) any $(i, j)^\star$-α-open set is$(i, j)^\star$-$\mathcal{S}$-open.

(ii) any $(i, j)^\star$-$\mathcal{T}$-set is $(i, j)^\star$-$\mathcal{S}$-open.

Proof. (i) Let H is be a $(i, j)^\star$-α-*open* set, $H \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}int(H))$. Then $\tau_{i,j}\text{-}cl(H) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$ and $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \subseteq \tau_{i,j}\text{-}cl(H) \subseteq \tau_{i,j}cl(\tau_{i,j}\text{-}int(H))$. Thus H is $(i, j)^\star$-$\mathcal{S}$-open.

(ii) Let H is a $(i, j)^\star$-$\mathcal{T}$-*set*, we have $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) = \tau_{i,j}\text{-}int(H) \subseteq H$. Then $\tau_{i,j}\text{-}int(\tau_{i,j}cl(H)) \subseteq \tau_{i,j}\text{-}int(H) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Thus H is $(i, j)^\star$-$\mathcal{S}$-open.

□

Proposition 3.4. A subset H of a space (X,τ_i,τ_j) is$(i, j)^\star$-semi-open ⟺$\tau_{i,j}$-cl(H) = $\tau_{i,j}$-cl$(\tau_{i,j}$-int(H)).

Proof. ⇒ Assuming that$H$ is $(i, j)^\star$-semi-open set, then $H \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$ and $\tau_{i,j}cl(H) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. But $\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)) \subseteq \tau_{i,j}\text{-}cl(H)$. Hence $\tau_{i,j}\text{-}cl(H) = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$.

Conversely, let the condition hold. We have $H \subseteq \tau_{i,j}\text{-}cl(H)$ and $\tau_{i,j}\text{-}cl(H) = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Therefore H is $(i, j)^\star$-semi-open. □

Theorem 3.5. For a subset of space (X,τ_i,τ_j), the following relations are receive:

(i) each $\tau_{i,j}$-open set is $(i, j)^\star$-semi-open.

(ii) each $(i, j)^\star$-α-open set is$(i, j)^\star$-semi-open.

Proof. (i) H is a $\tau_{i,j}$-open set ⇒$H = \tau_{i,j}\text{-}int(H) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. So H is $(i, j)^\star$-semi-open.

(ii) H is an $(i, j)^\star$-α-open set ⇒ $H \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Thus H is $(i, j)^\star$-semi-open.

□

Proposition 3.6. Let H be a subset of a space (X,τ_i,τ_j). Then H is$(i, j)^\star$-β-closed ⟺ $\tau_{i,j}$int$(\tau_{i,j}$-cl$(\tau_{i,j}$-int(H))) = $\tau_{i,j}$-int(H).

Proof. ⟹ Since H is $(i, j)^\star$-β-closed set, $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) \subseteq H$ and then $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) \subseteq \tau_{i,j}\text{-}int(H)$. But $\tau_{i,j}\text{-}int(H) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)))$. Thus we have $\tau_{i,j}\text{-}int(H) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)))$.

⟸let the condition hold. We have $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = \tau_{i,j}\text{-}int(H) \subseteq H$. Therefore H is $(i, j)^\star$-β-closed. □

Theorem 3.7. For a subset H of a space (X,τ_i,τ_j), the following relations are equivalent:

(i) H is $(i, j)^\star$-semi-closed.

(ii) H is $(i, j)^\star$-β-closed and $(i, j)^\star$-$\mathcal{S}$-closed.

Proof. (1)⇒ (2): Let H be $(i, j)^\star$-$\mathcal{S}$-semi-closed. By Proposition 2.2, H is $(i, j)^\star$-$\mathcal{S}$-β-closed. Since H is $(i, j)^\star$-$\mathcal{S}$-semi-closed, $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \subseteq H$ and $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \subseteq \tau_{i,j}\text{-}int(H)$. It gives that $\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H))) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Thus $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}cl(H))) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$ and so H is $(i, j)^\star$-$\mathcal{S}$-closed.

(2)⇒(1): Since H is $(i, j)^{\star}$-$\mathcal{S}$-closed, $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$ and $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)))$. Since H is $(i, j)^{\star}$-β-closed, $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))) \subseteq H$. Then $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq H$ and so H is $(i, j)^{\star}$-semi-closed. □

Remark: In a space (X,τ_i,τ_j), then the notions of $(i, j)^{\star}$-β-closed sets and the notions of $(i, j)^{\star}$-$\mathcal{S}$-closed sets are independent.

Example 3.8. Let $X = \{j_1,j_2,j_3,j_4\}$ with $\tau_i = \{\phi,X,\{j_2\},\{j_3,j_4\}\}$ and $\tau_j = \{\phi,X,\{j_2,j_3,j_4\}\}$ then the $\tau_{i,j} = \{\phi,X,\{j_2\},\{j_3,j_4\},\{j_2,j_3,j_4\}\}$. In the space (X,τ_i,τ_j), then

(i) the subset $\{j_2,j_3,j_4\}$ is $(i, j)^{\star}$-β-closed but not $(i, j)^{\star}$-$\mathcal{S}$-closed.

(ii) the subset $\{j_1,j_3\}$ is $(i, j)^{\star}$-$\mathcal{S}$-closed but not $(i, j)^{\star}$-β-closed.

Theorem 3.9.

In a space (X,τ_i,τ_j). Then a subset of X is $(i, j)^{\star}$-α-open ⇔ it is both $(i, j)^{\star}$-$\mathcal{S}$-open and $(i, j)^{\star}$-pre-open.

Proof. ⟹ Let H be an $(i, j)^{\star}$-α-open set. Then $H \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)))$. It implies that $\tau_{i,j}$-$cl(H) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$ and $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))) \subseteq \tau_{i,j}cl(\tau_{i,j}$-$int(H))$. Hence, H is a $(i, j)^{\star}$-$\mathcal{S}$-open set. On the other hand, since H is an $(i, j)^{\star}$-α-open set, H is a $(i, j)^{\star}$-pre-open set.

⟸ Let H be both $(i, j)^{\star}$-$\mathcal{S}$-open and $(i, j)^{\star}$-pre-open. Since H is $(i, j)^{\star}$-$\mathcal{S}$-open, we have $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$ and hence $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)))$. Since H is $(i, j)^{\star}$-pre-open, we have $H \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(H))$. Therefore we obtain that $H \subseteq \tau_{i,j}int(\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)))$ which proves that H is an $(i, j)^{\star}$-α-open set. □

Remark. In a space (X,τ_i,τ_j), then the notions of $(i, j)^{\star}$-$\mathcal{S}$-open sets and the notions of $(i, j)^{\star}$-pre-open sets are independent.

Example 3.10. Let $X= \{j_1,j_2,j_3,j_4\}$ with $\tau_i= \{\phi,X,\{j_1\},\{j_2,j_4\}\}$ and $\tau_j=\{\phi,X,\{j_1,j_2,j_4\}\}$ then the $\tau_{i,j} = \{\phi,X,\{j_1\},\{j_2,j_4\},\{j_1,j_2,j_4\}\}$. In the space, then

(i) the subset $\{j_1,j_3\}$ is $(i, j)^{\star}$-$\mathcal{S}$-open but not $(i, j)^{\star}$-pre-open.

(ii) the subset $\{j_1,j_2,j_4\}$ is $(i, j)^{\star}$-pre-open but not $(i, j)^{\star}$-$\mathcal{S}$-open.

Proposition 3.11. Two subsets H and K of a space (X,τ_i,τ_j). If $H \subseteq K \subseteq \tau_{i,j}$-cl(H) and H is $(i, j)^{\star}$-$\mathcal{S}$-open ⟹ K is $(i, j)^{\star}$-$\mathcal{S}$-open.

Proof. Assuming that $H \subseteq K \subseteq \tau_{i,j}$-$cl(H)$ and H is $(i, j)^{\star}$-$\mathcal{S}$-open in X. Then, we have $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$. Since $H \subseteq K$, $\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(K))$ and $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(K))$. Since $K \subseteq \tau_{i,j}$-$cl(H)$, we have $\tau_{i,j}$-$cl(K) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$cl(H)) = \tau_{i,j}$-$cl(H)$ and $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(K)) \subseteq \tau_{i,j}$-$int(\tau_{i,j}$-$cl(H))$. Therefore $\tau_{i,j}$-$int(\tau_{i, j}$-$cl(K)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(K))$. This shows that K is a $(i, j)^{\star}$-$\mathcal{S}$-open set. □

Corollary 3.12. For a space (X,τ_i,τ_j). If $H \subseteq X$ is $(i, j)^{\star}$-$\mathcal{S}$-open and $(i, j)^{\star}$-dense in (X,τ_i,τ_j), then each subset of X containing H is $(i, j)^{\star}$-$\mathcal{S}$-open.

Proof. It is obvious by Proposition 3.11. □

Proposition 3.13. In a space (X,τ_i,τ_j), each $\tau_{i,j}$-closed set is a $(i, j)^{\star}$-$\mathcal{T}$-set.

Proof. Let H be a $\tau_{i,j}$-closed set. Then $H = \tau_{i,j}$-$cl(H)$ and we have $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) = \tau_{i,j}$-$int(H)$ which proves that H is a $(i, j)^{\star}$-$\mathcal{T}$-set. □

Remark. The reverse part of Proposition 3.13 is need not true from the following Example.

Example 3.14. In Example 3.8, then the subset $\{j_1\}$ is $(i, j)^{\star}$-$\mathcal{T}$-set but not $\tau_{1,2}$-closed.

Theorem 3.15.

A subset H of a space (X,τ_i,τ_j) is $(i, j)^{\star}$-semi-closed ⇔ H is a $(i, j)^{\star}$-$\mathcal{T}$-set.

Proof. Let H be a $(i, j)^\star$-semi-closed set in X. Then $X-H$ is$(i, j)^\star$-semi-open. By Proposition 3.4, we have $\tau_{i,j}$-$cl(X-H) = \tau_{i,j}$-$cl(\tau_{i,j}$-$int(X-H))$. It follows that $X - \tau_{i,j}$-$int(H) = \tau_{i,j}$-$cl(X - \tau_{i,j}$-$cl(H)) = X - \tau_{i,j}$-$int(\tau_{i,j}$-$cl(H))$. Thus, $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) = \tau_{i,j}$-$int(H)$ and hence H is a $(i, j)^\star$-$\mathcal{T}$-set in X.

On the other hand side, let H be a$(i, j)^\star$-$\mathcal{T}$-set. Then $\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H)) = \tau_{i,j}$-$int(H) \subseteq H$. Therefore H is $(i, j)^\star$-semi-closed. □

4. $(i, j)^\star$-$\mathcal{S}_{\mathcal{T}}$-set

Definition 4.1.

A subset H of a space(X, τ_i, τ_j) issaid to be a$(i, j)^\star$-$\mathcal{S}_{\mathcal{T}}$-setif H is$(i, j)^\star$-semi-open and a $(i, j)^\star$-$\mathcal{T}$-set.

Theorem 4.2. Let H be a subset of a space(X,τ_i,τ_j). Then H is $(i, j)^\star$-$\mathcal{S}_{\mathcal{T}}$-set$\Leftrightarrow$H is both $(i, j)^\star$-β-open and $(i, j)^\star$-semi-closed.

Proof. H is $(i, j)^\star$-$\mathcal{S}_{\mathcal{T}}$-set$\Rightarrow H$ is both $(i, j)^\star$-semi-open and a $(i, j)^\star$-$\mathcal{T}$-*set*. Since every$(i, j)^\star$-semi-open set is $(i, j)^\star$-β-open, H is both $(i, j)^\star$-β-open and a $(i, j)^\star$- $\mathcal{T}$-*set*. By Theorem 3.15, we obtain the result.

Conversely, let H be $(i, j)^\star$-semi-closed and$(i, j)^\star$-β-open. Since H is a $(i, j)^\star$-semi-closed, by Theorem 3.18 H is a $(i, j)^\star$-$\mathcal{T}$-*set*. Since H is $(i, j)^\star$-β-open, $H \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(\tau_{i,j}$-$cl(H))) = \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$. Therefore H is $(i, j)^\star$-semi-open. Since H is both $(i, j)^\star$-semi-open and a $(i, j)^\star$- $\mathcal{T}$-*set*, H is $(i, j)^\star$-$\mathcal{S}_{\mathcal{T}}$-set. □

Remark. In a space (X,τ_i,τ_j), then the notions of$(i, j)^\star$-β-open and the notions of $(\mathbf{i}, \mathbf{j})^\star$-semi-closed are independent.

Example 4.3. In Example 3.10, then
(i) the subset $\{j_2\}$ is $(i, j)^\star$-β-open but not $(i, j)^\star$-semi-closed.
(ii) the subset $\{j_3\}$ is $(i, j)^\star$-semi-closed but not $(i, j)^\star$-β-open.

5. $(i, j)^\star$-$\mathcal{H}$-closed sets

Definition 5.1.

A subset H of a space (X,τ_i,τ_j) is called $(i, j)^\star$-$\mathcal{H}$-closedif $H = \tau_{i,j}$-cl$(\tau_{i,j}$-int(H)). $H^c = X-H$ is $(i, j)^\star$-$\mathcal{H}$-open.

Theorem 5.2. Let H a subset of a space (X,τ_i,τ_j). Then the following relations are equivalent.

(i) $H = \phi$ is $(i, j)^\star$-$\mathcal{H}$-closed.
(ii) There exists a non-empty $\tau_{i,j}$-open set F : $F \subseteq H = \tau_{i,j}$-cl(F).
(iii) There exists a non-empty $\tau_{i,j}$-open set F : $H = F \cup (\tau_{i,j}$-cl(F) $-F)$.

Proof. (1) $\Longrightarrow$ (2). Suppose $H = \phi$is a $(i, j)^\star$-$\mathcal{H}$-closed set. Then $H = \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$. Let $F = \tau_{i,j}$-$int(H)$. F is the required $\tau_{i,j}$-open set. $F \subseteq H = \tau_{i,j}$-$cl(F)$.
(2)$\Rightarrow$(3). Since$H = \tau_{i,j}$-$cl(F) = F \cup (\tau_{i,j}$-$cl(F) - F)$ where F is a nonempty $\tau_{1,2}$-open set,(3) follows.
(3)$\Rightarrow$(1).$H = F \cup (\tau_{i,j}$-$cl(F) - F)$ implies that $H = \tau_{i,j}$-$cl(F) = \tau_{i,j}$-$cl(\tau_{i,j}$-$int(F)) \subseteq \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$, since F is $\tau_{i,j}$-open and $F \subseteq H$. Again $\tau_{i,j}$-$int(H) \subseteq H$ implies that $\tau_{i,j}$-$cl(\tau_{i,j}$-$int(H)) \subseteq \tau_{i,j}$-$cl(H) = \tau_{i,j}$-$cl(F) = H$. Therefore $H = \tau_{i,j}$-$cl(\tau_{i,j}$-$int(H))$. H is $(i, j)^\star$-$\mathcal{H}$-closed. □

Theorem 5.3.

Let H be a subset of a space (X,τ_i,τ_j). If H is $(i, j)^\star$-β-open, then $\tau_{i,j}$-cl(H) is $(i, j)^\star$-$\mathcal{H}$-closed.

Proof. Assuming that H is $(i, j)^\star$-β-open. Then $H \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)))$ and so $\tau_{i,j}\text{-}cl(H) \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H))) \subseteq \tau_{i,j}\text{-}cl(H)$ which implies that $\tau_{i,j}\text{-}cl(H) = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)))$. Therefore $\tau_{i,j}\text{-}cl(H)$ is $(i, j)^\star$-$\mathcal{H}$-closed. □

Theorem 5.4. For a subset H of a space (X,τ_i,τ_j). Then the following properties are equivalent.

(i) H is $(i, j)^\star$-$\mathcal{H}$-closed.

(ii) H is $(i, j)^\star$-semi-open and $\tau_{i,j}$-closed.

(iii) H is $(i, j)^\star$-β-open and $\tau_{i,j}$-closed.

Proof. (1) ⟹ (2). H is $(i, j)^\star$-$\mathcal{H}$-closed ⟹ $H = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$ and $\tau_{i,j}\text{-}cl(H) = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Since $H \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$, H is $(i, j)^\star$-semi-open. Also, $H = \tau_{i,j}\text{-}cl(H)$ and so H is $\tau_{i,j}$-closed.

(2)⇒(3). It follows from the fact that every $(i, j)^\star$-semi-open set is a $(i, j)^\star$-β-open.

(3)⇒(1). H is $(i, j)^\star$-β-open and $(i, j)^\star$-closed ⟹ $H \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)))$ and $H = \tau_{i,j}\text{-}cl(H)$. Now $\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)) \subseteq \tau_{i,j}\text{-}cl(H) = H$. Also, $H \subseteq \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Therefore $H = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$ which implies that H is $(i, j)^\star$-$\mathcal{H}$-closed. □

Remark. In a space (X,τ_i,τ_j), then

(i) the notions of $(i, j)^\star$-semi-open sets and the notions of $\tau_{i,j}$-closed sets are independent.

(ii) the notions of $(i, j)^\star$-β-open sets and the notions of $\tau_{i,j}$-closed sets are independent.

Example 5.5. In Example 3.10, then (i) the subset $\{j_1\}$ is $(i, j)^\star$-semi-open but not $\tau_{i,j}$-closed. (ii) the subset $\{j_3\}$ is $\tau_{i,j}$-closed but not $(i, j)^\star$-semi-open.

6. $(i,j)^\star$-$\alpha^\star$-sets and $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-sets

Definition 6.1. A subset H of a space (X,τ_i,τ_j) is called

(i) $(i,j)^\star$-$\alpha^\star$-set if $\tau_{i,j}$-cl($\tau_{i,j}$-int($\tau_{i,j}$-cl(H))) = $\tau_{i,j}$-int(H).

(ii) $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set if H = L ∩ M, where L is $\tau_{i,j}$-open and M is $(i,j)^\star$-$\alpha^\star$-set.

Proposition 6.2. For a subset H of a space (X,τ_i,τ_j), the following are equivalent.

(i) H ∈ $(i,j)^\star$-$\alpha^\star(X,\tau_i,\tau_j)$.

(ii) H is $(i,j)^\star$-semi-pre closed.

(iii) $\tau_{i,j}$-int(H) is $(\mathbf{i},\mathbf{j})^\star$-regular open.

Proof. Obvious. □

Proposition 6.3. In a space (X,τ_i,τ_j), then

(i) H is a $(i,j)^\star$-$\mathcal{T}$-set⇒H ∈ $(i,j)^\star$-$\alpha^\star$ (X, τ_i, τ_j).

(ii) H is $(i,j)^\star$-semi-open and $(i,j)^\star$-$\mathcal{T}$-set ⟺ H ∈ $(i,j)^\star$-$\alpha^\star$ (X, τ_i, τ_j).

(iii) H is $(i,j)^\star$-α-open and H ∈ $(i,j)^\star$-$\alpha^\star$ (X, τ_i, τ_j)⟺H is $(i,j)^\star$-regular-open.

Proof. (i) Let H be a $(i,j)^\star$-$\mathcal{T}$-set, then $\tau_{i,j}\text{-}int(H) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H))$ and $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) = \tau_{i,j}\text{-}int(H)$. Therefore, H is an $(i,j)^\star$-$\alpha^\star$-set.

(ii) Let H be $(i,j)^\star$-semi-open and $H \in (i,j)^\star\text{-}\alpha^\star$ (X, τ_i, τ_j)). Since H is $(i,j)^\star$-semi-open, $\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)) = \tau_{i,j}\text{-}cl(H)$ and hence $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = \tau_{i,j}\text{-}int(H)$. Therefore, H is a $(i,j)^\star$-$\mathcal{T}$-set.

(iii) Let H be an $(i,j)^\star$-α-*open* set and $H \in (i,j)^\star\text{-}\alpha^\star$ (X, τ_i, τ_j). By Proposition 6.2 and the Definition of $(i,j)^\star$-α-*open* set, we have $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = H$ and hence $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = H$. The converse is obvious.

□

Remark. In a space (X,τ_i,τ_j), the union of two $(i,j)^\star$-$\alpha^\star$-sets but not$(i,j)^\star$-$\alpha^\star$-set. As shown in the following Example.

Example 6.4. Let $X = \{j_1,j_2,j_3\}$ with $\tau_i = \{\phi,X,\{j_1\}\}$ and $\tau_j = \{\phi,X,\{j_3\}\}$ then the $\tau_{i,j} = \{\phi,X,\{j_1\},\{j_3\},\{j_1,j_3\}\}$. In the space (X,τ_i,τ_j), then the subsets $A = \{j_1\}$ and $B = \{j_3\}$ are $(i,j)^\star$-$\alpha^\star$-set. But $A \cup B = \{j_1,j_3\}$ is not$(i,j)^\star$-$\alpha^\star$-set.

Proposition 6.5. In a space (X,τ_i,τ_j), then $(\mathbf{i},\mathbf{j})^\star$-$\alpha^\star$-$(X,\tau_i,\tau_j) \subseteq (i,j)^\star$-$\mathcal{B}_{\alpha^\star}(X,\tau_i,\tau_j)$ and $\tau_{i,j}$-open $\subseteq (i,j)^\star$-$\mathcal{B}_{\alpha^\star}(X,\tau_i,\tau_j)$.

Proof. Since $X \in \tau_{i,j}$-open $\cap (\boldsymbol{i},\boldsymbol{j})^\star$-$\alpha^\star$-$(X, \tau_i, \tau_j)$, the inclusions are obvious. □

Lemma 6.6. In a space (X,τ_i,τ_j). If either H,K is $(\mathbf{i},\mathbf{j})^\star$-semi-open, then $\tau_{i,j}$-int($\tau_{i,j}$-cl$(H \cap K)$) = $\tau_{i,j}$-int($\tau_{i,j}$-cl(H)) $\cap$ $\tau_{i,j}$-int($\tau_{i,j}$-cl(K)).

Proof. For any subset $H,K \subseteq X$, we generally have $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H \cap K)) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(K))$. Assume that H is $(\boldsymbol{i},\boldsymbol{j})^\star$-semi-open. Then we have $\tau_{i,j}\text{-}cl(H) = \tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))$. Therefore $\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(K)) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H)) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(K)))) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}cl(H) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(K)))) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H)) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(K)))) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H) \cap \tau_{i,j}\text{-}cl(K))) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H) \cap K)) \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(H \cap K))$.□

Proposition 6.7.

A subset H is $\tau_{i,j}$-open in a space (X,τ_i,τ_j)) $\Leftrightarrow$ it is a $(\mathbf{i},\mathbf{j})^\star$-$\alpha$-open set and a$(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set.

Proof. It is obvious that every $\tau_{i,j}$-open set is a $(\boldsymbol{i},\boldsymbol{j})^\star$-$\alpha$-open set and a $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set. let H be a $(\boldsymbol{i},\boldsymbol{j})^\star$-$\alpha$-open set and a $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set. Since H is a $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set, there exist $G \in \tau_{i,j}$-open and $Q \in (i,j)^\star$-$\alpha^\star(X,\tau_i,\tau_j)$ such that $H = G \cap Q$. Since H is a $(\boldsymbol{i},\boldsymbol{j})^\star$-$\alpha$-open set, by using Lemma 6.7, we have $H \subseteq \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(H))) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(G \cap Q))) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(G)) \cap \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(\tau_{i,j}\text{-}int(Q))) = \tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(G)) \cap \tau_{i,j}\text{-}int(Q)$ and hence $H = G \cap H \subseteq G \cap (\tau_{i,j}\text{-}int(\tau_{i,j}\text{-}cl(G)) \cap \tau_{i,j}\text{-}int(Q)) = G \cap \tau_{i,j}\text{-}int(Q) \subseteq H$. Consequently, we obtain $H = G \cap \tau_{i,j}\text{-}int(H)$ and H is $\tau_{i,j}$-open. □

Remark. The following example shows that the family of $(i,j)^\star$-α-open set and the family of $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set are independent.

Example 6.8. Let $X= \{j_1,j_2,j_3,j_4\}$ with $\tau_i = \{\phi,X,\{j_1\}\}$ and $\tau_j = \{\phi,X,\{j_3\}\}$ then the $\tau_{i,j} = \{\phi,X,\{j_1\},\{j_3\},\{j_1,j_3\}\}$. In the space (X,τ_i,τ_j), then

(i) $\{j_1,j_2,j_3\}$ is $(i,j)^\star$-α-open set but not $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set.

(ii) $\{j_2\}$ is $(i,j)^\star$-$\mathcal{B}_{\alpha^\star}$-set but not $(i,j)^\star$-α-open set.

References

[1] K. M. Dharmalingam and M. Ramaboopathi, *New forms of generalized closed sets in bitopological spaces*, Journal of Applied Science and Computations, VI(III)(2019), 712-718.

[2] J. C. Kelly, *Bitopological spaces*, Proc. London. Math Soc., 3(13)(1963), 71-89.

[3] M. Lellis Thivagar, *Generalization of pairwise α-continuous functions*, Pure and Applied Mathematicka Sciences., 28(1991), 55-63.

[4] M. Lellis Thivagar and O. Ravi, *On stronger forms of* (1,2)*-*quotient mappings in bitopological spaces*, Int. J. Math. Game theory and Algebra. Vol. 14, No. 6, (2004), 481-492.

[5] M. Lellis Thivagar and O. Ravi, *A bitopological* $(1,2)^*$-*semi generalized continuous mappings*, Bull. Malayasian Math. Soc., 29(1) (2006), 79-88.

[6] C. W. Patty, *Bitopological spaces*, Duke Math. J., 34 (1967), 387-392.

[7] O. Ravi, E. Ekici and M. Lellis Thivagar, *On* $(1,2)^*$-*sets and decompositions of bitopological* $(1,2)^*$-*continuous mappings*, Kochi J. Math., 3 (2008), 181-189

[8] O. Ravi and M. Lellis Thivagar, *A bitopological* $(1,2)^*$-*semi-generalized continuous maps*, Bull. Malaysian Math. Sci. Soc., (2)(29)(1)(2006), 76-88.

[9] O. Ravi and M. Lellis Thivagar, *On stronger forms of* (1,2)*-*quotient mappings in bitopological spaces*, Internat. J. Math. Game Theory and Algebra., 14(6)(2004), 481-492.

[10] M. Sheik John and P. Sundaram, *g*-closed sets in bitopological spaces*, Indian J. Pure Appl. Math., 35(1)(2004), 71-80.