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PERFECT DOMINATING SETS AND PERFECT DOMINATION POLYNOMIAL OF A CYCLE

[1] A.M. Anto, [2] P. Paul Hawkins and T. Shyla Isac Mary

[1] Assistant Professor, Department of Mathematics,

Malankara Catholic College, Mariagiri, Tamil Nadu, India.

antoalexam@gmail.com

[2]Research Scholar,Reg.no 18223112091013,

Research Department of Mathematics,Nesamony Memorial Christian College,

Marthandam, Tamil Nadu, India. hawkinspaul007@gmail.com

[3]A Assistant Professor, Research Department of Mathematics,

*Nesamony Memorial Christian College, Marthandam, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627012,Tamilnadu,India.*shylaiasscmary@yahoo.in

Abstract Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a perfect dominating set of G, if every vertex u in $V - S$ is adjacent to exactly one vertex in S. Let $C_{pf}(n, i)$ be the family of all perfect dominating sets of a Cycle C_n with cardinality i, and let $d_{pf}(C_n, i) = |C_{pf}(n, i)|$. In this paper, we construct $C_{pf}(n, i)$ for a Cycle C_n with n vertices and obtain a recursive formula of $d_{pf}(C_n, i)$. Using this recursive formula, we consider the polynomial $D_{pf}(C_n, x) = \sum_{i=\gamma_{pf}(C_n)}^{n} d_{pf}(C_n, i)\, x^i$, which we call perfect domination polynomial of Cycle and obtain some properties of this polynomial.

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V | uv \in E\}$. A set $S \subseteq V$ is a dominating set of G, if every vertex $u \in V$ is a element of S or is adjacent to an element of S [8]. The dominating set S is a perfect dominating set if $|N(u) \cap S| = 1$ for each $u \in V - S$[8], or equivalently, if every vertex u in $V - S$ is adjacent to exactly one vertex in S. The perfect domination number γ_{pf} is the minimum cardinality of a perfect dominating set in G. A closed trail whose origin and internal vertices are distinct is called cycle [3]. Let C_n be a cycle with n vertices. Let $C_{pf}(n, i)$ be the family of perfect dominating sets of a Cycle C_n with cardinality i and let $d_{pf}(C_n, i) = |C_{pf}(n, i)|$. We call the polynomial $D_{pf}(C_n, x) = \sum_{i=\gamma_{pf}(C_n)}^{n} d_{pf}(C_n, i)\, x^i$, the perfect domination polynomial of the cycle C_n. We use $\lceil n \rceil$, for the smallest integer greater than or equal to n. In this article we denote the set $\{1,2,\ldots,n\}$ simply by $[n]$.

2. Perfect Dominating sets of a Cycle

Let $C_{pf}(n, i)$ be the family of perfect dominating sets of the cycle C_n with cardinality i. We are going to investigate perfect dominating sets of cycle C_n.
We required the following lemmas to prove our main results in this article.

Lemma 2.1

i) $\gamma_{pf}(C_{3n}) = n$ for $n \in N$
ii) $\gamma_{pf}(C_{3n+1}) = \left\lceil \frac{3n+1}{3} \right\rceil$ for $n \in N$
iii) $\gamma_{pf}(C_{3n+2}) = n + 2$ for $n \in N$

By Lemma 2.1 and the definition of perfect domination number, we have the following lemma.

Lemma 2.2

For $n \geq 3$, $C_{pf}(n, i) = \emptyset$, if and only if $i > n$

Proof: Let C_n be a cycle with n vertices and any member of $C_{pf}(n, i)$ contains atmost n vertices. Therefore $C_{pf}(n, i) = \emptyset$ for $i > n$. Conversely, if $i > n$ then by definition of perfect dominating set of cycle C_n we have $C_{pf}(n, i) = \emptyset$.

Lemma 2.3

If G be a Cycle of length $3k - 1$, then every perfect dominating set of G must contains at least $k + 1$ vertices of G.

Lemma 2.4 For $n \geq 3$,

i) $C_{pf}(n, n-1) = \emptyset$
ii) $C_{pf}(n, n) = \{[n]\}$
iii) $C_{pf}(n, n-2) = \{\{2,3,4, \ldots, n-1\}, \{[n] - \{i, i+1\} | i = 1,2,3, \ldots n-1\}$

Proof: i) Let $G = (V, E)$ be a cycle of n vertices.
Suppose to the contrary we assume that $C_{pf}(n, n-1) \neq \emptyset$ then let T is a Perfect dominating set of G with cardinality $n - 1$. Let $v \in G - T$, then the neighborhood of v is adjacent to two vertices of T.
Which implies $|N(v) \cap T| \neq 1$. Which contradicts to the definition of Perfect dominating set. Hence, $C_{pf}(n, n-1) = \emptyset$
$ii)$ We know that for any $G = (V, E)$, $V(G)$ is always a perfect dominating set of G. Hence, $C_{pf}(n, n) = \{[n]\}$
$iii)$ Here, $C_{pf}(n, n-2)$ is the family of perfect dominating sets with cardinality $n - 1$ in a cycle of n vertices. Clearly, $\{2,3,4, \ldots, n-1\}$ and $\{[n] - \{i, i+1\} | i = 1,2,3, \ldots n-1\}$ are the possible perfect dominating sets with cardinality $n - 2$.

Lemma 2.5

For $n \in N$, $C_{pf}(3n, n) = \{\{1,4,7, \ldots, 3n-2\} \cup \{2,5,8, \ldots, 3n-1\} \cup \{3,6,9, \ldots, 3n\}\}$

Lemma 2.6

If $C_{pf}(n, i)$ be a family of perfect dominating sets with cardinality i then,

$$|C_{pf}(n, i)| = |C_{pf}(n-1, i-1)| + |C_{pf}(n-3, i-1)|$$

3. Perfect Domination Polynomial of a Cycle

Definition 3.1

Let C_n be a Cycle with n vertices. Let $C_{pf}(n,i)$ be the family of perfect dominating sets of a Cycle C_n with cardinality i and $d_{pf}(C_n,i) = |C_{pf}(n,i)|$. Then the Perfect dominating polynomial of a cycle C_n is given by $D_{pf}(C_n,x) = \sum_{i=\gamma_{pf}(C_n)}^{n} d_{pf}(C_n,i)\, x^i$.

Example 3.2 Consider a cycle C_5 in Fig 1.

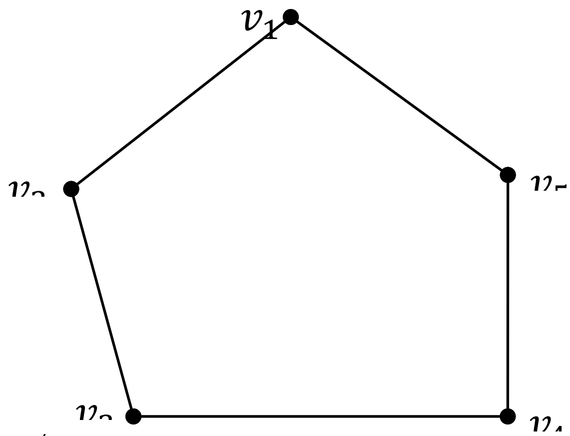

Fig 1

Here, $C_{pf}(5,1) = \emptyset$

$C_{pf}(5,2) = \emptyset$

$C_{pf}(5,3) = \{\{v_1,v_2,v_3\},\{v_1,v_2,v_5\},\{v_2,v_3,v_4\},\{v_3,v_4,v_5\},\{v_1,v_4,v_5\}\}$

$C_{pf}(5,4) = \emptyset$

$C_{pf}(5,5) = \{\{v_1,v_2,v_3,v_4,v_5\}\}$

Therefore, $d_{pf}(C_5,1) = 0$, $d_{pf}(C_5,2) = 0$, $d_{pf}(C_5,3) = 5$,
$d_{pf}(C_5,4) = 0$ and $d_{pf}(C_5,5) = 1$.

Hence, $D_{pf}(C_5,x) = 5x^3 + x^5$ is a perfect dominating set of cycle C_5.

Theorem 3.3

If $C_{pf}(n,i)$ be a family of perfect dominating sets with cardinality i then,
For every $n \geq 6$, $D_{pf}(C_n,x) = x[D_{pf}(C_{n-1},x) + D_{pf}(C_{n-3},x)]$ with initial values $D_{pf}(C_3,x) = 3x + x^3$, $D_{pf}(C_4,x) = 4x^2 + x^4$, $D_{pf}(C_5,x) = 5x^3 + x^5$.

Proof : It follows from the definition of perfect domination polynomial and Lemma 2.6

Using the above theorem we obtain $d_{pf}(C_n,i)$ for $3 \leq n \leq 15$ as shown in the following table

i / n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3	0	1												
4	0	4	0	1											
5	0	0	5	0	1										
6	0	3	0	6	0	1									
7	0	0	7	0	7	0	1								
8	0	0	0	12	0	8	0	1							

9	0	0	3	0	18	0	9	0	1						
10	0	0	0	10	0	25	0	10	0	1					
11	0	0	0	0	22	0	33	0	11	0	1				
12	0	0	0	3	0	40	0	42	0	12	0	1			
13	0	0	0	0	13	0	65	0	52	0	13	0	1		
14	0	0	0	0	0	35	0	98	0	63	0	14	0	1	
15	0	0	0	0	3	0	75	0	140	0	75	0	15	0	1

Table 1

Example 3.4. Consider a cycle C_6 with 6 vertices. We construct a perfect dominating polynomial $\mathrm{D}_{pf}(C_6, x)$ using Theorem 3.2 and Table 1 . Then the Perfect dominating polynomial of a cycle C_6 is given by $D_{pf}(C_6, x) = \sum_{i=\gamma_{pf}(C_6)}^{6} d_{pf}(C_6, i)\, x^i$ From the Table 1 we have $d_{pf}(C_6, 2) = 1$, $d_{pf}(C_6, 3) = 4$, $d_{pf}(C_6, 4) = 4$, $d_{pf}(C_6, 5) = 2$, $d_{pf}(C_6, 6) = 1$. Hence, $D_{pf}(C_6, x) = x^2 + 4x^3 + 4x^4 + 2x^5 + x^6$.

Theorem 3.5

The coefficients of $D_{pf}(C_n, x)$ have the following properties

$i)$ $d_{pf}(C_{3n}, n) = 3$ for every $n \in N$

$ii)$ $d_{pf}(C_n, n) = 1$ for every $n \geq 3$

$iii)$ $d_{pf}(C_n, n-1) = 0$ for every $n \geq 3$

$iv)$ $d_{pf}(C_n, n-2) = n$ for every $n \geq 3$

$v)$ $d_{pf}(C_n, n-3) = 0$ for every $n \geq 4$

$vi)$ $d_{pf}(C_n, n-4) = \frac{n(n-5)}{2}$ for every $n \geq 5$

$vii)$ $d_{pf}(C_n, n-5) = 0$ for every $n \geq 6$

$viii)$ $d_{pf}(C_n, n-6) = \frac{n(n-7)(n-8)}{6}$ for every $n \geq 7$

$ix)$ $d_{pf}(C_n, n-7) = 0$ for every $n \geq 8$

$x)$ $d_{pf}(C_n, n-8) = \frac{n(n-9)(n-10)(n-11)}{24}$ for every $n \geq 9$

$xi)$ $d_{pf}(C_n, n-9) = 0$ for every $n \geq 10$

$xii)$ $d_{pf}(C_n, n-10) = \frac{n(n-11)(n-12)(n-13)(n-14)}{120}$ for every $n \geq 11$

Proof: $i)$ By lemma 2.5 we have $d_{pf}(C_{3n}, n) = \left|\mathrm{C}_{pf}(3n, n)\right| = 3$ for every $n \in N$

$ii)$ We have by Lemma 2.4 (ii) $d_{pf}(C_n, n) = \left|\mathrm{C}_{pf}(n, n)\right| = 1$. Hence the result.

$iii)$ It follows from the definition of perfect dominating set.

$iv)$ We prove this result by induction on n. The result is true for $n = 3$. Because, $\mathrm{C}_{pf}(3,1) = \{\{1\},\{2\},\{3\}\}$. So, $d_{pf}(C_3, 1) = \left|\mathrm{C}_{pf}(3,1)\right| = 3$.

Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.6 $d_{pf}(C_n, n-2) = d_{pf}(C_{n-1}, n-3) + d_{pf}(C_{n-3}, n-3)$.

Then, by induction hypothesis $d_{pf}(C_{n-1}, n-3) = n-1$.

Also by part (ii) we have $d_{pf}(C_{n-3}, n-7) = 1$, Therefore, $d_{pf}(C_n, n-6) = n-1+1 = n$

$v)$ It follows from the definition of perfect dominating set.

vi) We prove this result by induction on n. The result is true for $n = 5$. Because, $\mathrm{C}_{pf}(5,1) = \{\emptyset\}$. So, $d_{pf}(C_5, 1) = |\mathrm{C}_{pf}(5,1)| = 0$.
Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.6 $d_{pf}(C_n, n-4) = d_{pf}(C_{n-1}, n-5) + d_{pf}(C_{n-3}, n-5)$.
Then, by induction hypothesis $d_{pf}(C_{n-1}, n-5) = \frac{(n-1)(n-6)}{2}$.
Also, by part (iv) we have $d_{pf}(C_{n-3}, n-5) = n-3$.
Therefore,
$$d_{pf}(C_n, n-6) = \frac{(n-1)(n-6)}{2} + n - 3 = \frac{(n-1)(n-6)+2(n-3)}{2} = \frac{n(n-5)}{2}$$
Hence, by the principle of induction, the result is true for all $n \geq 5$
vii) It follows from the definition of perfect dominating set.
$viii$) We prove this result by induction on n. The result is true for $n = 7$. Because, $\mathrm{C}_{pf}(7,1) = \{\emptyset\}$. So, $d_{pf}(C_7, 1) = |\mathrm{C}_{pf}(7,1)| = 0$.
Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.6 $d_{pf}(C_n, n-6) = d_{pf}(C_{n-1}, n-7) + d_{pf}(C_{n-3}, n-7)$.
Then, by induction hypothesis $d_{pf}(C_{n-1}, n-7) = \frac{(n-1)(n-8)(n-9)}{6}$.
Also by part (vi) we have $d_{pf}(C_{n-3}, n-7) = \frac{(n-3)(n-8)}{2}$.
Therefore,
$$d_{pf}(C_n, n-6) = \frac{(n-1)(n-8)(n-9)}{6} + \frac{(n-3)(n-8)}{2} = \frac{(n-1)(n-8)(n-9)+3(n-3)(n-8)}{6} = \frac{n(n-7)(n-8)}{6}$$
Hence, by the principle of induction, the result is true for all $n \geq 7$.
ix) It follows from the definition of perfect dominating set.
x) We prove this result by induction on n. The result is true for $n = 9$. Because, $\mathrm{C}_{pf}(9,1) = \{\emptyset\}$. So, $d_{pf}(C_9, 1) = |\mathrm{C}_{pf}(9,1)| = 0$.
Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.6 $d_{pf}(C_n, n-8) = d_{pf}(C_{n-1}, n-9) + d_{pf}(C_{n-3}, n-9)$.
Then, by induction hypothesis $d_{pf}(C_{n-1}, n-9) = \frac{(n-1)(n-10)(n-11)(n-12)}{24}$.
Also by part $(viii)$ we have $d_{pf}(C_{n-3}, n-9) = \frac{(n-3)(n-10)(n-11)}{6}$.
Therefore,
$$d_{pf}(C_n, n-8) = \frac{(n-1)(n-10)(n-11)(n-12)}{24} + \frac{(n-3)(n-10)(n-11)}{6}. = \frac{(n-1)(n-10)(n-11)(n-12)+4(n-3)(n-10)(n-11)}{24} = \frac{n(n-9)(n-10)(n-11)}{24}$$
Hence, by the principle of induction, the result is true for all $n \geq 9$
xi) It follows from the definition of perfect dominating set.
xii) We prove this result by induction on n. The result is true for $n = 11$. Because, $\mathrm{C}_{pf}(11,1) = \{\emptyset\}$. So, $d_{pf}(C_{11}, 1) = |\mathrm{C}_{pf}(11,1)| = 0$.
Now, assume that the result is true for all natural numbers less than n. For n, we have by lemma 2.6 $d_{pf}(C_n, n-10) = d_{pf}(C_{n-1}, n-11) + d_{pf}(C_{n-3}, n-11)$.
Then, by induction hypothesis $d_{pf}(C_{n-1}, n-11) = \frac{(n-1)(n-12)(n-13)(n-14)(n-15)}{120}$.
Also by part (x) we have $d_{pf}(C_{n-3}, n-11) = \frac{(n-3)(n-12)(n-13)(n-14)}{24}$.

Therefore, $d_{pf}(C_n, n-8) = \frac{(n-1)(n-12)(n-13)(n-14)(n-15)}{120} + \frac{(n-3)(n-12)(n-13)(n-14)}{24}$.

$$= \frac{(n-1)(n-12)(n-13)(n-14)(n-15)+5(n-3)(n-12)(n-13)(n-14)}{120}$$

$$= \frac{n(n-11)(n-12)(n-13)(n-14)}{120}$$

Hence, by the principle of induction, the result is true for all $n \geq 11$

Theorem 3.5

For every $n \in N$ and $\gamma_{pf}(C_n) \leq i \leq n$. $|C_{pf}(n,i)|$ is the coefficient of $u^n v^i$ in the expansion of the function: $f(u,v) = \frac{u^3v(1+3u^2v+u^3v^3+uv+3uv+uv^3)}{(1-uv-u^3v)}$

Proof: First we set $f(u,v) = \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|C_{pf}(n,i)|u^n v^i$. By using the recursive formula for $|C_{pf}(n,i)|$ we can write $f(u,v)$ as follows:

$$f(u,v) = \sum_{n=3}^{\infty}\sum_{i=1}^{\infty}\left(|C_{pf}(n-1,i-1)| + |C_{pf}(n-3,i-1)|\right)u^n v^i$$

$$= uv\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|C_{pf}(n-1,i-1)|u^{n-1}v^{i-1} + u^3v\sum_{n=3}^{\infty}\sum_{i=1}^{\infty}|C_{pf}(n-3,i-1)|u^{n-3}v^{i-1}$$

$= uv\big(|C_{pf}(2{,}0)|u^2 + |C_{pf}(2{,}1)|u^2v + |C_{pf}(2{,}2)|u^2v^2 + |C_{pf}(3{,}0)|u^3 + |C_{pf}(3{,}1)|u^3v + |C_{pf}(3{,}2)|u^3v^2 + |C_{pf}(3{,}3)|u^3v^3\big) + uvf(u,v) + u^3v\big(|C_{pf}(0{,}0)| + |C_{pf}(1{,}0)|u + |C_{pf}(1{,}1)|uv + |C_{pf}(2{,}0)|u^2 + |C_{pf}(2{,}1)|u^2v + |C_{pf}(2{,}2)|u^2v^2 + |C_{pf}(3{,}0)|u^3 + |C_{pf}(3{,}1)|u^3v + |C_{pf}(3{,}2)|u^3v^2 + |C_{pf}(3{,}3)|u^3v^3\big) + u^3vf(u,v)$.

Substituting the values from Table 1 also for $|C_{pf}(n,0)| = 0$ for all $n \in N$ and $|C_{pf}(0{,}0)| = 1$ we have:

$f(u,v) = uv(3u^3v + u^3v^3) + uvf(u,v) + u^3v(1 + 3u^3v + u^3v^3) + u^3vf(u,v)$

$f(u,v) = u^3v(1 + 3u^2v + u^3v^3 + uv + 3uv + uv^3) + uvf(u,v) + u^3vf(u,v)$

$f(u,v)(1 - uv - u^3v) = u^3v(1 + 3u^2v + u^3v^3 + uv + 3uv + uv^3)$

Hence, $f(u,v) = \frac{u^3v(1+3u^2v+u^3v^3+uv+3uv+uv^3)}{(1-uv-u^3v)}$.

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