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# SOME SUMS FORMULA FOR PRODUCTS OF TERMS OF JACOBSTHAL AND JACOBSTHAL - LUCAS NUMBERS

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**Abstract**. In this paper, we derive some sums formula for certain products of terms of the Jacobsthal and Jacobsthal - Lucas numbers. Also, we present generalized identities on the product of Jacobsthal and Jacobsthal - Lucas numbers to establish connection formulas between them with the help of Binet's formula.

**Keywords:** Jacobsthal Numbers, Jacobsthal – Lucas Numbers, Binet's formula.

**MSC**: 11B99

## 1. Introduction

Numbers is the essence of mathematical calculations. Varieties of numbers have variety of range and richness. Many numbers exhibit fascinating properties, they form sequences, they form pattern and so on [1, 2, 3]. Generally sequence stimulates students intellectual curiosity and sharpen their mathematical skills [4, 5]. Here, we make an attempt to study interesting characteristics of Jacobsthal and Jacobsthal – Lucas sequences.

In this paper, we derive some sums formula for certain products of terms of the Jacobsthal and Jacobsthal - Lucas numbers. Also, we present generalized identities on the product of Jacobsthal and Jacobsthal - Lucas numbers to establish connection formulas between them with the help of Binet's formula.

## 2. Preliminaries

The Jacobsthal sequence $\{J_n\}$, can be defined as

$$J_n = J_{n-1} + 2J_{n-2}\ ;\ n \geq 1$$

with initial conditions $J_0 = 0,\ J_1 = 1$. The Jacobsthal – Lucas sequences $\{j_n\}$, can be defined as

$$j_n = j_{n-1} + 2j_{n-2};\ n \geq 1$$

with initial conditions $j_0 = 2,\ j_1 = 1$. The Binet's formula for Jacobsthal and Jacobsthal – Lucas numbers are:

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, j_n = \alpha^n + \beta^n$$

where $\alpha = 2$, $\beta = -1$. Also, $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$.

**Method of Analysis**

The following are the sum formulas for these sequences:

$$\sum_{k=0}^{n-1} j_{2k} = \frac{j_{2n} + 3n - 2}{3}$$
$$\sum_{k=0}^{n-1} j_{2k+1} = \frac{j_{2n+1} + 3n - 1}{3}$$

***Proposition 2.1***

If $J_n$, $j_n$ are the nth Jacobsthal and Jacobsthal- Lucas numbers, then

$\sum_{i=1}^{n} J_i J_{i+k} = \frac{1}{27}\left(j_{2n+2p+1} - j_{2p+1} - 3n + 3\delta\right)$,if k is odd

$\sum_{i=1}^{n} J_i J_{i+k} = \frac{1}{27}\left(j_{2n+2p+2} - j_{2p+2} + 3n + 3\delta\right)$,if k is even

Where $\delta = \sum_{i=1}^{n} (-2)^{i+1} j_k$ .

**Proof:** Using the equation $J_n J_{n+k} = \frac{j_{2n+k} - (-2)^{n+1} j_k}{9}$, we can write the following equations.

$$J_1 J_{k+1} = \frac{1}{9}(j_{k+2} + 2^2 j_k)$$
$$J_2 J_{k+2} = \frac{1}{9}(j_{k+4} + 2^3 j_k)$$
$$J_3 J_{k+3} = \frac{1}{9}(j_{k+6} + 2^4 j_k)$$
$$J_n J_{n+k} = \frac{1}{9}(j_{2n+k} + (-2)^{n+1} j_k)$$

Then, we obtain that:

$$J_1 J_{k+1} + J_2 J_{k+2} + \cdots + J_n J_{n+k} = \frac{1}{9}\left(j_{k+2} + j_{k+4} + \cdots + j_{2n+k} + \sum_{i=1}^{n} (-2)^{i+1} j_k\right)$$
$$= \frac{1}{9}(j_{k+2} + j_{k+4} + \cdots + j_{2n+k} + \delta)$$

Where $\delta = \sum_{i=1}^{n} (-2)^{i+1} j_k$.

If k is an odd integer such thatk $= 2p - 1$, $p \in Z$, then

$$\sum_{i=1}^{n} J_i J_{i+k} = \frac{1}{9}\left(j_{2p+1} + j_{2p+3} + \cdots + j_{2n+2p-1} + \delta\right)$$
$$= \frac{1}{9}\left(\sum_{i=1}^{n+p-1} j_{2i+1} - \sum_{i=1}^{P-1} j_{2i+1} + \delta\right)$$
$$= \frac{1}{9}\left(\frac{j_{2n+2p+1} - 3(n+p) - 4}{3} - \frac{j_{2p+1} + 3p - 4}{3} + \delta\right)$$
$$= \frac{1}{27}\left(j_{2n+2p+1} - j_{2p+1} - 3n + 3\delta\right).$$

Consider k as an even integer such thatk $= 2p$, $p \in Z$, then:

$$\sum_{i=1}^{n} J_i J_{i+k} = \frac{1}{9}\left(j_{2p+2} + j_{2p+4} + \cdots + j_{2n+2p} + \delta\right)$$
$$= \frac{1}{9}\left(\sum_{i=1}^{n+p} j_{2i} - \sum_{i=1}^{P} j_{2i} + \delta\right)$$
$$= \frac{1}{9}\left(\frac{j_{2n+2p+2} - 3(n+p+1) - 8}{3} - \frac{j_{2p+2} + 3(p+1) - 8}{3} + \delta\right)$$

$$= \frac{1}{27}\left(j_{2n+2p+2} - j_{2p+2} + 3n + 3\delta\right).$$

## 3. Product of Jacobsthal and Jacobsthal – Lucas Numbers

***Theorem 3. 1***

$J_{2n}j_{2n} = J_{4n}$, where $n \geq 1$.

**Proof:**

$$J_{2n}j_{2n} = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)(\alpha^{2n} + \beta^{2n}) = \frac{\alpha^{4n} - \alpha^{2n}\beta^{2n} + \alpha^{2n}\beta^{2n} - \beta^{4n}}{\alpha - \beta} = \frac{\alpha^{4n} - \beta^{4n}}{\alpha - \beta} = J_{4n}$$

***Theorem 3.2***

$J_{2n}j_{2n+2} = J_{4n+2} - (2)^{2n}$, where $n \geq 1$.

**Proof:**

$$J_{2n}j_{2n+2} = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)(\alpha^{2n+2} + \beta^{2n+2}) = \frac{\alpha^{4n+2} - \alpha^{2n+2}\beta^{2n} + \alpha^{2n}\beta^{2n+2} - \beta^{4n+2}}{\alpha - \beta}$$

$$= \frac{\alpha^{4n+2} - \beta^{4n+2}}{\alpha - \beta} - \alpha^{2n}\beta^{2n}\left(\frac{\alpha^2 - \beta^2}{\alpha - \beta}\right) = J_{4n+2} - (2)^{2n}$$

***Theorem 3.3***

$J_{2n}j_{2n+1} = J_{4n+1} - (2)^{2n}$, where $n \geq 1$.

**Proof:**

$$J_{2n}j_{2n+1} = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)(\alpha^{2n+1} + \beta^{2n+1}) = \frac{\alpha^{4n+1} - \alpha^{2n+1}\beta^{2n} + \alpha^{2n}\beta^{2n+1} - \beta^{4n+1}}{\alpha - \beta}$$

$$= \frac{\alpha^{4n+1} - \beta^{4n+1}}{\alpha - \beta} - \alpha^{2n}\beta^{2n}\left(\frac{\alpha - \beta}{\alpha - \beta}\right) = J_{4n+1} - (2)^{2n}$$

***Theorem 3.4***

$J_{2n}j_{2n+3} = J_{4n+3} - (2)^{2n}(3)$, where $n \geq 1$.

**Proof:**

$$J_{2n}j_{2n+3} = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)(\alpha^{2n+3} + \beta^{2n+3}) = \frac{\alpha^{4n+3} - \alpha^{2n+3}\beta^{2n} + \alpha^{2n}\beta^{2n+3} - \beta^{4n+3}}{\alpha - \beta}$$

$$= \frac{\alpha^{4n+3} - \beta^{4n+3}}{\alpha - \beta} - \alpha^{2n}\beta^{2n}\left(\frac{\alpha^3 - \beta^3}{\alpha - \beta}\right) = J_{4n+3} - (-2)^{2n}\left(\frac{(\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta)}{\alpha - \beta}\right)$$

$$= J_{4n+3} - (2)^{2n}(j_2 - 2) = J_{4n+3} - (2)^{2n}(3)$$

***Theorem 3.5***

$J_{2n-1}j_{2n+1} = J_{4n} + (2)^{2n-1}$, where $n \geq 1$.

**Proof:**

$$J_{2n-1}j_{2n+1} = \left(\frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta}\right)(\alpha^{2n+1} + \beta^{2n+1}) = \frac{\alpha^{4n} - \alpha^{2n+1}\beta^{2n-1} + \alpha^{2n-1}\beta^{2n+1} - \beta^{4n}}{\alpha - \beta}$$

$$= \frac{\alpha^{4n} - \beta^{4n}}{\alpha - \beta} + \alpha^{2n}\beta^{2n}\left(\frac{\beta/\alpha - \alpha/\beta}{\alpha - \beta}\right) = J_{4n} + (-2)^{2n}\left(\frac{(\beta^2 - \alpha^2)}{(\alpha - \beta)(\alpha\beta)}\right) = J_{4n} + 2^{2n-1}$$

***Theorem 3.6***

$J_{2n+1}j_n = J_{4n+1} + (2)^{2n}$, where $n \geq 1$.

**Proof:**

$$J_{2n+1}j_n = \left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}\right)(\alpha^{2n} + \beta^{2n}) = \frac{\alpha^{4n+1} - \alpha^{2n}\beta^{2n+1} + \alpha^{2n+1}\beta^{2n} - \beta^{4n+1}}{\alpha - \beta}$$

$$= \frac{\alpha^{4n+1} - \beta^{4n+1}}{\alpha - \beta} - \alpha^{2n}\beta^{2n}\left(\frac{\beta - \alpha}{\alpha - \beta}\right) = J_{4n+1} + (2)^{2n}$$

**Generalized Identities on the products of Jacobsthal and Jacobsthal – Lucas Numbers:**

***Theorem 3. 7***

$J_m j_n = J_{m+n} - (-2)^m J_{n-m}$, where $n \geq 1,\ m \geq 0$.

**Proof:**

$$J_m j_n = \left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right)(\alpha^n + \beta^n) = \frac{\alpha^{m+n} - \alpha^n\beta^m + \alpha^m\beta^n - \beta^{m+n}}{\alpha - \beta}$$
$$= J_{m+n} - (\alpha\beta)^m\left(\frac{\alpha^{n-m} - \beta^{n-m}}{\alpha - \beta}\right) = J_{m+n} - (-2)^m J_{n-m}$$

***Theorem 3.8***

$J_n j_{2n+m} = J_{3n+m} - (-2)^n J_{n+m}$, where $n \geq 1,\ m \geq 0$.

**Proof:**

$$J_n j_{2n+m} = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)(\alpha^{2n+m} + \beta^{2n+m}) = \frac{\alpha^{3n+m} - \alpha^{2n+m}\beta^n + \alpha^n\beta^{2n+m} - \beta^{3n+m}}{\alpha - \beta}$$
$$= J_{3n+m} - (\alpha\beta)^n\left(\frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta}\right) = J_{3n+m} - (-2)^n J_{n+m}$$

***Theorem 3. 9***

$J_{2n+m} j_n = J_{3n+m} + (-2)^n J_{n+m}$, where $n \geq 1,\ m \geq 0$.

**Proof:**

$$J_{2n+m} j_n = \left(\frac{\alpha^{2n+m} - \beta^{2n+m}}{\alpha - \beta}\right)(\alpha^n + \beta^n) = \frac{\alpha^{3n+m} - \alpha^n\beta^{2n+m} + \alpha^{2n+m}\beta^n - \beta^{3n+m}}{\alpha - \beta}$$
$$= J_{3n+m} + (\alpha\beta)^n\left(\frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta}\right) = J_{3n+m} + (-2)^n (J_{n+m})$$

***Theorem 3. 10***

$J_{2n} j_{2n+m} = J_{4n+m} - (2)^{2n} J_m$, where $n \geq 1,\ m \geq 0$.

**Proof:**

$$J_{2n} j_{2n+m} = \left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right)(\alpha^{2n+m} + \beta^{2n+m}) = \frac{\alpha^{4n+m} - \alpha^{2n+m}\beta^{2n} + \alpha^{2n}\beta^{2n+m} - \beta^{4n+m}}{\alpha - \beta}$$
$$= J_{4n+m} - (\alpha\beta)^{2n}\left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = J_{4n+m} - (2)^{2n}(J_m)$$

***Theorem 3.11***

$J_{2n+m} j_{2n} = J_{4n+m} + (2)^{2n} J_m$, where $n \geq 1,\ m \geq 0$.

**Proof:**

$$J_{2n+m} j_{2n} = \left(\frac{\alpha^{2n+m} - \beta^{2n+m}}{\alpha - \beta}\right)(\alpha^{2n} + \beta^{2n}) = \frac{\alpha^{4n+m} - \alpha^{2n}\beta^{2n+m} + \alpha^{2n+m}\beta^{2n} - \beta^{4n+m}}{\alpha - \beta}$$
$$= J_{4n+m} + (\alpha\beta)^{2n}\left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) = J_{4n+m} + (2)^{2n}(J_m)$$

## 4. Conclusion

One may search for complex number sequences for Jacobsthal and Jacobsthal – Lucas Numbers.

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