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LATTICE OF FUZZY FILTERS IN A LATTICE

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Abstract. The family of all fuzzy filters of a lattice FFF(L) is discussed and some of their properties are studied. The concept of union, intersection of two fuzzy filters and complement of a fuzzy filter are defined and it is proved that intersection of two fuzzy filters is also a fuzzy filter. As an immediate consequence of intersection, the study of "union of fuzzy filters" of a lattice is carried out. Further, the complement of a fuzzy filter is developed and it is shown that the complement of a fuzzy filter of a lattice need not be a fuzzy filter of that lattice. Further shown that the set of all fuzzy filters FFF(L) of a lattice L is a complete lattice under the relation "$\subseteq$". The sup and inf of any subfamily $\{\mu_i / i\in\Omega\}$ of fuzzy filters are $\langle\cup \{\mu_i/i\in\Omega\}\rangle$ and $\cap\{\mu_i/i\in\Omega\}$ respectively.

Keywords: fuzzy filter, level filter, lattice of fuzzy filters.

1. Introduction

The notion of fuzzy sets was introduced by Lofti. A. Zadeh in 1965. Zadeh had initiated fuzzy set theory as a modification of the ordinary set theory. In1971, Rosenfeld introduced fuzzy sets in the realm of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, researchers in various disciplines of mathematics have been trying to extend their ideas to the broader framework of the fuzzy setting. In 1982, Liu developed the concept of fuzzy subrings as well as fuzzy filters in ring. The concept of lattice was first defined by Dedekind in 1897, and then developed by Birkhoff G. in 1933.Here we made an attempt to study the algebraic nature of fuzzy subset of lattice especially fuzzy filters, by using the ideas of fuzzy theory and lattice theory.

2. Preliminaries

Some well – known definitions and preliminary results are recalled here.

Definition 2.1

A relation defined on a set S which is reflexive, antisymmetric and transitive is called a partial ordering on S. A set S with a partial ordering ρ defined on it is called a partially ordered set or a poset and is denoted by (S, ρ).

Definition 2.2

Let $(P, \leq)$ be a poset. Let A be a non–empty subset of P. An element $u \in P$ is called an upper bound of A if $a \leq u$ for all $a \in A$. An element $u \in P$ is called the least upper bound (l.u.b.) of A if

(i) u is an upper bound of A.

(ii) if v is any other upper bound of A, then $u \leq v$.

An element $l \in P$ is called the lower bound of A if $l \leq a$ for all $a \in A$. An element $l \in P$ is called the greatest lower bound (g.l.b.) of A if

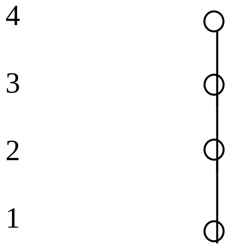
(i) l is a lower bound of A.

(ii) if m is any other lower bound of A, then $m \leq l$.

Definition 2.3

A lattice is a poset in which any two elements have a g.l.b. and al.u.b. We denote the l.u.b. of $a \vee b$ and g.l.b. by $a \wedge b$.

Example 2.4 Consider the poset {1, 2, 3, 4} with the usual $\leq$. Here $1 \leq 2 \leq 3 \leq 4$ and 2 covers 1 and 3 covers 2 and 4 covers 3. Hence we obtain the diagram for this poset. This is also a lattice.



Definition 2.5

Let $(L, \vee, \wedge)$ be a lattice. A non–empty subset S of L is called a filter of L if it satisfies the following conditions:

(i) $x, y \in S \Rightarrow x \wedge y \in S$.

(ii) $x \in S$ and $r \in L$ with $r \geq x \Rightarrow r \in S$.

Definition 2.6

Let X be a non–empty set. A mapping $\mu: X \to [0, 1]$is called a fuzzy subset of X.

Definition 2.7

Let μ be any fuzzy subset of a set X and $\mu = \{(x_i, t_i) / i = 1 \text{ to } n \text{ and } t_i \in [0, 1]\}$. Then, $\{t_i / i = 1 \text{ to } n\}$ is called the image set of μ and is denoted by Im μ.

Definition 2.8

Let μ be any fuzzy subset of a set X and $t \in \text{Im}\, \mu$. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the level subset of μ.

Clearly, $\mu_t \subseteq \mu_s$ whenever $t > s$.

Definition 2.9

A fuzzy subset μ of a lattice L, is called a fuzzy sub lattice of L, if the following conditions are satisfied: For all $x, y \in L$,

(i) $\mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$.

Definition 2.10

A fuzzy subset μ of a lattice L, is called a fuzzy lattice filter or fuzzy filter of L if, for all x, $y \in L$ the following conditions are satisfied:

(i) $\mu(x \vee y) \geq \max \{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min \{\mu(x), \mu(y)\}$

Theorem 2.11 *[Characterization Theorem]*

A fuzzy subset μ of a lattice L, is a fuzzy filter of L if and only if, the level subsets μ_t, $t \in \text{Im}\,\mu$ are filters of L.

Proposition 2.12 Let μ and θ be any two fuzzy filter of L.
If $\mu(x) \leq \theta(x)$ and $\mu(y) \leq \theta(y)$, then $\mu(x \vee y) \leq \theta(x \vee y)$ for some $x, y \in L$.

Proposition 2.13 Let μ be any fuzzy filter of L. $\mu(x) \geq \mu(y)$ whenever $x \leq y$, where $x, y \in L$.

3. Lattice of fuzzy filters in a lattice

Here, FFF(L)denotes the family of all fuzzy filters of a lattice L and the properties of FFF(L)are established. The necessary and sufficient condition for the equality of two fuzzy filters is proved. It is shown that the intersection of two fuzzy filters of a lattice is also a fuzzy filter. Further, the study of union of fuzzy filters of a lattice is carried out. The lattice of all fuzzy filters of a lattice is also presented.

Proposition 3.1 Two fuzzy filters μ and θ of a lattice L such that Card Im $\mu < \infty$, Card Im$\theta < \infty$ are equal if and only if, Im μ = Imθ and $F^f_\mu = F^f_\theta$.

Definition 3.2

The union of two fuzzy filters μ and θ of a lattice L, denoted by $\mu \cup \theta$ is a fuzzy subset of L defined by $(\mu \cup \theta)(x) = \max\{\mu(x), \theta(x)\}$, for all $x \in L$.

The intersection of two fuzzy filters μ and θ of a lattice L, denoted by $\mu \cap \theta$is a fuzzy subset of L defined by $(\mu \cap \theta)(x) = \min\{\mu(x), \theta(x)\}$, for all $x \in L$.

Definition 3.3

The complement of a fuzzy filter μ of a lattice L, denoted by (~μ) is defined by:
$(\sim\mu)(x) = 1 - \mu(x)$, $\forall\, x \in L$.

Definition 3.4

Let μ and θ be any two fuzzy filters of a lattice L. Then μ is said to be contained inθ, denoted by $\mu \subseteq \theta$ if $\mu(x) \leq \theta(x)$, $\forall\, x \in L$. If $\mu(x) = \theta(x)$ for all $x \in L$, then μ and θ are said to be equal and we write μ = θ.

Remark: Let μ and θ be any two fuzzy filters of a lattice L. If $\mu \subseteq \theta$, then $\mu \cup \theta = \theta$ and $\mu \cap \theta = \mu$.

Remark: Let μ and θ be any two fuzzy filters of a lattice L. Then $\mu \cup \theta \supseteq \mu \cap \theta$.

Theorem 3.5

Intersection of any two fuzzy filters of a lattice L, is again a fuzzy filter of L.

Proof: Assume that μ and θ are any two fuzzy filters of L. Then,

(i) $\mu(x \vee y) \geq \max\{\mu(x), \mu(y)\}$
(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$ -----------------→ (1)

and

(iii) $\theta(x \vee y) \geq \max\{\theta(x), \theta(y)\}$
(iv) $\theta(x \wedge y) \geq \min\{\theta(x), \theta(y)\}$ ------------------→ (2)

We have to prove $\mu \cap \theta$ is also a fuzzy filter. Clearly $\mu \cap \theta$ is a fuzzy subset of L.
Let Im $(\mu \cap \theta) = \{u_1, u_2, \ldots, u_p\}$. It is enough to prove that $(\mu \cap \theta)_{u_k}$ are filters of L, k = 1 to p.
That is to prove:

(i) x, y $\in (\mu \cap \theta)_{u_k} \Rightarrow$ x$\wedge$y $\in (\mu \cap \theta)_{u_k}$

(ii) x$\in (\mu \cap \theta)_{u_k}$ and $r_1 \geq$ x, then $r_1 \in (\mu \cap \theta)_{u_k}$

For (i):

Let x, y $\in (\mu \cap \theta)_{u_k}$ be arbitrary.

$\Rightarrow (\mu\cap\theta)(x) \geq u_k$ and $(\mu\cap\theta)(y) \geq u_k$ --------------------------→ (3)

By the definition, $(\mu\cap\theta)(x) = \min\{\mu(x), \theta(x)\}$ and $(\mu\cap\theta)(y) = \min\{\mu(y), \theta(y)\}$ -----------→ (4)

Now, $(\mu\cap\theta)(x\wedge y) = \min \{(\mu(x\wedge y), \theta(x\wedge y)\}$ ----------------→ (5)

Case (i) Let $\min \{\mu(x), \theta(y)\} = \mu(x)$ and $\min \{\mu(y), \theta(y)\} = \mu(y)$

$\Rightarrow \mu(x) \leq \theta(x)$ and $\mu(y) \leq \theta(y)$

$\Rightarrow \mu(x\wedge y) \leq \theta(x\wedge y)$, by Proposition 2.12

$\Rightarrow \min\{\mu(x\wedge y), \theta(x\wedge y)\} = \mu(x\wedge y)$ -------------------→ (6)

Now (4) $\Rightarrow (\mu\cap\theta)(x) = \mu(x)$ and $\mu\cap\theta)(y) = \mu(y)$, (3) $\Rightarrow \mu(x) \geq u_k$ and $\mu(y) \geq u_k$

Again (1) $\Rightarrow \mu(x\wedge y) \geq \min \{\mu(x), \mu(y)\} \geq u_{k,}$.

Now (6) $\Rightarrow \min \{\mu(x\wedge y), \theta(x\wedge y)\} \geq u_{k,}$ (5) $\Rightarrow (\mu\cap\theta)(x\wedge y) \geq u_k \Rightarrow x\wedge y \in (\mu \cap \theta)_{u_k}$.

Case (ii) Let $\min \{\mu(x), \theta(y)\} = \theta(x)$ and $\min \{\mu(y), \theta(y)\} = \theta(y)$

$\Rightarrow \theta(x) \leq \mu(x)$ and $\theta(y) \leq \mu(y)$

$\Rightarrow \theta(x\wedge y) \leq \mu(x\wedge y)$, by Proposition 2.12

$\Rightarrow \min \{\mu(x\wedge y), \theta(x\wedge y)\} = \theta(x\wedge y)$ ---------------→ (7)

Now (4) $\Rightarrow (\mu\cap\theta)(x) = \theta(x)$ and $\mu\cap\theta)(y) = \theta(y)$, (3) $\Rightarrow \theta(x) \geq u_k$ and $\theta(y) \geq u_k$

Again (2) $\Rightarrow \theta(x\wedge y) \geq \min \{\theta(x), \theta(y)\} \geq u_k$

Now (7) $\Rightarrow \min \{\mu(x\wedge y), \theta(x\wedge y)\} \geq u_k$, (5) $\Rightarrow (\mu\cap\theta)(x\wedge y) \geq u_k \Rightarrow x\wedge y \in (\mu \cap \theta)_{u_k}$

Case (iii) Let $\min \{\mu(x), \theta(x)\} = \mu(x)$ and $\min \{\mu(y), \theta(y)\} = \theta(y)$

$\Rightarrow \mu(x) \leq \theta(x)$ and $\theta(y) \leq \mu(y)$ --------------------------------------→ (8)

(4) $\Rightarrow (\mu\cap\theta)(x) = \mu(x)$ and $(\mu\cap\theta)(y) = \theta(y)$

(3) $\Rightarrow \mu(x) \geq u_k$ and $\theta (y) \geq u_k$ ---------------------------------------→ (9)

From (8) and (9), we have $\theta(x) \geq u_k$ and $\mu(y) \geq u_k$ ---------------→ (10)

$\Rightarrow \min \{\mu(x), \mu(y)\} \geq u_k$ and $\min \{\theta(x), \theta(y)\} \geq u_k$, by (9) and (10)

Now (1) $\Rightarrow \mu(x\wedge y) \geq u_k$, and (2) $\Rightarrow \theta(x\wedge y) \geq u_k$ .Therefore,

$\min \{\mu(x\wedge y), \theta(x\wedge y)\} \geq u_k$, (5) $\Rightarrow (\mu\cap\theta)(x\wedge y) \geq u_k$ $\Rightarrow x\wedge y \in (\mu \cap \theta)_{u_k}$

Case (iv) Let $\min \{\mu(x), \theta(y)\} = \theta(x)$ and $\min \{\mu(y), \theta(y)\} = \mu(y)$

$\Rightarrow \theta(x) \leq \mu(x)$ and $\mu(y) \leq \theta(y)$ --------------------------------------→ (11)

(4) $\Rightarrow (\mu\cap\theta)(x) = \theta(x)$ and $(\mu\cap\theta)(y) = \mu(y)$

(3) $\Rightarrow \theta(x) \geq u_k$ and $\mu(y) \geq u_k$ --→ (12)

From (11) and (12), we have $\mu(x) \geq u_k$ and $\theta(y) \geq u_k$ ------------→ (13)

Hence, $\min \{\mu(x), \mu(y)\} \geq u_k$ and $\min \{\theta(x), \theta(y)\} \geq u_k$, by (12) and (13)

Now (1) $\Rightarrow \mu(x\wedge y) \geq u_k$, and (2) $\Rightarrow \theta(x\wedge y) \geq u_k$

Therefore, $\min \{\mu(x\wedge y), \theta(x\wedge y)\} \geq u_k$, (5) $\Rightarrow (\mu\cap\theta)(x\wedge y) \geq u_k \Rightarrow x\wedge y \in (\mu \cap \theta)_{u_k}$

For (ii): Let x $\in (\mu \cap \theta)_{u_k}$ and $u_k \in \mathrm{Im}\ (\mu\cap\theta)$; $r_1 \in$ R and $r_1 \leq$ x.

To prove $r_1 \in (\mu \cap \theta)_{u_k}$ That is, to prove $(\mu\cap\theta)(r_1) \geq u_k$. Here x $\in (\mu \cap \theta)_{u_k} \Rightarrow (\mu\cap\theta)(x) \geq u_k$

$\Rightarrow \min \{\mu(x), \theta(x)\} \geq u_k$ -------→ (14)

As μ and θ are fuzzy filters of L, and $r_1 \geq$ x, by Proposition 2.13, we have,

$\mu(x) \leq \mu(r_1)$ ------→ (15)

$\theta(x) \leq \theta(r_1)$ ------→ (16)

Case (i) Let $\min \{\mu(x), \theta(x)\} = \mu(x)$ and $\min \{\mu(r_1), \theta(r_1)\} = \mu(r_1)$

Now (14) $\Rightarrow \mu(x) \geq u_k$ ---------→ (17)

Again $(\mu\cap\theta)(r_1) = \min \{\mu(r_1), \theta(r_1)\}$, by Definition 3.2

$= \mu(r_1)$, by the assumption in this case $\geq \mu(x)$, by (15)

$\Rightarrow$ $(\mu\cap\theta)(r_1) \geq u_k$, by (17), $\Rightarrow r_1 \in (\mu \cap \theta)_{u_k}$

Case (ii) Let min $\{\mu(x), \theta(x)\} = \mu(x)$ and min $\{\mu(r_1), \theta(r_1)\} = \theta(r_1)$

Now (14) $\Rightarrow \mu(x) \geq u_k$ ----------------------------→ (18)

By assumption in this case $\mu(x) \leq \theta(x)$ ---------→ (19)

Again $(\mu\cap\theta)(r_1)$ = min $\{\mu(r_1), \theta(r_1)\}$

= $\theta(r_1)$, by the assumption in this case $\geq \theta(x)$, by (16)

$\geq \mu(x)$, by (19) $\geq u_k$, by (18) $\Rightarrow (\mu\cap\theta)(r_1) \geq u_k \Rightarrow r_1 \in (\mu \cap \theta)_{u_k}$

Case (iii)Let min $\{\mu(x), \theta(x)\} = \theta(x)$ and min $\{\mu(r_1), \theta(r_1)\} = \mu(r_1)$

Now (14) $\Rightarrow \theta(x) \geq u_k$ --------------------------------→ (20)

By the assumption in this case, $\mu(x) \geq \theta(x)$ --------→ (21)

Again $(\mu\cap\theta)(r_1)$ = min $\{\mu(r_1), \theta(r_1)\} = \mu(r_1)$, by the assumption in this case

$\geq \mu(x)$, by (15) $\geq \theta(x)$, by (21) $\geq u_k$, by (20) $\Rightarrow r_1 \in (\mu \cap \theta)_{u_k}$

Case (iv)Let min $\{\mu(x), \theta(x)\} = \theta(x)$ and min $\{\mu(r_1), \theta(r_1)\} = \theta(r_1)$

Now (14) $\Rightarrow \theta(x) \geq u_k$ ----------------------------------→ (22)

Again $(\mu\cap\theta)(r_1)$ = min $\{\mu(r_1), \theta(r_1)\} = \theta(r_1)$, by the assumption in this case

$\geq \theta(x)$, by (16) $\geq u_k$, by (22) $\Rightarrow (\mu\cap\theta)(r_1) \geq u_k \Rightarrow r_1 \in (\mu \cap \theta)_{u_k}$

Thus $r_1 \in (\mu \cap \theta)_{u_k}$ in all the cases.

Hence $\mu\cap\theta$ is a fuzzy filter of L.

Thus the intersection two fuzzy filters of L, is again a fuzzy filter of L.

Remark: Intersection of any family of fuzzy filters of a lattice L is also a fuzzy filter of L.

Proposition 3.6 (FFF(L), $\cap$) is a semi lattice.

Now, the study of "union of fuzzy filters" of a lattice L is carried out. Also the complement of a fuzzy filter is discussed.

Remark 3.10:Let μ and θ be any fuzzy filters of a lattice L. Then $\mu\cup\theta$ need not be a fuzzy filter of L.

Proof: We prove this by giving a counter example. Let L be any lattice. Let A and B be two filters of L such that $A \cup B$ is not a filter of L. Now define the fuzzy subsets μ and θ by

$$\mu(x) = \begin{cases} 0 \ if \ \ x \in A \\ .9 \ if \ \ x \in L \sim A \end{cases} \qquad \theta(x) = \begin{cases} 0 \ if \ \ x \in B \\ .7 \ if \ \ x \in L \sim B \end{cases}$$

Clearly μ and θ are fuzzy filters of L.

Now, $(\mu \cup\theta)(x) = \begin{cases} 0 \ if \ \ x \in A \\ .7 \ if \ \ x \in B \cap A^c \\ .9 \ \ otherwise \end{cases}$

Here t = $.7 \in \mathrm{Im}\,(\mu \cup\theta)$

And $(\mu \cup\theta)_t = \{x \,/\, (\mu \cup\theta)(x) \geq .7\}$

$= \{x \,/\, x \in A \text{ and } x \in (B \cap A^c)\}$

$= \{x \,/\, x \in A \cup (B \cap A^c)\}$

$= \{x \,/\, x \in A \cup B\}$, which is not a filter of L, since $A \cup B$ is not a filter of L.

Then, $(\mu \cup\theta)$ is not a fuzzy filter of L.

Proposition 3.7 The complement of a fuzzy filter need not be a fuzzy filter of L.

Proof: We prove this by giving a counter example. Consider the lattice defined in Example 2.4

Now define the fuzzy subset μ of L by

$$\mu(x) = \begin{cases} .9 & if \quad x = 1 \\ .6 & if \quad x = 2 \\ .4 & if \quad x = 3, 4 \end{cases}$$

Then μ is a fuzzy filter of L.

Here $(\sim\mu)(x) = \begin{cases} .1 & if \quad x = 1 \\ .4 & if \quad x = 2 \\ .6 & if \quad x = 3, 4 \end{cases}$

Let t = .6 $\in$Im (~μ)

Then $(\sim\mu)_t = \{x \,/\, (\sim\mu)(x) \geq .6\} = \{x \,/\, x = 3, 4\}$, which is not a filter of L

$\Rightarrow$ (~μ) is not a fuzzy filter of L.

Proposition 3.8 Let μ be a fuzzy filter of a lattice L. Then (~μ) is also a fuzzy filter of L if and only if μ is a constant function. Hence (~μ) is also a constant function.

The following Remark shows the deviation from the classical theory.

Remark: Let μ be a fuzzy filter of a lattice L. Then $\mu\cap(\sim\mu) \neq ¥_\phi$ and $\mu\cup(\sim\mu) \neq ¥_L$.

Proof: We prove these by giving an example.

Consider the lattice defined in Example 1.4. Now define the fuzzy subset μ of L by

$$\mu(x) = \begin{cases} .9 & if \quad x = 1 \\ .6 & if \quad x = 2 \\ .4 & if \quad x = 3, 4 \end{cases}$$

Then μ is a fuzzy filter of L.

Here $(\sim\mu)(x) = \begin{cases} .1 & if \quad x = 1 \\ .4 & if \quad x = 2 \\ .6 & if \quad x = 3, 4 \end{cases}$

$$\Rightarrow \quad [\mu \cup (\sim\mu)](x) = \begin{cases} .9 & if \quad x = 1 \\ .6 & if \quad x = 2 \\ .6 & if \quad x = 3, 4 \end{cases}$$

$$\text{And } [\mu \cap (\sim\mu)](x) = \begin{cases} .1 & if \quad x = 1 \\ .4 & if \quad x = 2 \\ .4 & if \quad x = 3, 4 \end{cases}$$

But $¥_\phi(x) = 0$, for all $x\in L$ and $¥_L(x) = 1$, for all $x\in L$.

Then $\mu\cap(\sim\mu) \neq ¥_\phi$ and $\mu\cup(\sim\mu) \neq ¥_L$.

Now, the lattice of all fuzzy filters of a lattice is presented in the following Theorem.

Theorem 3.9

The set of all fuzzy filters FFF(L) of a lattice L is a complete lattice under the relation "$\subseteq$". The sup and inf of any subfamily $\{\mu i \,/ i\in\Omega\}$ of fuzzy filters are $\langle\cup \{\mu_i/i\in\Omega\}\rangle$ and $\cap\{\mu_i/i\in\Omega\}$ respectively.

Proof: Let μ, θ and $\sigma\in$FFF(L) be arbitrary, let $x \in L$ be arbitrary.

(1) Reflexive

$\mu(x) \leq \mu(x)$ for all $x \in L$

$\Rightarrow \mu \subseteq \mu$.

(2) Antisymmetric

Let $\mu \subseteq \theta$ and $\theta \subseteq \mu$.

$\Rightarrow \mu(x) \leq \theta(x)$ and $\theta(x) \leq \mu(x)$, for all $x \in L$.

(3) Transitive

Let $\mu \subseteq \theta$ and $\theta \subseteq \sigma$.

$\Rightarrow \mu(x) \leq \theta(x)$ and $\theta(x) \leq \sigma(x)$

$\Rightarrow \mu(x) \leq \theta(x) \leq \sigma(x)$

$\Rightarrow \mu \subseteq \sigma$.

(4) Least upper bound

Now $(\mu \cup \theta)(x) = \max \{\mu(x), \theta(x)\}$

Then $\mu(x) \leq (\mu \cup \theta)(x)$ and $\theta(x) \leq (\mu \cup \theta)(x)$

$\Rightarrow \mu \subseteq \mu \cup \theta$ and $\theta \subseteq \mu \cup \theta$

$\Rightarrow \mu \cup \theta$ is the upper bound of μ and θ.

Suppose σ is another upper bound of μ and θ

Then $\mu \subseteq \sigma$ and $\theta \subseteq \sigma$

$\Rightarrow \mu(x) \leq \sigma(x)$ and $\theta(x) \leq \sigma(x)$

$\Rightarrow \max \{\mu(x), \theta(x)\} \leq \sigma(x)$.

$\Rightarrow (\mu \cup \theta)(x) \leq \sigma(x)$

$\Rightarrow \mu \cup \theta \subseteq \sigma$

Hence $\mu \cup \theta$ is the least upper bound of μ and θ.

(5) Greatest lower bound

Now $(\mu \cap \theta)(x) = \min \{\mu(x), \theta(x)\}$

Then $\mu(x) \leq (\mu \cap \theta)(x)$ and $\theta(x) \leq (\mu \cap \theta)(x)$

$\Rightarrow \mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$

$\Rightarrow \mu \cap \theta$ is the lower bound of μ and θ.

Suppose γ is another lower bound of μ and θ. Then $\gamma \subseteq \mu$ and $\gamma \subseteq \theta$.

$\Rightarrow \gamma(x) \leq \mu(x)$ and $\gamma(x) \leq \theta(x)$

$\Rightarrow \gamma(x) \leq \min \{\mu(x), \theta(x)\}$

$\Rightarrow \gamma(x) \leq (\mu \cap \theta)(x) \Rightarrow \gamma \subseteq (\mu \cap \theta)$

Hence $\mu \cap \theta$ is the greatest lower bound of μ and θ.

(6) Completeness

Consider any subfamily $\{\mu_i / i \in \Omega\}$ of the family of fuzzy filters FFF(L).Then the sup of $\{\mu_i / i \in \Omega\}$ is $\langle \cup \{\mu_i / i \in \Omega\} \rangle$ and the inf of $\{\mu_i / i \in \Omega\}$ is $\cap \{\mu_i / i \in \Omega\}$.

Thus the set of all fuzzy filters of a lattice L is a complete lattice under the relation "$\subseteq$".

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