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EINSTEIN OPERATIONS ON PYTHAGOREAN FUZZY MATRICES

T. M. Selvarajan[1] and R. S. Ramya[2]

[1]*Department of Mathematics, Noorul Islam University,*

*Kanyakumari - 629 175,India.*selvarajan1967@gmail.com

[2]*Department of Mathematics, Noorul Islam University,*

Kanyakumari - 629 175,India.

Abstract. In this paper, the authors defined the Einstein operations of Pythagorean fuzzy matrices and proved several properties of them.

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1. Introduction

It is well known that matrices play major role in various areas such as Mathematics, Physics, Statistics, Engineering, Social sciences and many others. However, we cannot successfully use classical matrices because of various types of uncertainties present in real world situations. Now a days probability, Fuzzy sets, Intuitionistic fuzzy sets, Vague sets are used as Mathematical tools for dealing uncertainties. A fuzzy matrix is a matrix over the fuzzy algebra $\mathcal{F}$ =[0,1] under the fuzzy operations formulated by Zadeh in 1965 [21]. Several authors presented a number of results on fuzzy matrices. In 1977, Thomoson [18] studied the behaviour of powers of fuzzy matrices using max-min composition. The theory of fuzzy matrices was systematically developed by Kim and Roush in [7] analogous to that of Boolean matrices. Ragab and Emam [11] studied some properties of the min-max compositions of fuzzy matrices; it can be regarded as the dual of max-min composition of fuzzy matrices. Among the well known operations which can be performed on fuzzy matrices are the operations $\vee$, $\wedge$ and complementation. In addition to these operations, the operations $\oplus$ and $\odot$are introduced by Shyamal and Pal [14]. Also several properties on $\oplus$ and $\odot$, some results on existing operators along with these operations are studied. The concept of Atanassov's intuitionistic fuzzy set [1] was introduced to generalize the concept of Zadeh's fuzzy set. Atanassov [2] defined some basic operations and relations of intuitionistic fuzzy sets and proved an equality between them. Im et al.[5] defined the notation of intuitionistic fuzzy matrix(IFM) as a generalization of fuzzy matrix. Pal[8] introduced the intuitionistic fuzzy determinant, studied some properties on it. Shyamal and pal[13] defined distances between two intuitionistic fuzzy matrices and proposed some relations among distances between intuitionistic fuzzy matrices.

Sriram and Boobalan [15] proved the set of all intuitionistic fuzzy matrices form a commutative monoid under arithmetic sum of intuitionistic fuzzy matrices as well as arithmetic product of intuitionistic product. Recently Emem and Fndh [3] defined some kinds of intuitionistic fuzzy matrices, the max-min and min-max compositions of intuitionistic fuzzy matrices. Also they derived several important results by these compositions and construct an idempotent intuitionistic fuzzy matrix from any given one through the min-max composition. Wang and Liu[19] introduced some Einstein operations of intuitionistic fuzzy sets and analyse some desirable properties of the proposed operations. Yager [20] introduced Pythagorean fuzzy set(PFS) characterized by a membership degree and a no membership degree satisfying the condition that the square sum of its membership and no membership degree is equal to or less than 1, has much stronger ability than intuitionistic fuzzy set to model such uncertain information in multi criteria decision making (MCDM) problems. Einstein operational laws have been proposed for Pythagorean fuzzy numbers by Garg [4]. Silambarasan and Sriram [15] introduced Pythagorean fuzzy matrix (PFM) and the operations algebraic sum and algebraic product of PFMs. Also they investigated its algebraic properties. Further, they defined some new operations for PFMs and discuss their algebraic properties with some existing operations.

In [12] Selvarajan et.al studied Einstein Operations of Intuitionistic fuzzy matrices and proved several properties of them. In this paper, we extend the Einstein operations to Pythagorean fuzzy matrix (PFM) and proved several properties of them.

Definition 1.1

A Pythagorean fuzzy matrix (PFM) is a pair $A = (\langle a_{ij}, a'_{ij}\rangle)$ of non negative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satisfying $0 \leq a_{ij}^2 + {a'_{ij}}^2 \leq 1$, for all i, j.

Definition 1.2

Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle)$ be Pythagorean fuzzy matrices of same size, operations for PFMs can be defined as follows:

$$\text{(i) } A \vee B = (\langle \max(a_{ij}, b_{ij}), \min(a'_{ij}, b'_{ij})\rangle)$$

$$\text{(ii) } A \wedge B = (\langle \min(a_{ij}, b_{ij}), \max(a'_{ij}, b'_{ij})\rangle)$$

$$\text{(iii)} A^c = (\langle a'_{ij}, a_{ij}\rangle)$$

(iv) $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$.

2. Einstein operations of Pythagorean fuzzy matrices

In this section, we shall introduce the Einstein operations on Pythagorean fuzzy matrices (PFMs) and analyse some desirable properties of these operations.

Definition 2.1

Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle)$ be Pythagorean fuzzy matrices of same size. Then

(i) $A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{a_{ij}^2 + b_{ij}^2}{1 + a_{ij}^2 b_{ij}^2}}, \frac{a'_{ij} b'_{ij}}{\sqrt{1 + (1 - {a'_{ij}}^2)(1 - {b'_{ij}}^2)}}\rangle\right)$ is called the Einstein sum of A and B.

(ii) $A \otimes_\varepsilon B = \left(\langle \frac{a_{ij} b_{ij}}{\sqrt{1 + (1 - a_{ij}^2)(1 - b_{ij}^2)}}, \sqrt{\frac{{a'_{ij}}^2 + {b'_{ij}}^2}{1 + {a'_{ij}}^2 {b'_{ij}}^2}}\rangle\right)$ is called the Einstein product of A and B.

These operations are constructed in such a way that they produce PFMs. Since it easy to prove that $\left(\sqrt{\frac{a_{ij}^2+b_{ij}^2}{1+a_{ij}^2 b_{ij}^2}}\right)^2 + \left(\frac{a'_{ij}b'_{ij}}{\sqrt{1+\left(1-a'^2_{ij}\right)\left(1-b'^2_{ij}\right)}}\right)^2 \le 1$. Using these Einstein operations the following equations are obtained for any integer $n = 1,2, \dots \dots$

$$nA = A \oplus_\varepsilon A \oplus_\varepsilon \dots\dots \oplus_\varepsilon A = \left(\langle \sqrt{\frac{\left(1+a_{ij}^2\right)^n - \left(1-a_{ij}^2\right)^n}{\left(1+a_{ij}^2\right)^n + \left(1-a_{ij}^2\right)^n}}, \frac{\sqrt{2}a'^n_{ij}}{\sqrt{\left(2-a'^2_{ij}\right)^n + \left(a'^2_{ij}\right)^n}} \rangle\right)$$

$$A^n = A \otimes_\varepsilon A \otimes_\varepsilon \dots\dots \otimes_\varepsilon A = \left(\langle \frac{\sqrt{2}a_{ij}^n}{\sqrt{\left(2-a_{ij}^2\right)^n + \left(a_{ij}^2\right)^n}}, \sqrt{\frac{\left(1+a'^2_{ij}\right)^n - \left(1-a'^2_{ij}\right)^n}{\left(1+a'^2_{ij}\right)^n + \left(1-a'^2_{ij}\right)^n}} \rangle\right)$$

Einstein operations have the following algebraic properties:

Theorem 2.1

Let $A = \left(\langle a_{ij}, a'_{ij} \rangle\right)$, $B = \left(\langle b_{ij}, b'_{ij} \rangle\right)$ and $C = \left(\langle c_{ij}, c'_{ij} \rangle\right)$ be Pythagorean fuzzy matrices of same size. Then:

(i) $A \oplus_\varepsilon B = B \oplus_\varepsilon A$

(ii) $A \otimes_\varepsilon B = B \otimes_\varepsilon A$

(iii) $(A \oplus_\varepsilon B) \oplus_\varepsilon C = A \oplus_\varepsilon (B \oplus_\varepsilon C)$

(iv) $(A \otimes_\varepsilon B) \otimes_\varepsilon C = A \otimes_\varepsilon (B \otimes_\varepsilon C)$

Proof:

Let $= \left(\langle a_{ij}, a'_{ij} \rangle\right)$, $B = \left(\langle b_{ij}, b'_{ij} \rangle\right)$ and $C = \left(\langle c_{ij}, c'_{ij} \rangle\right)$ be Pythagorean fuzzy matrices of same size.

$$\text{(i) } A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{a_{ij}^2+b_{ij}^2}{1+a_{ij}^2 b_{ij}^2}}, \frac{a'_{ij}b'_{ij}}{\sqrt{1+\left(1-a'^2_{ij}\right)\left(1-b'^2_{ij}\right)}} \rangle\right)$$

$$= \left(\langle \sqrt{\frac{b_{ij}^2+a_{ij}^2}{1+b_{ij}^2 a_{ij}^2}}, \frac{b'_{ij}a'_{ij}}{\sqrt{1+\left(1-b'^2_{ij}\right)\left(1-a'^2_{ij}\right)}} \rangle\right)$$

$$= B \oplus_\varepsilon A.$$

(ii) It can be proved analogously.

$$\text{(iii) } A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{a_{ij}^2+b_{ij}^2}{1+a_{ij}^2 b_{ij}^2}}, \frac{a'_{ij}b'_{ij}}{\sqrt{1+\left(1-a'^2_{ij}\right)\left(1-b'^2_{ij}\right)}} \rangle\right) = \left(\langle d_{ij}, d'_{ij} \rangle\right)$$

$$B \oplus_\varepsilon C = \left(\langle \sqrt{\frac{b_{ij}^2+c_{ij}^2}{1+b_{ij}^2 c_{ij}^2}}, \frac{b'_{ij}c'_{ij}}{\sqrt{1+\left(1-b'^2_{ij}\right)\left(1-c'^2_{ij}\right)}} \rangle\right) = \left(\langle e_{ij}, e'_{ij} \rangle\right)$$

$$(A \oplus_\varepsilon B) \oplus_\varepsilon C = \left(\langle d_{ij}, d'_{ij} \rangle\right) \oplus_\varepsilon \left(\langle c_{ij}, c'_{ij} \rangle\right) = \left(\langle \sqrt{\frac{d_{ij}^2+c_{ij}^2}{1+d_{ij}^2 c_{ij}^2}}, \frac{d'_{ij}c'_{ij}}{\sqrt{1+\left(1-d'^2_{ij}\right)\left(1-c'^2_{ij}\right)}} \rangle\right)$$

$$A \oplus_\varepsilon (B \oplus_\varepsilon C) = \left(\langle a_{ij}, a'_{ij} \rangle\right) \oplus_\varepsilon \left(\langle e_{ij}, e'_{ij} \rangle\right) = \left(\langle \sqrt{\frac{a_{ij}^2+e_{ij}^2}{1+a_{ij}^2 e_{ij}^2}}, \frac{a'_{ij}e'_{ij}}{\sqrt{1+\left(1-a'^2_{ij}\right)\left(1-e'^2_{ij}\right)}} \rangle\right)$$

Then we can compute the final results of $(A \oplus_\varepsilon B) \oplus_\varepsilon C$ and $A \oplus_\varepsilon (B \oplus_\varepsilon C)$ as:

$d_{ij}^2 + c_{ij}^2 = \frac{a_{ij}^2+b_{ij}^2}{1+a_{ij}^2 b_{ij}^2} + c_{ij}^2 = \frac{a_{ij}^2+b_{ij}^2+c_{ij}^2+a_{ij}^2 b_{ij}^2 c_{ij}^2}{1+a_{ij}^2 b_{ij}^2}$,

$$1 + d_{ij}^2 c_{ij}^2 = 1 + c_{ij}^2 \frac{a_{ij}^2 + b_{ij}^2}{1 + a_{ij}^2 b_{ij}^2} = \frac{a_{ij}^2 b_{ij}^2 + b_{ij}^2 c_{ij}^2 + a_{ij}^2 c_{ij}^2}{1 + a_{ij}^2 b_{ij}^2}$$

$$\frac{d_{ij}^2 + c_{ij}^2}{1 + d_{ij}^2 c_{ij}^2} = \frac{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}{a_{ij}^2 b_{ij}^2 + b_{ij}^2 c_{ij}^2 + a_{ij}^2 c_{ij}^2}$$

$d'_{ij} c'_{ij} = \frac{a'_{ij} b'_{ij} c'_{ij}}{\sqrt{1+\left(1-a'^2_{ij}\right)\left(1-b'^2_{ij}\right)}}$,

$$1 + \left(1 - d'^2_{ij}\right)\left(1 - c'^2_{ij}\right) = \frac{4 - 2a'^2_{ij} - 2b'^2_{ij} - 2c'^2_{ij} + a'^2_{ij} b'^2_{ij} + b'^2_{ij} c'^2_{ij} + a'^2_{ij} c'^2_{ij}}{1 + \left(1 - a'^2_{ij}\right)\left(1 - b'^2_{ij}\right)}$$

$$\frac{d'_{ij} c'_{ij}}{\sqrt{1 + \left(1 - d'^2_{ij}\right)\left(1 - c'^2_{ij}\right)}} = \frac{a'_{ij} b'_{ij} c'_{ij}}{\sqrt{4 - 2a'^2_{ij} - 2b'^2_{ij} - 2c'^2_{ij} + a'^2_{ij} b'^2_{ij} + b'^2_{ij} c'^2_{ij} + a'^2_{ij} c'^2_{ij}}}$$

$$(A \oplus_\varepsilon B) \oplus_\varepsilon C = \left(\left\langle \sqrt{\frac{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}{a_{ij}^2 b_{ij}^2 + b_{ij}^2 c_{ij}^2 + a_{ij}^2 c_{ij}^2}}, \frac{a'_{ij} b'_{ij} c'_{ij}}{\sqrt{4 - 2a'^2_{ij} - 2b'^2_{ij} - 2c'^2_{ij} + a'^2_{ij} b'^2_{ij} + b'^2_{ij} c'^2_{ij} + a'^2_{ij} c'^2_{ij}}} \right\rangle\right)$$

Similarly, we have

$$A \oplus_\varepsilon (B \oplus_\varepsilon C) = \left(\left\langle \sqrt{\frac{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}{a_{ij}^2 b_{ij}^2 + b_{ij}^2 c_{ij}^2 + a_{ij}^2 c_{ij}^2}}, \frac{a'_{ij} b'_{ij} c'_{ij}}{\sqrt{4 - 2a'^2_{ij} - 2b'^2_{ij} - 2c'^2_{ij} + a'^2_{ij} b'^2_{ij} + b'^2_{ij} c'^2_{ij} + a'^2_{ij} c'^2_{ij}}} \right\rangle\right)$$

Hence, $(A \oplus_\varepsilon B) \oplus_\varepsilon C = A \oplus_\varepsilon (B \oplus_\varepsilon C)$.

(iv) It can be proved analogously.

Theorem 2. 3

Let $A = (\langle a_{ij}, a'_{ij} \rangle)$ be a PFM, then A^n is also PFM, for a positive integer $n > 0$.

Proof: As $n > 0$ be any positive integer and A be a PFM, then $0 \le a_{ij} \le 1, 0 \le a'_{ij} \le 1$ and $a_{ij}^2 + a'^2_{ij} \le 1, 1 - a_{ij}^2 \ge a'^2_{ij}, 1 - a'^2_{ij} \ge a_{ij}^2, (1 - a_{ij}^2)^n \ge \left(a'^2_{ij}\right)^n$ and then we have,

$$\frac{\sqrt{2} a_{ij}^n}{\sqrt{\left(2 - a_{ij}^2\right)^n + \left(a_{ij}^2\right)^n}} \le \frac{\sqrt{2} a_{ij}^n}{\sqrt{\left(1 + a'^2_{ij}\right)^n + \left(a_{ij}^2\right)^n}}$$

and $\sqrt{\frac{\left(1+a'^2_{ij}\right)^n - \left(1-a'^2_{ij}\right)^n}{\left(1+a'^2_{ij}\right)^n + \left(1-a'^2_{ij}\right)^n}} \le \sqrt{\frac{\left(1+a'^2_{ij}\right)^n - \left(a_{ij}^2\right)^n}{\left(1+a'^2_{ij}\right)^n + \left(a_{ij}^2\right)^n}}$

$$\left(\frac{\sqrt{2} a_{ij}^n}{\sqrt{\left(2 - a_{ij}^2\right)^n + \left(a_{ij}^2\right)^n}}\right)^2 + \left(\sqrt{\frac{\left(1 + a'^2_{ij}\right)^n - \left(1 - a'^2_{ij}\right)^n}{\left(1 + a'^2_{ij}\right)^n + \left(1 - a'^2_{ij}\right)^n}}\right)^2 \le 1$$

Thus A^n is PFM for any positive integer $n > 0$.

Theorem 2.4

Let $A = (\langle a_{ij}, a'_{ij} \rangle)$ be a PFM, then nA is also PFM, for a positive integer $n > 0$.

Proof: As $n > 0$ be any positive integer and A be a PFM, then $0 \le a_{ij} \le 1, 0 \le a'_{ij} \le 1$ and $a_{ij}^2 + a'^2_{ij} \le 1, 1 - a_{ij}^2 \ge a'^2_{ij}, 1 - a'^2_{ij} \ge a_{ij}^2, (1 - a_{ij}^2)^n \ge \left(a'^2_{ij}\right)^n$ and then we have,

$$\sqrt{\frac{\left(1+a_{ij}^2\right)^n-\left(1-a_{ij}^2\right)^n}{\left(1+a_{ij}^2\right)^n+\left(1-a_{ij}^2\right)^n}} \leq \sqrt{\frac{\left(1+a_{ij}^2\right)^n-\left(a_{ij}'^2\right)^n}{\left(1+a_{ij}^2\right)^n+\left(a_{ij}'^2\right)^n}}$$

and $\frac{\sqrt{2}a_{ij}'^n}{\sqrt{\left(2-a_{ij}'^2\right)^n+\left(a_{ij}'^2\right)^n}} \leq \frac{\sqrt{2}a_{ij}'^n}{\sqrt{\left(1+a_{ij}^2\right)^n+\left(a_{ij}'^2\right)^n}}$

$$\left(\sqrt{\frac{\left(1+a_{ij}^2\right)^n-\left(1-a_{ij}^2\right)^n}{\left(1+a_{ij}^2\right)^n+\left(1-a_{ij}^2\right)^n}}\right)^2+\left(\frac{\sqrt{2}a_{ij}'^n}{\sqrt{\left(2-a_{ij}'^2\right)^n+\left(a_{ij}'^2\right)^n}}\right)^2 \leq 1.$$

Thus nA is also PFM, for a positive integer $n > 0$.

Theorem 2.5

Let $A = \left(\langle a_{ij}, a_{ij}'\rangle\right), B = \left(\langle b_{ij}, b_{ij}'\rangle\right)$ be PFMs, then for positive integer $n > 0$,

(i) $n(A \oplus_\varepsilon B) = nA \oplus_\varepsilon nB$

(ii) $(A \otimes_\varepsilon B)^n = A^n \otimes_\varepsilon B^n$

Proof: $A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{a_{ij}^2+b_{ij}^2}{1+a_{ij}^2 b_{ij}^2}}, \frac{a_{ij}' b_{ij}'}{\sqrt{1+\left(1-a_{ij}'^2\right)\left(1-b_{ij}'^2\right)}}\rangle\right)$

is equivalent to $A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{\left(1+a_{ij}^2\right)\left(1+b_{ij}^2\right)-\left(1-a_{ij}^2\right)\left(1-b_{ij}^2\right)}{\left(1+a_{ij}^2\right)\left(1+b_{ij}^2\right)+\left(1-a_{ij}^2\right)\left(1-b_{ij}^2\right)}}, \frac{\sqrt{2}a_{ij}' b_{ij}'}{\sqrt{\left(2-a_{ij}'^2\right)\left(2-b_{ij}'^2\right)+a_{ij}'^2 b_{ij}'^2}}\rangle\right)$

Take $r = \left(1+a_{ij}^2\right)\left(1+b_{ij}^2\right),\ s = \left(1-a_{ij}^2\right)\left(1-b_{ij}^2\right),\ t = a_{ij}'^2 b_{ij}'^2,\ u = \left(2-a_{ij}'^2\right)\left(2-b_{ij}'^2\right)$

$$A \oplus_\varepsilon B = \left(\langle \sqrt{\frac{r-s}{r+s}}, \frac{\sqrt{2t}}{\sqrt{u+t}}\rangle\right)$$

$$n(A \oplus_\varepsilon B) = \left(\langle \sqrt{\frac{\left(1+\frac{r-s}{r+s}\right)^n-\left(1-\frac{r-s}{r+s}\right)^n}{\left(1+\frac{r-s}{r+s}\right)^n+\left(1-\frac{r-s}{r+s}\right)^n}}, \frac{\sqrt{2}\left(\frac{\sqrt{2t}}{\sqrt{u+t}}\right)^n}{\sqrt{\left(2-\frac{2t}{u+t}\right)^n+\left(\frac{2t}{u+t}\right)^n}}\rangle\right)$$

$$= \left(\langle \sqrt{\frac{r^n-s^n}{r^n+s^n}}, \frac{\sqrt{2t^n}}{\sqrt{u^n+t^n}}\rangle\right)$$

$$= \left(\langle \sqrt{\frac{\left(1+a_{ij}^2\right)^n\left(1+b_{ij}^2\right)^n-\left(1-a_{ij}^2\right)^n\left(1-b_{ij}^2\right)^n}{\left(1+a_{ij}^2\right)^n\left(1+b_{ij}^2\right)^n+\left(1-a_{ij}^2\right)^n\left(1-b_{ij}^2\right)^n}}, \frac{\sqrt{2}a_{ij}'^n b_{ij}'^n}{\sqrt{\left(2-a_{ij}'^2\right)^n\left(2-b_{ij}'^2\right)^n+\left(a_{ij}'^2\right)^n\left(b_{ij}'^2\right)^n}}\rangle\right)$$

On the other hand, $nA = \left(\sqrt{\frac{\left(1+a_{ij}^2\right)^n-\left(1-a_{ij}^2\right)^n}{\left(1+a_{ij}^2\right)^n+\left(1-a_{ij}^2\right)^n}}, \frac{\sqrt{2}a_{ij}'^n}{\sqrt{\left(2-a_{ij}'^2\right)^n+\left(a_{ij}'^2\right)^n}}\right)$

$$= \left(\langle \sqrt{\frac{r_1-s_1}{r_1+s_1}}, \frac{\sqrt{2t_1}}{\sqrt{u_1+t_1}}\rangle\right)$$

and

$$nB = \left(\sqrt{\frac{\left(1+b_{ij}^2\right)^n-\left(1-b_{ij}^2\right)^n}{\left(1+b_{ij}^2\right)^n+\left(1-b_{ij}^2\right)^n}}, \frac{\sqrt{2}b_{ij}'^n}{\sqrt{\left(2-b_{ij}'^2\right)^n+\left(b_{ij}'^2\right)^n}}\right)$$

$$= \left(\langle \sqrt{\frac{r_2-s_2}{r_2+s_2}}, \frac{\sqrt{2t_2}}{\sqrt{u_2+t_2}}\rangle\right)$$

Where $r_1 = (1+a_{ij}^2)^n, s_1 = (1-a_{ij}^2)^n, t_1 = (a_{ij}'^2)^n, u_1 = (2-a_{ij}'^2)^n, r_2 = (1+b_{ij}^2)^n, s_2 = (1-b_{ij}^2)^n, t_2 = (b_{ij}'^2)^n, u_2 = (2-b_{ij}'^2)^n$

So, $nA \oplus_\varepsilon nB = \left(\langle \sqrt{\frac{r_1r_2 - s_1s_2}{r_1r_2 + s_1s_2}}, \sqrt{\frac{2t_1t_2}{t_1t_2 + t_1t_2}} \rangle\right)$

$$= \left(\langle \sqrt{\frac{(1+a_{ij}^2)^n(1+b_{ij}^2)^n - (1-a_{ij}^2)^n(1-b_{ij}^2)^n}{(1+a_{ij}^2)^n(1+b_{ij}^2)^n + (1-a_{ij}^2)^n(1-b_{ij}^2)^n}}, \frac{\sqrt{2}a_{ij}'^n b_{ij}'^n}{\sqrt{(2-a_{ij}'^2)^n(2-b_{ij}'^2)^n + (a_{ij}'^2)^n(b_{ij}'^2)^n}} \rangle\right)$$

Hence, $n(A \oplus_\varepsilon B) = nA \oplus_\varepsilon nB$

(ii) It can be proved analogously.

Theorem 2.6

Let $A = (\langle a_{ij}, a'_{ij} \rangle)$, be PFM, then for positive integers $m > 0, n > 0$,

(i) $mA \oplus_\varepsilon nA = (m+n)A$

(ii) $A^m \otimes_\varepsilon A^n = A^{m+n}$

Proof: Similar to Theorem 2.5.

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