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COMMON FIXED POINT THEOREMS IN PARTIAL FUZZY METRIC SPACES USING CONTRACTIVE CONDITION

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Abstract: In this paper, we extend the concept of partial fuzzy metric space by using contractive condition. We also prove that common fixed point theorems for four self mappings which are weakly compatible satisfying some contractive conditions in partial fuzzy metric spaces. Our work extends and generalize the several known results in the literature.

Keywords: Fixed point theorem, Partial fuzzy metric spaces, Contraction mappings, Weakly compatible

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Partial metric space introduced by Matthews [8] is a generalization of the notion of metric space in which the points are allowed to have "self-distance" $[d(x,x) \geq 0]$and proved the Banach contraction mapping theorem in partial metric space. Then, Valero[13], Oltra and Valero[10] and Altunet al.[2] gave some generalization of the result of Matthews. In 2009 Romaguera[11] proved the Caristi type fixed point theorems on this space. Shaban Sedghi, Nabi and Altun[12] introduced the concept of partial fuzzy metric space and proved some of the fixed point results. In 1986, Jungck[5] introduced the concept of compatible mappings and proved that weakly commuting mappings are compatible mappings.After that, Jungck [6], generalized the notion of compatibility by introducing the weakly compatibility. Later Abbas et al.[1] introduced the generalized condition (B) to prove common fixed points for two self mappings.

In this paper, we prove the existence of common fixed point theorems for four self mappings which are weakly compatible satisfying some contractive conditions on partial fuzzy metric spaces for four self mappings.

2. Preliminaries

Definition 2.1 *[2]*

Let X be a metric space. A mapping $F : X \to X$ is said to satisfy a generalized condition (B) associated with self mapping f on X if there exists $\delta \in (0,1)$ and $L \geq 0$ such that

$$d(Fx,Fy) \leq \delta M(x,y) + L\, min\{d(fx,Fx),d(fy,Fy),d(fx,Fy),d(fy,Fx)\}$$

for all x, y ∈ X, where

$$M(x,y) = max\{d(fx,fy), d(fx,Fx), d(fy,Fy), \frac{1}{2}[d(fx,Fy) + d(fy,Fx)]\}$$

Definition 2.2 *[8]*

A partial metric on a non empty set X is a function $p : X \times X \to \Re_+$ such that for all x,y, z ∈ X.

(i) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$
(ii) $p(x,x) \leq p(x,y)$
(iii) $p(x,y) = p(y,x)$
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

The pair (X, p) is called a partial metric space in short (PMS) and p is a partial metric on X. For each partial metric p on X, the function p: $X \times X \to \Re_+$ on family of p-open balls defined by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

is a usual metric on X.

Definition 2.3 *[12]*

A partial fuzzy metric on a non-empty set X is a function $P_M : X \times X \times (0,\infty) \to [0,1]$ such that for all $x,y,z \in X$ and $t,s > 0$

(i) (PM-1) $x = y \Leftrightarrow P_M(x,x,t) = P_M(x,y,t) = P_M(y,y,t)$
(ii) (PM-2) $P_M(x,x,t) \geq P_M(x,y,t)$
(iii) (PM-3) $P_M(x,y,t) = P_M(y,x,t)$
(iv) (PM-4) $P_M(x,y,max\{t,s\}) * P_M(z,z,max\{t,s\}) \geq P_M(x,z,t) * P_M(z,y,s)$
(v) (PM-5) $P_M(x,y,.) : (0,\infty) \to [0,1]$ is continuous

A partial fuzzy metric space is a 3-tuple $(X,P_M,*)$ such that X is a non-empty set and P_M is a partial fuzzy metric on X. It is clear that, if $P_M(x,y,t) = 1$, then from (PM-1) and (PM-2), x = y. But if $x = y$, $P_M(x,y,t)$ may not be 1.
A basic example of a partial fuzzy metric space is the 3-tuple $(\Re_+,P_M,*)$ where

$$P_M(x,y,t) = \frac{t}{t + max\{x,y\}}$$

for all $t > 0$, $x,y \in \Re_+$ and $a * b = ab$.
From (PM-4) for all x,y,z ∈ X and $t > 0$, we have

$$P_M(x,y,t) * P_M(z,z,t) \geq P_M(x,z,t) * P_M(z,y,t)$$

Let (X, M, $*$) and (X, P_M, $*$) be a fuzzy metric space and partial fuzzy metric space respectively. Then mappings $P_{Mi}(x,y,t)$: $X \times X \times (0,\infty) \to [0,1]$, $i \in \{1,2\}$ defined by

$$P_{M1}(x,y,t) = M(x,y,t) * P_M(x,y,t)$$

and $P_{M2}(x,y,t) = M(x,y,t) * a$, are partial fuzzy metrics on X, where $0 < a < 1$.

Example 2.4 *[12]* Let (X, p) is a partial metric space in the sense of Matthews [08] and $P_M : X \times X \times (0,\infty) \to [0,1]$ be a mapping defined as

$$P_M(x,y,t) = \frac{t}{t + p(x,y)}$$

(or)

$$P_M(x,y,t) = exp\left(-\frac{p(x,y)}{t}\right)$$

If $a * b = ab$ for all $a,b \in [0,1]$, then clearly P_M is a partial fuzzy metric, but it may not be a fuzzy meric.

Definition 2.5.*[12]*

Let$(X,P_M,*)$ be a partial fuzzy metric space.

(i) A sequence $\{x_n\}$ in $(X,P_M,*)$ converges to x if and only if $P_M(x,x,t) = \lim_{n\to\infty} P_M(x_n,x,t)$ for every $t > 0$.

(ii) A sequence $\{x_n\}$ in $(X,P_M,*)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ exists.

(iii) If every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, then the partial fuzzy metric space $(X,P_M,*)$ is called complete.

Suppose that $\{x_n\}$ is a sequence in partial fuzzy metric space $(X,P_M,*)$, then we define $L(x_n) = \{x \in X : x_n \to x\}$. The following example shows that every convergent sequence $\{x_n\}$ in a partial fuzzy metric space $(X,P_M,*)$ fails to satisfy Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not unique.

Example 2.6 *[12]* Let $x = [0,\infty)$ and $P_M(x,y,t) = \frac{t}{t+max\{x,y\}}$, then it is clear that $(X,P_M,*)$ is a partial fuzzy metric space where $a * b = ab$ for all $a,b \in [0,1]$. Let $\{x_n\} = \{1,2,1,2,...\}$. Then clearly it is a convergent sequence and for every $x \geq 2$ we have:

$$\lim_{n\to\infty} P_M(x_n,x,t) = P_M(x,x,t).$$

Therefore, $L(x_n) = \{x \in X : x_n \to x\} \to [2,\infty)$, but $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ is not exist, that is $\{x_n\}$ is not Cauchy sequence.

3. Main Results

Theorem 3.1

Let $(X,P_M,*)$ be a complete partial fuzzy metric space. Suppose that *p, q, P and Q* are self mappings on X satisfying the following conditions:

(i) $p(X) \subseteq q(X)$ and $P(X) \subseteq Q(X)$

(ii) There exists $\delta > 0$ and $L \geq 0$ with $\delta + 2L < 1$ such that

$$P_M(Pu,pv,t) \geq \delta M(u,v,t) + L max\{P_M(qu,Pu,t), P_M(Qv,pv,t), P_M(qu,Qv,t), P_M(Qv,Pu,t)\} for all u,v \in X \quad (3.1)$$

where

$$M(u,v,t) = min\{P_M(qu,Qv,t), P_M(qu,Pu,t), P_M(Qv,pv,t), \frac{1}{2}[P_M(qu,pv,t) + P_M(Qv,Pu,t)]\}$$

,

(iii) $p(X)$ or $q(X)$ is closed.

If $\{p,Q\}$ and $\{q,P\}$ are weakly compatible, then p, q, P *and* Q have a unique common fixed point in X.

Proof: Suppose that u_0 is an arbitrary point in X. Since $p(X) \subseteq q(X)$ and $P(X) \subseteq Q(X)$, we can construct a sequence $\{v_n\}$ in X satisfying $v_n = Pu_n = Qu_{n+1}$ and $v_{n+1} = pu_{n+1} = qu_{n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. By applying (3.1) we have

$$P_M(Pu_n,pu_{n+1},t) \geq \delta M(u_n,u_{n+1},t) + L\, max\{P_M(qu_n,Pu_n,t),P_M(Qu_{n+1},pu_{n+1},t), P_M(qu_n,Qu_{n+1},t),P_M(Qu_{n+1},Pu_n,t)\}$$

Since,

$$M(u_n,u_{n+1},t) = min\{P_M(qu_n,Qu_{n+1},t),P_M(qu_n,Pu_n,t),P_M(Qu_{n+1},pu_{n+1},t) \frac{1}{2}[P_M(qu_n,pu_{n+1},t)+P_M(Qu_{n+1},Pu_n,t)]\}$$

$$= min\{P_M(v_{n-1},v_n,t),P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t), \frac{1}{2}P_M(v_{n-1},v_{n+1},t)+P_M(v_n,v_n,t)]\}$$

$$\geq min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t), \frac{1}{2}[P_M(v_{n-1},v_n,t)+P_M(v_n,v_{n+1},t)]\}$$

$$\geq min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\}$$

and $max\{P_M(qu_n,Pu_n,t),P_M(Qu_{n+1},pu_{n+1},t),P_M(qu_n,Qu_{n+1},t),P_M(Qu_{n+1},Pu_n,t)\}$

$$= max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t),P_M(v_{n-1},v_n,t),P_M(v_n,v_n,t)\}$$
$$= max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t),P_M(v_n,v_n,t)\}$$
$$= \max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} .$$

We obtain that

$$P_M(v_n,v_{n+1},t) = P_M(Pu_n,u_{n+1},t) \geq \delta max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} + L max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\}.$$

We separate the proof into following cases.

Case I: If $\max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_{n-1},v_n,t)$ and $\min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_{n-1},v_n,t)$ then

$$P_M(v_n,v_{n+1},t)\}) = P_M(Pu_n,pu_{n+1},t) \geq \delta P_M(v_{n-1},v_n,t) + LP_M(v_{n-1},v_n,t), \geq (\delta + L)P_M(v_{n-1},v_n,t)$$

Let $k_1 = \delta + L$. Since $\delta + 2L < 1$, we have $k_1 < 1$. Therefore,

$$P_M(v_n,v_n+1,t) \geq k_1 P_M(v_n-1,v_n,t).$$

Case II: If $\max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_{n-1},v_n,t)$ and $\min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_n,v_{n+1},t)$ then

$$P_M(v_n,v_{n+1},t) \geq \delta P_M(v_{n-1},v_n,t) + LP_M(v_n,v_{n+1},t)$$

$$P_M(v_n,v_{n+1},t) \geq \frac{\delta}{1-L}P_M(v_{n-1},v_n,t) ,$$

Let, $k_2 = \frac{\delta}{1-L}$. Since $\delta + 2L < 1$, we have $k_2 < 1$. Therefore

$$P_M(v_n,v_{n+1},t) \geq k_2 P_M(v_{n-1},v_n,t).$$

Case III: If $\max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_n,v_{n+1},t)$ and $\min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_{n-1},v_n,t)$ then

$$P_M(v_n,v_{n+1},t) \geq \delta P_M(v_n,v_{n+1},t) + LP_M(v_{n-1},v_n,t)$$

$$P_M(v_n,v_{n+1},t) \geq \frac{L}{1-\delta}P_M(v_{n-1},v_n,t) ,$$

Let, $k_3 = \frac{L}{1-\delta}$. Since $\delta + 2L < 1$, we have $k_3 < 1$. Therefore

$$P_M(v_n,v_{n+1},t) \geq k_3 P_M(v_{n-1},v_n,t).$$

Case IV: If $\max\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_n,v_{n+1},t)$ and $\min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t)\} = P_M(v_n,v_{n+1},t)$ then

$$P_M(v_n,v_{n+1},t) \geq \delta P_M(v_n,v_{n+1},t + LP_M(v_n,v_{n+1},t),$$

This implies that:

$$P_M(v_n, v_{n+1}, t) \geq \frac{L}{1-\delta} P_M(v_{n-1}, v_n, t)$$

Let, $k_4 = \frac{L}{1-\delta}$. Since $\delta + 2L < 1$, we have $k_4 < 1$. Therefore

$$P_M(v_n,v_{n+1},t) \geq k_4 P_M(v_{n-1},v_n,t).$$

Choose $k = \max\{k_1,k_2,k_3,k_4\}$. Therefore $0 < k < 1$. For each $n \in \mathbb{N}$ we obtain that

$$P_M(v_n,v_{n+1},t) \geq k^n P_M(v_0,v_1,t) \quad (3.2)$$

We will prove that $\{v_n\}$ is a Cauchy sequence in $(X,P_M^S,*)$. Let m,n $\in \mathbb{N}$, with m >n.
By applying (3.2) we have

$$\begin{aligned}
P_M(v_m, v_n, t) &\geq [P_M(v_n, v_{n+1}, t) + P_M(v_{n+1}, v_{n+2}, t) + \dots + P_M(v_{m-1}, v_m, t)] \\
&\geq [P_M(v_n, v_{n+1}, t) + P_M(v_{n+1}, v_{n+2}, t) + \dots + P_M(v_{m-1}, v_m, t)] \\
&\geq [k^n + k^{n+1} + \dots + k^{m-1}] p_M(v_0, v_1, t) \\
&\geq k^n [1 + k + \dots + k^{m-2}] P_M(v_0, v_1, t) \\
&\geq \frac{k^n}{1-k} P_M(v_0, v_1, t)
\end{aligned}$$

It follows that $\lim_{n,m\to\infty} P_M(v_m,v_n,t) = 1$. (3.3)

We have

$$\begin{aligned}
P^s(v_m,v_n,t) &= 2P_M(v_m,v_n,t) - P_M(v_m,v_m,t) - P_M(v_n,v_n,t) \\
&\geq 2P_M(v_m,v_n,t) - 1 - 1
\end{aligned}$$

We obtain that $\lim_{n,m\to\infty} P^s(v_m,v_n,t) = 1$.

This implies that $\{v_n\}$ is a Cauchv sequence in $(X, P_M^S, *)$. Since X is complete, we have

$$\lim_{n\to\infty} v_n = z$$

for some $z \in X$.

By Definition 1.4 we obtain that

$$P_M(z,z,t) = \lim_{n\to\infty} P_M(v_n,z,t) = \lim_{n,m\to\infty} P_M(v_m,v_n,t) \quad (3.4)$$

From (3.3) and (3.4), we can conclude that $P_M(z,z,t) = 1$. Assume that q(X) is closed.Therefore, there exists a point $u \in X$ such that $z = qu$. Using (3.1) this yields

$$
\begin{aligned}
P_M(z,Pu,t) &\geq P_M(z,v_{n+1},t) * P_M(v_{n+1},Pu,t) * P_M(v_{n+1},v_{n+1},t)\\
&\geq P_M(z,v_{n+1},t) * P_M(Pu,pu_{n+1},t) * 1\\
&\geq \begin{cases} P_M(z,v_{n+1},t) * \delta max\{P_M(qu,Qu_{n+1},t), P_M(qu,Pu,t),\\ P_M(Qu_{n+1},pu_{n+1},t), \frac{1}{2}P_M(qu,pu_{n+1},t) + P_M(Qu_{n+1},Pu,t)]\}\\ +Lmin\{P_M(qu,Pu,t), P_M(Qu_{n+1},pu_{n+1},t),\\ P_M(qu,Qu_{n+1},t), P_M(Qu_{n+1},Pu,t)\} \end{cases}\\
&= \begin{cases} P_M(z,v_{n+1},t) * \delta max\{P_M(z,v_n,t), P_M(z,Pu,t),\\ P_M(v_n,v_{n+1},t), \frac{1}{2}[P_M(z,v_{n+1},t) + P_M(v_n,Pu,t)]\}\\ +Lmin\{P_M(z,Pu,t), P_M(v_n,v_{n+1},t), P_M(z,v_n,t), P_M(v_n,Pu,t) \end{cases}\\
&\geq \begin{cases} P_M(z,v_{n+1},t) * \delta max\{P_M(z,v_n,t), P_M(z,Pu,t), P_M(v_n,z,t)\\ *P_M(z,v_{n+1},t) * P_M(z,z,t), \frac{1}{2}[P_M(z,v_{n+1},t) + P_M(v_n,z,t),\\ *P_M(z,Pu,t) * P_M(z,z,t)]\} + Lmin\{P_M(z,Pu,t), P_M(v_n,z,t)\}\\ *P_M(z,v_n,t) * P_M(z,z,t), P_M(z,v_n,t),\\ P_M(v_n,z,t) * P_M(z,Pu,t) * P_M(z,z,t) \end{cases}\\
&\geq \begin{cases} P_M(z,v_{n+1},t) * \delta max\{P_M(z,v_n,t) * P_M(z,Pu,t),\\ P_M(v_n,z,t), P_M(z,v_{n+1},t), \frac{1}{2}[P_M(z,v_{n+1},t) + P_M(v_n,z,t),\\ *P_M(z,Pu,t)]\} + L\ min\{P_M(z,Pu,t), P_M(v_n,z,t), P_M(z,v_n,t),\\ P_M(z,v_n,t), P_M(v_n,z,t) * P_M(z,Pu,t)\} \end{cases}
\end{aligned}
$$

Taking the limit as $n \to \infty$ and using the fact that $p(z,z,t) = 1$. We have

$$
\begin{aligned}
P_M(z,Pu,t) &\geq \delta P_M(z,Pu,t) + LP_M(z,Pu,t)\\
&\geq (\delta + L)P_M(z,Pu,t)
\end{aligned}
$$

It follows that $P_M(z,Pu,t) = 1$ and so $Pu = z = qu$. Since P and q are weakly compatible, we obtain that $qPu = Pqu$. Therefore $qz = Pz$.
Since $P(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.Applying (3.1) we have

$$
\begin{aligned}
P_M(z,pv,t) &= P_M(Pu,,pv,t)\\
&\geq \begin{cases} \delta max\{P_M(qu,Qv,t), P_M(qu,Pu,t), P_M(Qv,pv,t), \frac{1}{2}[P_M(qu,pv,t)\\ +P_M(Qv,Pu,t)]\} + Lmin\{P_M(qu,Pu,t), P_M(Qv,pv,t), P_M(qu,Qv,t),\\ P_M(Qv,Pu,t)\} \end{cases}\\
&= \begin{cases} \delta max\{P_M(z,z,t), P_M(z,z,t), P_M(z,pv,t), \frac{1}{2}[P_M(z,pv,t)\\ +P_M(z,z,t)\} + Lmin\{P_M(z,z,t), P_M(z,pv,t), P_M(z,z,t), P_M(z,z,t)\} \end{cases}\\
&\geq \delta max\{1, 1, P_M(z,pv,t), \frac{1}{2}[P_M(z,pv,t), 1]\} + Lmin\{1, P_M(z,pv,t), 1, 1\}\\
&\geq \delta P_M(z,pv,t)
\end{aligned}
$$

This implies that $P_M(z,pv,t) = 1$ and so $pv = z = Qv$. Since Q and p are weakly compatible.
We obtain that $pQv = Qpv$. Therefore $pz = Qz$.
We next prove that z is a common fixed point of *p, q, P and Q* .Using (3.1) this yields

$$
\begin{aligned}
P_M(Pz,z,t) &= P_M(Pz,pv,t)\\
&\geq \left\{ \begin{array}{c} \delta max\{P_M(qz,Qv,t), P_M(qz,Pz,t), P_M(Qv,pv,t), \frac{1}{2}\left[P_M(qz,pv,t)\right.\\ \left.+P_M(Qv,Pz,t)\right]\} + Lmin\{P_M(qz,Pz,t), P_M(Qv,pv,t),\\ P_M(qz,Qv,t), P_M(Qv,Pz,t)\} \end{array} \right\}\\
&= \left\{ \begin{array}{c} \delta max\{P_M(Pz,z,t), P_M(Pz,Pz,t), P_M(z,z,t), \frac{1}{2}[P_M(Pz,z,t)\\ +P_M(z,Pz,t)\} + Lmin\{P_M(Pz,Pz,t), P_M(z,z,t), P_M(Pz,z,t), P_M(z,Pz,t)\} \end{array} \right\}\\
&\geq \left\{ \begin{array}{c} \delta max\{P_M(Pz,z,t), P_M(Pz,Pz,t), 1, \frac{1}{2}[P_M(Pz,z,t) + P_M(z,Pz,t)]\}+\\ Lmin\{P_M(Pz,Pz,t), 1, P_M(Pz,z,t), P_M(z,Pz,t)\} \end{array} \right\}\\
&\geq \delta P_M(Pz,z,t)
\end{aligned}
$$

Hence $P_M(Pz,z,t) = 1$ and so $qz = Pz = z$. Similarly, applying (3.1) we obtain that

$$
\begin{aligned}
P_M(z,pz,t) &= P_M(Pz,,pz,t)\\
&\geq \left\{ \begin{array}{c} \delta max\{P_M(qz,Qz,t), P_M(qz,Pz,t), P_M(qz,pz,t), \frac{1}{2}\left[P_M(qz,pz,t)\right.\\ \left.+P_M(Qz,Pz,t)\right]\} + Lmin\{P_M(qz,Pz,t), P_M(Qz,pz,t), P_M(qz,Qz,t),\\ P_M(Qz,Pz,t)\} \end{array} \right\}\\
&= \left\{ \begin{array}{c} \delta max\{P_M(z,pz,t), P_M(z,z,t), P_M(pz,pz,t), \frac{1}{2}\left[P_M(z,pz,t) + P_M(pz,z,t)\right\}+\\ Lmin\{P_M(z,z,t), P_M(pz,pz,t), P_M(z,pz,t), P_M(pz,z,t)\} \end{array} \right\}\\
&\geq \left\{ \begin{array}{c} \delta max\{P_M(z,pz,t), P_M(z,z,t), P_M(pz,z,t)\frac{1}{2}[P_M(z,pz,t) + P_M(pz,z,t)]\}+\\ Lmin\{1, 1, P_M(z,pz,t), 1, P_M(pz,z,t)\} \end{array} \right\}\\
&\geq \delta P_M(z,pz,t)
\end{aligned}
$$

This implies that $P_M(z,pz,t) = 1$ and so $Qz = pz = z$.
Therefore z is a common fixed point of *p,q,P* and *Q*.
We will prove the uniqueness of a common fixed point of p, q, P and Q.
Let w be any common fixed point of p, q, P and Q.
By applying (3.1) it follows that

$$
\begin{aligned}
P_M(z,w,t) &= P_M(Pz,pw,t)\\
&\geq \left\{ \begin{array}{c} \delta max\{P_M(qz,Qw,t), P_M(qz,Pz,t), P_M(Qw,pw,t), \frac{1}{2}\left[P_M(qz,pw,t)\right.\\ \left.+P_M(Qw,Pz,t)\right]\} + Lmin\{P_M(qz,Pz,t), P_M(Qw,pw,t), P_M(qz,Qw,t),\\ P_M(Qw,Pz,t)\} \end{array} \right\}\\
&= \left\{ \begin{array}{c} \delta max\{P_M(z,w,t), P_M(z,z,t), P_M(w,w,t), \frac{1}{2}[P_M(z,w,t) + P_M(w,z,t)\}+\\ Lmin\{P_M(z,z,t), P_M(w,w,t), P_M(z,w,t), P_M(w,z,t)\} \end{array} \right\}\\
&\geq \delta P_M(z,w,t)
\end{aligned}
$$

This implies that $P_M(z,w,t) = 1$ and so $z= w$. Hence p, q, P and Q have a unique common fixed point in X.
Letting P = p and Q = q in theorem 3.1 we immediately obtain the following corollary.

Corollary 3.2

Let $(X,P_M,*)$ be a complete partial fuzzy metric space. Suppose that p and q are self mappings on X satisfying the following conditions:

(i) $p(X) \subseteq q(X)$

(ii) There exists $\delta > 0$ and $L \geq 0$ with $\delta + 2L < 1$ such that

$P_M(pu,pv,t) \geq \delta M(u,v,t) + L\min\{P_M(qu,pu,t),P_M(qv,pv,t),P_M(qu,qv,t),P_M(qv,pu,t)\}$ for all $u,v \in X$, where

$$M(u,v,t) = max\{P_M(qu,qv,t), P_M(qu,pu,t), P_M(qv,pv,t), \frac{1}{2}[P_M(qu,pv,t) + P_M(qv,pu,t)]\}$$

(iii) $p(X)$ or $q(X)$ is complete

If {p,q} are weakly compatible, then p,q have a unique common fixed point in X.

Theorem 3.3

Let $(X,P_M,*)$ be a complete partial fuzzy metric space. Suppose that p,q, P and Q are self mappings on X satisfying the following conditions:

(i) $p(X) \subseteq q(X)$ and $P(X) \subseteq Q(X)$

(ii) There exists $\delta > 0$ and $L \geq 0$ with $\delta + 2L < 1$ such that

$P_M(Pu,pv,t) \geq \delta M(u,v,t) + L\min\{P_M(qu,Pu,t),P_M(Qv,pv,t),P_M(qu,Qv,t),P_M(Qv,Pu,t)\}$ for all $u,v \in X$ where

$$M(u,v,t) = max\{P_M(qu,Qv,t), \frac{1}{2}\left[P_M(qu,Pu,t) + P_M(Qv,pv,t)\right]\frac{1}{2}[P_M(qu,pv,t) + P_M(Qv,Pu,t)]\}$$

(iii) $p(X)$ or $q(X)$ is closed

If $\{p,Q\}$ and $\{q,P\}$ are weakly compatible, then p, q, P and Q have a unique common fixed point in X.

Proof: Since the inequality implies the equality (3.1) , we have the result obtain from theorem 3.1

Theorem 3.4

Let $(X,P_M,*)$ be a complete partial fuzzy metric space. Suppose that p,q, P and Q are self mappings on X satisfying the following conditions:

(i) $p(X) \subseteq q(X)$ and $P(X) \subseteq Q(X)$

(ii) There exists $\delta > 0$ and $L \geq 0$ with $\delta + L < \frac{1}{2}$ such that

$P_M(Pu,pv,t) \geq \delta M(u,v,t) + L\min\{P_M(qu,Pu,t),P_M(Qv,pv,t),P_M(qu,Qv,t),P_M(Qv,Pu,t)\}$ for all $u,v \in X$, where $M(u,v,t) = max\{P_M(qu,Qv,t),P_M(qu,Pu,t),P_M(Qv,pv,t),P_M(qu,pv,t),P_M(Qv,Pu,t)\}$

(iii) $p(X)$ or $q(X)$ is closed

If {p,Q} and {q,P} are weakly compatible, then p, q, P and Q have a unique common fixed point in X.

Proof: Suppose that u_0 is an arbitrary point in U. Since $p(X) \subseteq q(X)$ and $P(X) \subseteq Q(X)$. We can construct a sequence $\{v_n\}$ in X satisfying $v_n = Pu_n = Qu_{n+1}$ and $v_{n+1} = pu_{n+1} = qu_{n+2}$ for all $n \in N \cup \{0\}$. This yields,

$$P_M(Pu_n, pu_{n+1}, t) \geq \begin{cases} \delta M(u_n, u_{n+1}, t) + Lmin\{P_M(qu_n, Pu_n, t), P_M(Qu_{n+1}, pu_{n+1}, t), \\ \qquad P_M(qu_n, pu_{n+1}, t), P_M(Qu_{n+1}, Pu_n, t) \end{cases}\}$$

$$\begin{aligned} M(u_n, u_{n+1}, t) &= \begin{cases} max\{P_M(qu_n, Qu_{n+1}, t), P_M(qu_n, Pu_n, t), P_M(Qu_{n+1}, pu_{n+1}, t), \\ \qquad P_M(qu_n, pu_{n+1}, t) P_M(Qu_{n+1}, Pu_n, t)\} \end{cases}\} \\ &= \begin{cases} max\{P_M(v_{n-1}, v_n, t), P_M(v_{n-1}, v_n, t), P_M(v_n, v_{n+1}, t), \\ \qquad P_M(v_{n-1}, v_{n+1}, t), P_M(v_n, v_n, t)\} \end{cases}\} \\ &\geq \begin{cases} max\{P_M(v_{n-1}, v_n, t), P_M(v_n, v_{n+1}, t), P_M(v_{n-1}, v_{n+1}, t), \\ \qquad P_M(v_n, v_{n+1}, t), P_M(v_n, v_n, t), 1\} \end{cases}\} \\ &\geq max\{P_M(v_{n-1}, v_n, t), P_M(v_n, v_{n+1}, t)\} \end{aligned}$$

And $\min\{P_M(qu_n,Pu_n,t),P_M(Qu_{n+1},pu_{n+1},t),P_M(qu_n,Qu_{n+1},t),P_M(Qu_{n+1},Pu_n,t)\}$

$= min\{P_M(v_{n-1},v_n,t),P_M(v_n,v_{n+1},t),P_M(v_{n-1},v_n,t),P_M(v_n,v_n,t)\}$

$= min\{P_M(v_{n-1},v_n,t),\}$

We obtain that

$$\begin{aligned} P_M(v_n, v_{n+1}, t) &= P_M(Pu_n, pu_{n+1}, t) \\ &\geq \delta max\{P_M(v_{n-1}, v_n, t), P_M(v_n, v_{n+1}, t)\} + LP_M(v_{n-1}, v_n, t), \} \\ (1-\delta)P_M(v_n, v_{n+1}, t) &\geq (\delta + L)P_M(v_{n-1}, v_n, t) \\ P_M(v_n, v_{n+1}, t) &\geq \frac{\delta + L}{1-\delta} P_M(v_{n-1}, v_n, t) \end{aligned}$$

Let k $=\frac{\delta+L}{1-\delta}$. Since $\delta + L < \frac{1}{2}$, we have k <1. Therefore $P_M(v_n,v_{n+1},t) \geq kP_M(v_{n-1},v_n,t)$. Therefore $0 < k < 1$, for each n ∈ N. Letting P = p and Q = q we immediately have the following result.

Corollary 3.5

Let $(X,P_M,*)$ be a complete partial fuzzy metric space. Suppose that p and q are self mappings on X satisfying the following conditions:

(i) p(X) ⊆ q(X)

(ii) There exists $\delta > 0$ and $L \geq 0$ with $\delta + L < \frac{1}{2}$ such that

$P_M(pu,pv,t) \geq \delta M(u,v,t) + Lmin\{P_M(qu,pu,t),P_M(qv,pv,t),P_M(qu,Qv,t),P_M(qv,pu,t)\}$ for all u,v ∈ X, where

$M(u,v,t) = max\{P_M(qu,qv,t),P_M(qu,pu,t),P_M(qv,pv,t),P_M(qu,pv,t),P_M(qv,pu,t)\}$

(iii) p(X) or q(X) is complete.

If $\{p,q\}$ are weakly compatible, then p,q have a unique common fixed point in X .

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