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PROPERTIES OF CONTRA $g^{\#}$-CONTINUOUS MAPS AND SEPARATION AXIOMS

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Abstract. In this paper, we introduce contra $g^{\#}$-continuous maps, study some of its properties and discuss its relationships with some topological maps and separation axioms.

Keywords: g-closed set, $g^{\#}$-closed set, contra α-continuous map, contra $g^{\#}$-continuous map, $g^{\#}$-irresolute map.

AMS Subject Classification: 54A05, 54D10.

1. Introduction

In the literature there are many types of continuities introduced by various authors. Quite recently, Jafari and Noiri introduced and investigated the notions of contra-precontinuity, contra-α-continuity, contra-g-continuity and contra-super-continuity as a continuation of research done by Dontchev [5], and Dontchev and Noiri [6] on the interesting notions of contra-continuity and contra-semi-continuity.

This paper devotes to introduce and investigate a new class of maps called contra $g^{\#}$-continuous maps which are weaker than contra-continuity and stronger than contra g-continuity, contra sg-continuity and contra gs- continuity. The main results of the paper are that several properties concerning contra $g^{\#}$-continuous maps. Furthermore, the relationships between the contra $g^{\#}$-continuity and some topological maps as well as separation axioms are also investigated.

Throughout this paper, (X,τ), (Y,σ) and (Z,ρ) (briefly X, Y and Z) represent non-empty topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of a space X, cl(A), int(A) and C(A) denotes the closure of A, the interior of A and the complement of A respectively.

2. Preliminaries

We recall the following definitions which are useful in the sequel.

Definition 2.1

A subset A of a space X is called

(i) a semi-open [10] if A $\subseteq$ cl (int A);

(ii) an α-open [14] if A $\subseteq$ int (cl (int A)) and

(iii) a semi-closed [10] (an α-closed [14]) if C(A) is a semi-open (an α-open).

The semi-closure (α-closure) of a subset A of X is denoted by scl(A) (αcl(A)) and is the intersection of all semi-closed (α-closed) sets containing A.

Definition 2.2

A subset A of a space X is called:

(i) g-closed [11] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(ii) sg-closed [2] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X;

(iii) gs-closed [1] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(iv) αg-closed [12] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(v) $g^{\#}$-closed [15] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is αg-open in X and

(vi) An g-open [11] (an sg-open [2], an gs-open [1], an αg-open [12] and an $g^{\#}$-open [15]) if C(A) is g-closed (sg-closed, gs-closed, αg-closed and $g^{\#}$-closed).

Definition 2.3

A map f: X $\rightarrow$ Y is said to be

(i) contra-continuous [5] if f^{-1} (V) is closed in X for every open set V of Y;

(ii) contra semi-continuous [6] if f^{-1} (V) is semi-closed in X for every open set V of Y;

(iii) contra α-continuous [8] if f^{-1}(V) is α-closed in X for every open set V of Y;

(iv) contra g-continuous [9] if f^{-1}(V) is g-closed in X for every open set V of Y;

(v) contra sg-continuous [6] if f^{-1}(V) is sg-closed in X for every open set V of Y;

(vi) contra gs-continuous [6] if f^{-1}(V) is gs-closed in X for every open set V of Y;

(vii) $g^{\#}$-irresolute [15] if f^{-1}(V) is $g^{\#}$-closed in X for every $g^{\#}$-closed set V of Y;

(viii) gc-irresolute [3] if f^{-1}(V) is g-closed in X for every g-closed set V of Y;

(ix) sg-irresolute [4] if f^{-1}(V) is sg-closed in X for every sg-closed set V of Y;

(x) pre $g^{\#}$-closed [15] if f(V) is $g^{\#}$-closed in Y for every $g^{\#}$-closed set V of X;

(xi) $g^{\#}$-continuous [15] if f^{-1}(V) is $g^{\#}$-closed in X for every closed set V of Y and

(xii) preclosed [7] if f(V) is preclosed in Y for every closed set V of X.

Theorem 2.4 *[15]*

In a topological space X,

(i) Every closed set is $g^{\#}$-closed.

(ii) Every $g^{\#}$-closed set is g-closed and hence sg-closed and gs-closed.

The converses of the above statements are not true in general.

Definition 2.5 *[15]*

A space X is called

(i) an $\alpha T_{1/2}^{\#}$ space if every $g^{\#}$-closed set in it is α-closed;

(ii) an $sT_b^{\#}$ space if every semi-closed set in it is $g^{\#}$-closed ;

(iii) an $\alpha T_c^{\#}$ space if every α-closed set in it is $g^{\#}$-closed and

(iv) a T_b space if every gs-closed set in it is closed.

***Definition 2.6** [13]*

A space X is called a locally indiscrete [13] if every open set in it is closed..

3. Properties of contra $g^{\#}$-continuous maps

Definition 3.1

A map f: X $\rightarrow$ Y is called contra $g^{\#}$-continuous if f^{-1} (V) is $g^{\#}$-closed set of X for every open set V of Y.

Theorem 3.2

(i) Every contra-continuous map is contra $g^{\#}$-continuous.

(ii) Every contra $g^{\#}$-continuous map is contra g-continuous and hence contra sg-continuous and contra gs-continuous.

Proof: It is obvious.

The converses of the above statements are not true as we see the following examples.

Example 3.3 Let X = Y = {a, b, c}. Let τ = {ϕ, X, {a, b}} and σ = {ϕ, Y, {a, c}}. Let f: X $\rightarrow$ Y be the identity map. Then f is contra $g^{\#}$-continuous map but it is not contra-continuous.

Example 3.4 Let X = Y = {a, b, c }. Let τ = {ϕ, X, {a}, {b, c}} and σ = {ϕ, Y, {b}}. Let f: X $\rightarrow$ Y be the identity map. Then f is contra g-continuous map and hence contra sg-continuous map and contra gs-continuous map but it is not contra $g^{\#}$-continuous.

Theorem 3.5

The composition of two contra $g^{\#}$-continuous maps need not be contra $g^{\#}$-continuous map.

The following example supports the above theorem.

Example 3.6 Let X = Y = Z = {a, b, c}. Let τ = {ϕ, X, {a, b}}, σ = {ϕ, Y, {a, c}}and ρ = {ϕ, Z, Z}}. Let f: X $\rightarrow$ Y be the identity map and g: Y $\rightarrow$ Z be the identity map. Then f is contra $g^{\#}$-continuous map and g is contra $g^{\#}$-continuous map. But their composition g o f: X $\rightarrow$ Z is not contra $g^{\#}$-continuous.

Theorem 3.7

Let f:X $\rightarrow$ Y be a map. Then the following statements are equivalent.

(i) f is contra $g^{\#}$-continuous.

(ii) The inverse image of each open set in Y is $g^{\#}$-closed in X.

(iii) The inverse image of each closed set in Y is $g^{\#}$-open in X.

Proof: (i) ═► (ii) : Let G be any open set in Y. By the assumption of (i), f^{-1}(G) is $g^{\#}$-closed in X.
(ii)═►(iii) : Let G be any closed set in Y. Then Y–G is open set in Y. By the assumption of (ii), f^{-1}(Y–G) = X–f^{-1}(G) is $g^{\#}$-closed in X.
Therefore f^{-1}(G) is $g^{\#}$-open in X.
(iii) ═► (i) : Let G be any open set in Y. Then Y–G is closed in Y. By the assumption of (iii), f^{-1}(Y–G) = X–f^{-1}(G) is $g^{\#}$-open in X.
Therefore, f^{-1}(G) is $g^{\#}$-closed in X. Thus f is contra $g^{\#}$-continuous map.

Theorem 3.8

Let f: X $\rightarrow$ Y be surjective, $g^{\#}$-irresolute and pre $g^{\#}$-closed, and g: Y $\rightarrow$ Z be any map. Then g o f: X $\rightarrow$ Z is contra $g^{\#}$-continuous if and only if g is contra $g^{\#}$-continuous.

Proof: Let g o f: X$\rightarrow$Z be contra $g^{\#}$-continuous map. Let F be an open subset of Z. Then

$(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a $g^{\#}$-closed subset of X. Since f is pre $g^{\#}$-closed, $f(f^{-1}(g^{-1}(F))) = g^{-1}(F)$ is $g^{\#}$-closed in Y. Thus g is contra $g^{\#}$-continuous map.

Conversely, let g: $Y \to Z$ be contra $g^{\#}$-continuous map. Let G be an open subset of Z. Since g is contra $g^{\#}$-continuous, $g^{-1}(G)$ is $g^{\#}$-closed in Y. Since f is $g^{\#}$-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $g^{\#}$-closed in X. Hence g o f is contra $g^{\#}$-continuous map.

4. Relation with other maps

Theorem 4.1

If f: $X \to Y$ is $g^{\#}$-irresolute map and g: $Y \to Z$ is contra-continuous map, then the composition gof: $X \to Z$ is contra $g^{\#}$-continuous map.

Proof: Let G be an open set in Z. Since g is contra-continuous, $g^{-1}(G)$ is closed in Y. It implies that $g^{-1}(G)$ is $g^{\#}$-closed in Y. Since f is $g^{\#}$-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $g^{\#}$-closed in X. Therefore, g o f is contra $g^{\#}$-continuous map.

Corollary 4.2

If f: $X \to Y$ is $g^{\#}$-irresolute map and g: $Y \to Z$ is contra $g^{\#}$-continuous map, then g o f: $X \to Z$ is contra $g^{\#}$-continuous map.

Theorem 4.3

If f: $X \to Y$ is gc-irresolute map and g: $Y \to Z$ is contra $g^{\#}$-continuous map, then g o f: $X \to Z$ is contra g-continuous map.

Proof: Let G be an open set in Z. Since g is contra $g^{\#}$-continuous map, $g^{-1}(G)$ is $g^{\#}$-closed in Y. It implies that g-closed in Y. Since f is gc-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is g-closed in X. Thus g o f is contra g-continuous map.

Corollary 4.4

If f: $X \to Y$ is sg-irresolute map and g: $Y \to Z$ is contra $g^{\#}$-continuous map, then g o f: $X \to Z$ is contra sg-continuous map.

Theorem 4.5

Let $\{X\lambda / \lambda \in \Omega\}$ be any family of topological spaces. If f: $X \to \Pi X\lambda$ is a contra $g^{\#}$-continuous map, then $Pr\lambda$ o f: $X \to X\lambda$ is contra $g^{\#}$-continuous for each $\lambda \in \Omega$, where $Pr\lambda$ is the projection of $\Pi X\lambda$ onto $X\lambda$.

Proof : We shall consider a fixed $\lambda \in \Omega$. Suppose $U\lambda$ is an arbitrary open set in $X\lambda$. Then $Pr\lambda^{-1}(U\lambda)$ is open in $\Pi X\lambda$. Since f is contra $g^{\#}$-continuous, we have by definition $f^{-1}(Pr\lambda^{-1}(U\lambda)) = (Pr\lambda \circ f)^{-1}(U\lambda)$ is $g^{\#}$-closed in X. Therefore, $Pr\lambda$ o f is contra $g^{\#}$-continuous.

Theorem 4.6

Let f: $X \to Y$ be a map and g: $X \to X \times Y$ the graph function of f, defined by g(x) = (x, f(x)) for every $x \in X$. If g is contra $g^{\#}$-continuous, then f is contra $g^{\#}$-continuous.

Proof: Let U be an open set in Y. Then X X U is an open set in X X Y. It follows from Theorem 2.7 that $f^{-1}(U) = g^{-1}(X \times U)$ is $g^{\#}$-closed in X. Thus, f is contra $g^{\#}$-continuous.

5. Relation with Separation Axioms

Theorem 5.1

Let f: $X \to Y$ be a contra $g^{\#}$-continuous map. If X is an $\alpha T_{1/2}^{\#}$ space, then f is contra α-continuous map.

Proof: Let V be an open set of Y. Since f is contra $g^{\#}$-continuous, $f^{-1}(V)$ is a $g^{\#}$-closed set of X. Since X is an $\alpha T½^{\#}$ space, $f^{-1}(V)$ is an α-closed set of X. Therefore f is a contra α-continuous map.

Theorem 5.2

Let f: X $\rightarrow$ Y be a contra semi-continuous map. If X is an sT b$^{\#}$ space, then f is contra $g^{\#}$-continuous map.

Proof: Let V be an open set of Y. Since f is contra semi-continuous, $f^{-1}(V)$ is a semi-closed set of X. Since X is an $sTb^{\#}$ space, $f^{-1}(V)$ is a $g^{\#}$-closed set of X. Therefore, f is a contra $g^{\#}$-continuous map.

Theorem 5.3

Let f: X $\rightarrow$ Y be a contra α-continuous map. If X is an $\alpha Tc^{\#}$ space, then f is contra $g^{\#}$-continuous map.

Proof: Let V be an open set of Y. Since f is contra α-continuous, $f^{-1}(V)$ is a α-closed set of X. Since X is an $\alpha Tc^{\#}$ space, $f^{-1}(V)$ is a $g^{\#}$-closed set of X. Therefore, f is contra $g^{\#}$-continuous map.

Theorem 5.4

Let f: X $\rightarrow$ Y be a contra gs-continuous map. If X is a Tb space, then f is contra $g^{\#}$-continuous map.

Proof : Let V be an open set of Y. Since f is contra gs-continuous, $f^{-1}(V)$ is gs-closed set of X. Since X is a Tb space, it is a closed set of X. It implies that $f^{-1}(V)$ is $g^{\#}$-closed set of X. Therefore, f is a contra $g^{\#}$-continuous map.

Theorem 5.5

Let f: X $\rightarrow$ Y be a surjective, preclosed, contra $g^{\#}$-continuous map and X be aTb space, then Y is locally indiscrete.

Proof: Suppose V is open set in Y. By hypothesis, f is contra $g^{\#}$-continuous map, $f^{-1}(V)$ is $g^{\#}$-closed and hence gs-closed in X. Since X is a Tb space, $f^{-1}(V)$ is closed in X. Since f is preclosed, V is preclosed in Y. Now we have $cl(V) = cl(int(V)) \subseteq V$. This means that V is closed in Y. Thus Y is locally indiscrete.

Theorem 5.6

Let X and Z be any topological spaces and Y be a Tb space. If f: X $\rightarrow$ Y is $g^{\#}$-continuous map and g: Y $\rightarrow$ Z is contra gs-continuous map, then g o f: X$\rightarrow$ Z is contra $g^{\#}$-continuous map.

Proof: Let G be an open set in Z. Since g is contra gs-continuous, $g^{-1}(G)$ is gs-closed in Y. But Y is a Tb space, $g^{-1}(G)$ is closed in Y. Since f is $g^{\#}$-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $g^{\#}$-closed in X. Therefore, gof is contra $g^{\#}$-continuous map.

Corollary 5.7

Let X and Z be any topological spaces and Y be a Tb space. If f: X $\rightarrow$ Y is $g^{\#}$-irresolute map and g: Y $\rightarrow$ Z is contra gs-continuous map, then g o f: X $\rightarrow$ Z is contra $g^{\#}$-continuous map.

Theorem 5.8

Let X and Z be any topological spaces and Y be a $sTb^{\#}$ space. If f: X $\rightarrow$ Y is $g^{\#}$-irresolute map and g: Y $\rightarrow$ Z is contra-continuous map, then g o f: X $\rightarrow$ Z is contra $g^{\#}$-continuous map.

Proof: Let G be an open set in Z. Since g is contra-continuous, $g^{-1}(G)$ is closed and hence semi-closed in Y. But Y is a $sTb^{\#}$ space, $g^{-1}(G)$ is $g^{\#}$-closed in Y. Since f is $g^{\#}$-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $g^{\#}$-closed in X. Therefore, g o f is contra $g^{\#}$-continuous map.

Corollary 5.9

Let X and Z be any topological spaces and Y be an $\alpha Tc^{\#}$ space. If f: X $\rightarrow$ Y is $g^{\#}$-irresolute map and g: Y $\rightarrow$ Z is contra-continuous map, then g o f: X $\rightarrow$ Z is contra $g^{\#}$-continuous map.

Definition 5.10

A space X is called $g^{\#}$-connected provided that X is not the union of two disjoint non-empty $g^{\#}$-open sets.

Theorem 5.11

If f: X $\rightarrow$ Y is contra $g^{\#}$-continuous surjection and X is $g^{\#}$-connected, then Y is connected.

Proof : Suppose that Y is not connected space. There exist non-empty disjoint open sets V1 and V2 such that Y = V1 $\cup$ V2. Therefore \, V1 and V2 are clopen in Y. Since f is contra $g^{\#}$-continuous, f^{-1}(V1) and f^{-1}(V2) are $g^{\#}$-open in X.

Moreover, f^{-1}(V1) and f^{-1}(V2) are non-empty disjoint and X = f^{-1}(V1) $\cup$ f^{-1}(V2). This shows that X is not $g^{\#}$-connected. This contradicts that Y is not connected assumed. Hence Y is connected.

Definition 5.12

A space X is said to be

(i) $g^{\#}$-compact (strongly S-closed [5]) if every $g^{\#}$-open (respectively closed) cover of X has a finite subcover;

(ii) countably $g^{\#}$-compact (strongly countably S-closed) if every countable cover of X by $g^{\#}$-open (respectively closed) sets has a finite subcover;

(iii) $g^{\#}$-Lindelöf (strongly S- Lindelöf) if every $g^{\#}$-open (respectively closed) cover of X has a countable subcover.

Theorem 5.13

The contra $g^{\#}$-continuous images of $g^{\#}$-compact ($g^{\#}$-Lindelöf, countably $g^{\#}$-compact) spaces are strongly S-closed (respectively strongly S- Lindelöf, strongly countably S-closed).

Proof : Suppose that f: X $\rightarrow$ Y is a contra $g^{\#}$-continuous surjection. Let $\{V\alpha : \alpha \in I\}$ be any closed cover of Y. Since f is contra $g^{\#}$-continuous, then $\{f^{-1}(V\alpha) : \alpha \in I\}$ is an $g^{\#}$-open cover of X and hence there exists a finite subset I0 of I such that $X = \cup \{f^{-1}(V\alpha) : \alpha \in I0\}$. Therefore, we have $Y = \cup \{V\alpha : \alpha \in I0\}$ and Y is strongly S-closed. The other proofs can be obtained similarly.

Definition 5.14

A space X is said to be :

(i) $g^{\#}$-closed-compact if every $g^{\#}$-closed cover of X has a finite subcover;

(ii) countably $g^{\#}$-closed-compact if every countable cover of X by $g^{\#}$-closed sets has a finite subcover;

(iii) $g^{\#}$-closed-Lindelöf if every $g^{\#}$-closed cover of X has a countable subcover.

Theorem 5.15

The contra $g^{\#}$-continuous images of $g^{\#}$-closed-compact ($g^{\#}$-closed-Lindelöf, countably $g^{\#}$-closed-compact) spaces are compact (respectively Lindelöf, countably compact).

Proof : Suppose that f: X $\rightarrow$ Y is a contra $g^{\#}$-continuous surjection. Let $\{V\alpha : \alpha \in I\}$ be any open cover of Y. Since f is contra $g^{\#}$-continuous, then $\{f^{-1}(V\alpha) : \alpha \in I\}$ is a $g^{\#}$-closed cover of X. Since X is $g^{\#}$-closed-compact, there exists a finite subset I0 of I such that $X = \cup \{f^{-1}(V\alpha) : \alpha \in I0\}$. Therefore, we have $Y = \cup \{V\alpha : \alpha \in I0\}$ and Y is compact.

The other proofs can be obtained similarly.

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