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RESULTS FOR DAMPED SECOND-ORDER NEUTRAL INTEGRAL EQUATION WITH IMPULSES AND INFINITE DELAY

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Abstract. In this study, we discuss the existence results for damped second order neutral integral equation with impulses and infinite delay in Banach spaces (BS). We use fixed point(fp) theorem of Sadovskii's combined with the theories of cosine function(fn).

1. Introduction

Differential equations from both theoretical and practical angles are a very significant mathematical topic. The practical significance is provided by the fact that differential equations describe the most significant time-dependent science, social and economic issues. The existence results for damped second order neutral integral equation with impulses and infinite delay have appeared over the previous several years as a popular explanation of identified evolution events of certain actual life issues. Some of the basic ideas from [3, 4, 12, 15, 19] will be used, and for more data on this paper we refer to [1, 2, 5, 9, 10, 13, 16, 17, 20, 21, 24].

Throughout this document, (X,k.k) is a BS and A´ is the infinitesimal generator of a strongly continuous(cts) cosine family $(C´(\iota))_{\iota\in R}$ of bounded linear operator(BLO) on X. We define sine fn $(S´(\iota))_{\iota\in R}$ by S´$(\iota)\omega = \int_0^\iota \acute{\mathfrak{C}}(s)\omega ds, \omega \in \mathfrak{X}, \iota \in \mathbb{R}$. Assume kC´($\iota$)k ≤ M´ , kS´($\iota$)k ≤ N´, ∀$\iota$ ∈J = [0,P] where M´ >0, N´ >0 and B is the phase space, for more information on the reference document [6, 7, 8, 20, 21, 22, 23, 24] .A mild solution(ms) is derived from the fixed-point theorem of the sadovskii [18].

2. Existence of solutions

2.1. Damped impulsive system

The damped integrodifferential equation of second order with impulses and infinite delay of the form

$$\omega''(\iota) = A\omega(\iota) + \mathcal{D}\omega'(\iota) + l_2\Big(\iota, \omega_\iota, \int_0^\iota b(\iota, s, \omega_s)ds\Big)$$

$$\iota \in \mathfrak{J} = [0, \mathfrak{P}], \ \iota \neq \iota_{\acute{r}}, \ \acute{r} = 1, 2, ..., n, \quad (2.1.1)$$

$\omega_0 = \phi \in B$, $\quad \omega'(0) = \xi \in X$, (2.1.2)

(2.1.3)

$\Delta\omega(\iota_{r'}) = I'r_{'}(\omega_{\iota r'})$, $\quad r' = 1,2,...,n,$

$\Delta\omega'(\iota_{r'}) = J'r_{'}(\omega_{\iota r'})$, $\quad r' = 1,2,...,n,$ (2.1.4)

D is a BLO on X along D(D) ⊂ D(A´). For $\iota \in J$, ω_ι denotes the fn $\omega_\iota : (-\infty,0] \to X$ defined by $\omega_\iota(\theta) = \omega(\iota+\theta)$, $-\infty < \theta \leq 0$ that belonging to B, $l_2 : J\times B\times X \to X$ and $b : J\times J\times B \to X$ are suitable fns. The moments of momentum $\{\iota_{r'}\}$ are given s.t. $0 = \iota_0 < \iota_1 < \cdots < \iota_n < \iota_{n+1} = P$,

$I''r_{'} : B \to X$, $J'r_{'} : B \to X$, $\Delta\xi(\iota)$ indicates the jump of a fn ξ at ι , that is explained by

$\Delta\xi(\iota) = \xi(\iota^+) - \xi(\iota^-)$, where $\xi(\iota^-)$ and $\xi(\iota^+)$ the left and the right limits of ξ at ι. The ms of (2.1.1)-(2.1.4).

Definition 2.1

A fn $\omega : (-\infty,P] \to X$ is called a ms of (2.1.1)-(2.1.4), if $\omega_0 = \phi \in B$, $\omega|_J \in PC$, the impulsive conditions (condn) $\Delta\omega(\iota_{r'}) = I''r_{'}(\omega_{\iota r'})$, $\Delta\omega'(\iota_{r'}) = J'r_{'}(\omega_{\iota r'})$, $r' = 1,2,...,n$, are fulfilled and

$$\omega(\iota) = \acute{\mathfrak{C}}(\iota)\varphi(0) + \acute{\mathfrak{S}}(\iota)(\xi) + \sum_{\acute{r}=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}+1})\mathcal{D}\omega(\iota_{\acute{r}+1}^-) - \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\mathcal{D}\omega(\iota_{\acute{r}}^+)\Big]$$
$$- \acute{\mathfrak{S}}(\iota - \iota_j)\mathcal{D}\omega(\iota_j^+) + \int_0^\iota \acute{\mathfrak{C}}(\iota - s)\mathcal{D}\omega(s)ds + \int_0^\iota \acute{\mathfrak{S}}(\iota - s)l_2\Big(s, \omega_s, \int_0^s b(s, \acute{\tau}, \omega_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+ \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}\acute{r}(\omega_{\iota_{\acute{r}}}) + \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}_i(\omega_{\iota_{\acute{r}}}), \quad \iota \in \mathfrak{J}, \forall, \iota \in [\iota_j, \iota_{j+1}].$$

(2.1.5)

Remark: We can write the $\omega(\iota)$ as

$$\omega(\iota) = \acute{\mathfrak{C}}(\iota)\varphi(0) + \acute{\mathfrak{S}}(\iota)(\xi) + \int_0^\iota \acute{\mathfrak{S}}(\iota - s)\mathcal{D}\omega'(s)ds + \int_0^\iota \acute{\mathfrak{S}}(\iota - s)l_2\Big(s, \omega_s, \int_0^s b(s, \acute{\tau}, \omega_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+ \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}r(\omega_{\iota_{\acute{r}}}) + \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}r(\omega_{\iota_{\acute{r}}}), \quad \iota \in \mathfrak{J}.$$

Now we use the integration by parts method to understand that $\omega(\cdot)$ is a ms of (2.1.1)-(2.1.4).

Remark 3.3. *In what follows, it is convenient to introduce the function* $\widetilde{\phi} : (-\infty, a] \to X$ *defined by*
$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & \text{if} \quad t \in (-\infty, 0], \\ C(t)\phi(0), & \text{if} \quad t \in J. \end{cases}$$

.

Also consider the assumptions stated below:

(H_1) The fn l_2 : J × B × X → X fulfilled:

(i) Let $\omega : (-\infty,P] \to X$ be s.t. $\omega_0 = \phi \in B$ and $\omega|_J \in PC$. For each $\iota \in J$,

$l_2(\iota,\cdot,\cdot) : \mathcal{B} \to \mathfrak{X}$ is cts, $\iota \to l_2\Big(\iota, \omega_\iota, \int_0^\iota b(\iota,s,\omega_s)ds\Big)$ is highly measurable.

(ii) The fn l_2 : [0,P] × B × X → X is completely(comp) cts.

(iii) ∃ a non-decreasing cts fn $\Omega : [0,\infty) \to (0,\infty)$ and an integrable fn $m : J \to (0,\infty)$ s.t.

$$\|l_2(\iota,v,w)\| \le m(\iota)\Omega(\|v\|_{\mathcal{B}} + \|w\|), \quad \lim\inf_{\xi\to\infty}\left(\frac{\xi + L_0\nu(\xi)}{\xi}\right) = \Lambda < \infty,$$

where $\iota \in J$, $(v,w) \in B \times X$.

(iv) ∃ a $\alpha_r \in L^1(J)$ s.t.

$\sup \|l_2(\iota,v,w)\| \le \alpha_r(\iota)$ for every $r > 0$.

(H_2) The impulsive fns satisfy:

(i) The maps $I_{\acute r}, J_{\acute r} : B \to X$, $\acute r = 1,2,...,n$ are comp cts and ∃ cts non-decreasing fns $\lambda_{\acute r}, \mu_{\acute r} : [0,\infty) \to (0,\infty)$, $\acute r = 1,2,...,n$, s.t.

$$\|I_{\acute r}(\acute\psi)\| \le \lambda_{\acute r}(\|\acute\psi\|_B), \quad \|J_{\acute r}(\acute\psi)\| \le \mu_{\acute r}(\|\acute\psi\|_B), \quad \acute\psi \in B.$$

(ii) ∃ a constants $K_1 > 0$, $K_2 > 0$ s.t.

$$\|I_{\acute r}(\acute\psi_1) - I_{\acute r}(\acute\psi_2)\| \le K_1\|\acute\psi_1 - \acute\psi_2\|_B, \quad \|J_{\acute r}(\acute\psi_1) - J_{\acute r}(\acute\psi_2)\| \le K_2\|\acute\psi_1 - \acute\psi_2\|_B, \quad \acute\psi_1, \acute\psi_2 \in B, \quad \acute r = 1,2,...,n.$$

Theorem 2.2 If the inferences (H1) − (H2) are fulfilled, then (2.1.1)-(2.1.4) has a ms on J on a particular condn

$$h(3N' + PM')\|D\| + K_P\left(\acute{\mathfrak{N}}\Lambda\int_0^{\mathfrak{P}} m(s)ds + \sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\right)\Big] < 1.$$

Proof. H(P) denotes the space H(P) = $\{y :]-\infty,P] \to X : y|J \in PC, y_0 = 0\}$ endowed with the uniform convergence topology, we define the operator Ψ : H(P) → H(P) defined by

$(\Psi y)_0 = 0$ and

$$\begin{aligned}\Psi y(\iota) &= \acute{\mathfrak{S}}(\iota)(\xi) + \sum_{\acute r=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota - \iota_{\acute r+1})\mathcal{D}(y(\iota_{\acute r+1}^-) + \tilde\nu(\iota_{\acute r+1}^-)) - \acute{\mathfrak{S}}(\iota - \iota_{\acute r})\mathcal{D}(y(\iota_{\acute r}^+) + \tilde\nu(\iota_{\acute r}^+))\Big] \\ &\quad - \acute{\mathfrak{S}}(\iota - \iota_j)\mathcal{D}(y(\iota_j^+) + \tilde\nu(\iota_j^+)) + \int_0^\iota \acute{\mathfrak{C}}(\iota - s)\mathcal{D}(y(s) + \tilde\nu(s))ds \\ &\quad + \int_0^\iota \acute{\mathfrak{S}}(\iota - s)l_2\Big(s, y_s + \tilde\nu_s, \int_0^s b(s,\acute\tau, y_{\acute\tau} + \tilde\nu_{\acute\tau})d\acute\tau\Big)ds + \sum_{0<\iota_{\acute r}<\iota}\acute{\mathfrak{C}}(\iota - \iota_{\acute r})I_{\acute r}(y_{\iota_{\acute r}} + \tilde\nu_{\iota_{\acute r}}) \\ &\quad + \sum_{0<\iota_{\acute r}<\iota}\acute{\mathfrak{S}}(\iota - \iota_{\acute r})J_{\acute r}(y_{\iota_{\acute r}} + \tilde\nu_{\iota_{\acute r}}), \ \iota \in [\iota_j, \iota_{j+1}], \ j = 0,...,n,\end{aligned}$$

has a fp $\omega(\cdot)$, which is a ms of (2.1.1)-(2.1.4). From the hypothesis that Ψ is cts and well defined.

Next we wish to show that $\exists\ r>0$ s.t. $\Psi(B_r(0,H(P))) \subseteq B_r(0,H(P))$. Suppose this statement is

$$+\acute{\mathfrak{N}}\int_0^{\iota^r} m(s)\Omega\Big(\|y_s^r+\widetilde{\varphi}_s\|+\|\int_0^s b(s,\acute{\tau},y_{\acute{\tau}}+\widetilde{\varphi}_{\acute{\tau}})d\acute{\tau}\|\Big)ds+\acute{\mathfrak{M}}\sum_{\acute{r}=1}^{n}[\|\acute{I}r(y_{\iota_{\acute{r}}}^r+\widetilde{\varphi}_{\iota_{\acute{r}}})-\acute{I}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|$$

$$+\|\acute{I}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|]+\acute{\mathfrak{N}}\sum_{\acute{r}=1}^{n}[\|\acute{J}r(y_{\iota_{\acute{r}}}^r+\widetilde{\varphi}_{\iota_{\acute{r}}})-\acute{J}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|+\|\acute{J}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|]$$

$\mathfrak{P}$

not true, then for each $r>0$, we can take $\omega^r \in B_r(0,H(P))$, $j=\{0,\ldots,n\}$ and $\iota^r \in [\iota_j,\iota_{j+1}]$ s.t. k$\Psi y^r(\iota^r)$k $> r$.

$$+\acute{\mathfrak{M}}\sum_{\acute{r}=1}(K_{\mathfrak{P}}K_1 r+\|\acute{I}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|)+\acute{\mathfrak{N}}\sum_{\acute{r}=1}(K_{\mathfrak{P}}K_2 r+\|\acute{J}r(\widetilde{\varphi}_{\iota_{\acute{r}}})\|)$$

Using the notation $\|y_\iota+\widetilde{\varphi}_\iota\|_{\mathcal{B}} \le K_{\mathfrak{P}}\|y_\iota\|+\|\widetilde{\varphi}_\iota\|_{\mathcal{B}}$, we get that:

$r <$k$\Psi y^r(\iota^r)$k $\le$ N´[kξk + (3N´ + PM´)kDk(r + kϕk$_P$)

$$\le \acute{\mathfrak{N}}[\|\xi\|+(3\acute{\mathfrak{N}}_n+\text{PM´})\|\mathcal{D}\|(r+\|\varphi\|_{\mathfrak{P}})+\acute{\mathfrak{N}}_n\int_0 m(s)\Omega\Big((1+\acute{\mathfrak{N}}_1)(K_{\mathfrak{P}}r+\|\widetilde{\varphi}_s\|_{\mathcal{B},\mathfrak{P}})+L_2\Big)ds$$

and hence $1 \le$ h(3N´ + PM´)kDk + K_P $\Big(\acute{\mathfrak{N}}\Lambda\int_0^{\mathfrak{P}} m(s)ds+\sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1+\acute{\mathfrak{N}}K_2)\Big)\Big]$

which is a contradiction.

Let $r>0$ be s.t. $\Psi(B_r(0,H(P))) \subset B_r(0,H(P))$. To prove Ψ is a condensing map on $B_r(0,H(P))$ into $B_r(0,H(P))$. The decomposition $\Psi=\Psi_1+\Psi_2$ where

$$\Psi_1\omega(\iota)=\acute{\mathfrak{S}}(\iota)(\xi)+\sum_{\acute{r}=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota-\iota_{\acute{r}+1})\mathcal{D}(y(\iota_{\acute{r}+1}^-)+\widetilde{\nu}(\iota_{\acute{r}+1}^-))-\acute{\mathfrak{S}}(\iota-\iota_{\acute{r}})\mathcal{D}(y(\iota_{\acute{r}}^+)+\widetilde{\nu}(\iota_{\acute{r}}^+))\Big]$$

$$-\acute{\mathfrak{S}}(\iota-\iota_j)\mathcal{D}(y(\iota_j^+)+\widetilde{\nu}(\iota_j^+))+\int_0^\iota \acute{\mathfrak{C}}(\iota-s)\mathcal{D}(y(s)+\widetilde{\nu}(s))ds$$

$$+\sum_{0<\iota_{\acute{r}}<\iota}\acute{\mathfrak{C}}(\iota-\iota_{\acute{r}})\acute{I}r(y_{\iota_{\acute{r}}}+\widetilde{\nu}_{\iota_{\acute{r}}})+\sum_{0<\iota_{\acute{r}}<\iota}\acute{\mathfrak{S}}(\iota-\iota_{\acute{r}})\acute{J}r(y_{\iota_{\acute{r}}}+\widetilde{\nu}_{\iota_{\acute{r}}}),$$

$$\Psi_2\omega(\iota)=\int_0^\iota \acute{\mathfrak{S}}(\iota-s)l_2\Big(s,y_s+\widetilde{\nu}_s,\int_0^s b(s,\acute{\tau},y_{\acute{\tau}}+\widetilde{\nu}_{\acute{\tau}})d\acute{\tau}\Big)ds.$$

From [[14], Lemma 3.1], we understand that Ψ_2 is comp cts. That is

k$\Psi_1 v-\Psi_1 w$k = (3N´ + PM´) $\|\mathcal{D}\|+K_{\mathfrak{P}}\Big(\mathfrak{P}\acute{\mathfrak{M}}(L_g(1+\acute{\mathfrak{N}}_1))+\sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1+\acute{\mathfrak{N}}K_2)\Big)\|u-v\|_{\mathfrak{P}}$,

together imply that Ψ is condensing on $B_r(0,H(P))$.

Finally, from the fp theorem of Sadovskii's we get a fp y of Ψ. Clearly, $\omega=y+v$ is a ms of the problem (2.1.1)-(2.1.4). Hence the proof.

Corollary 2.3 If all condns of Theorem 2.2 true except that (H2) replaced by the following one, (C1) : ∃ constants $a_{r'} > 0$, $b_{r'} > 0$, $c_{r'} > 0$, $d_{r'} > 0$ and constants $\theta_{r'}, \delta_{r'} \in (0,1)$, $r' = 1,2,...,n$ s.t. for each $\nu \in X$,

$$\|I'_{r'}(\nu)\| \le a_{r'} + b_{r'}(\|\nu\|)^{\theta_{r'}}, \quad r' = 1,2,...,n.$$

and

$$\|J'_{r'}(\nu)\| \le c_{r'} + d_{r'}(\|\nu\|)^{\delta_{r'}}, \quad r' = 1,2,...,n.$$

then the system (2.1.1)-(2.1.4) is a ms on J on condn that

$$(3N' + PM')\|D\| + K_P \Big(\acute{\mathfrak{N}}\Lambda \int_0^{\mathfrak{P}} m(s)ds + \sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\Big) < 1.$$

2.2. Damped impulsive neutral system

The damped neutral integrodifferential equation of second order with impulses and infinite delay of the form

$$\frac{d}{d\iota}\Big[\omega'(\iota) - l_1\Big(\iota, \omega_\iota, \int_0^\iota a(\iota, s, \omega_s)ds\Big)\Big] = \acute{\mathfrak{A}}\omega(\iota) + \mathcal{D}\omega'(\iota) + l_2\Big(\iota, \omega_\iota, \int_0^\iota b(\iota, s, \omega_s)ds\Big)$$
$$\iota \in \mathfrak{P} = [0, \mathfrak{P}], \ \iota \ne \iota_{\acute{r}}, \ \acute{r} = 1, 2, ..., n, \quad (2.2.1)$$

$$\omega_0 = \varphi \in \mathcal{B}, \quad \omega'(0) = \xi \in \mathfrak{X}, \quad (2.2.2)$$

$$\Delta\omega(\iota_{\acute{r}}) = \acute{I}\acute{r}(\omega_{\iota_{\acute{r}}}), \quad \acute{r} = 1, 2, ..., n, \quad (2.2.3)$$

$$\Delta\omega'(\iota_{\acute{r}}) = \acute{J}\acute{r}(\omega_{\iota_{\acute{r}}}), \quad \acute{r} = 1, 2, ..., n, \quad (2.2.4)$$

where $l_1 : J \times B \times X \to X$, and $a : J \times J \times B \to X$ the remaining fns defined in previous session.

The ms of (2.2.1)-(2.2.4).

Definition 2.4

A fn $\omega : (-\infty, P] \to X$ is called a ms of the abstract Cauchy problem (2.2.1)-(2.2.4), if $\omega_0 = \phi \in B$, $\omega|_J \in PC$, the impulsive condns $\Delta\omega(\iota_{r'}) = I'_{r'}(\omega_{\iota r'})$, $\Delta\omega'(\iota_{r'}) = J'_{r'}(\omega_{\iota r'})$, $r' = 1,2,...,n$, are fulfilled and

$$\begin{aligned}\omega(\iota) &= \acute{\mathfrak{C}}(\iota)\varphi(0) + \acute{\mathfrak{S}}(\iota)\Big(\xi - l_1(0, \varphi, 0)\Big) + \int_0^\iota \acute{\mathfrak{C}}(\iota - s)l_1\Big(s, \omega_s, \int_0^s a(s, \acute{\tau}, \omega_{\acute{\tau}})d\acute{\tau}\Big)ds \\ &+ \sum_{\acute{r}=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}+1})\mathcal{D}\omega(\iota_{\acute{r}+1}^-) - \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\mathcal{D}\omega(\iota_{\acute{r}}^+)\Big] - \acute{\mathfrak{S}}(\iota - \iota_j)\mathcal{D}\omega(\iota_j^+) \\ &+ \int_0^\iota \acute{\mathfrak{C}}(\iota - s)\mathcal{D}\omega(s)ds + \int_0^\iota \acute{\mathfrak{S}}(\iota - s)l_2\Big(s, \omega_s, \int_0^s b(s, \acute{\tau}, \omega_{\acute{\tau}})d\acute{\tau}\Big)ds \\ &+ \sum_{0<\iota_{\acute{r}}<\iota}\acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}\acute{r}(\omega_{\iota_{\acute{r}}}) + \sum_{0<\iota_{\acute{r}}<\iota}\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}_i(\omega_{\iota_{\acute{r}}}), \quad \iota \in \mathfrak{J},\end{aligned} \quad (2.2.5)$$

for all $\iota \in [\iota_j, \iota_{j+1}]$ *and every* $j = 0,...,n$.

Remark: $\omega(\iota)$ can be written:

$$\omega(\iota) = \acute{\mathfrak{C}}(\iota)\varphi(0) + \acute{\mathfrak{S}}(\iota)\Big(\xi - l_1(0,\varphi,0)\Big) + \int_0^\iota \acute{\mathfrak{C}}(\iota - s) l_1\Big(s, \omega_s, \int_0^s a(s,\acute{\tau},\omega_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+ \int_0^\iota \acute{\mathfrak{S}}(\iota - s)\mathcal{D}\omega'(s)ds + \int_0^\iota \acute{\mathfrak{S}}(\iota - s) l_2\Big(s, \omega_s, \int_0^s b(s,\acute{\tau},\omega_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+ \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}\acute{r}(\omega_{\iota_{\acute{r}}}) + \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}\acute{r}(\omega_{\iota_{\acute{r}}}), \quad \iota \in \mathfrak{J}.$$

as

Now we use integration by parts method to understandd that $\omega(\cdot)$ is a ms of (2.2.1)-(2.2.4).

Remark 3.3. *In what follows, it is convenient to introduce the function $\widetilde{\phi} : (-\infty, a] \to X$ defined by* $\widetilde{\phi}(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ C(t)\phi(0), & \text{if } t \in J. \end{cases}$

Also consider the assumptions stated below:

(H_3) ∃ a constant $N'_1 > 0$ s.t.

$$\left\| \int_0^\iota [a(\iota,s,\omega) - a(\iota,s,y)]ds \right\| \leq \acute{\mathfrak{N}}_1 \|\omega - y\|_{\mathcal{B}}, \text{ for } \iota, s \in \mathfrak{J},\ \omega, y \in \mathcal{B}$$

and $L_2 = P\sup_{(\iota,s)\in J\times J} \|a(\iota,s,0)\|$.

(H_4) ∃ a positive constant L_g s.t.

$$\|l_1(\iota,v_1,w_1) - l_1(\iota,v_2,w_2)\| \leq L_g(\|v_1 - v_2\|_B + \|w_1 - w_2\|),$$

where $0 < L_g < 1$, $(\iota,v_i,w_i) \in J \times B \times X$, $i = 1,2$ and $\|g(\iota,u,v)\| \leq L_g(\|u\|_B + \|v\|) + L_1$ and $L_1 = \sup_{\iota\in J} \|g(\iota,0,0)\|$.

Theorem 2.5

If the inferences (H1)−(H4) are true, then the system (2.2.1)-(2.2.4) has a ms on J on the condn that

$$\left[K_{\mathfrak{P}}\Big(\mathfrak{P}\acute{\mathfrak{M}}(L_g(1+\acute{\mathfrak{N}}_1)) + \frac{1}{K}(3\acute{\mathfrak{N}} + PM'_P)\ \|\mathcal{D}\| + \acute{\mathfrak{N}}\Lambda \int_0^{\mathfrak{P}} m(s)ds + \sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\Big)\right] < 1$$

.

Proof. H(P) denotes the space H(P) = $\{y :]-\infty, P] \to X : y|J \in PC, y_0 = 0\}$ endowed with the uniform convergence topology, we define the operator $\Psi : H(P) \to H(P)$ by $(\Psi y)_0 = 0$ and

$$\Psi y(\iota) = \acute{\mathfrak{S}}(\iota)[\xi - l_1(0,\varphi,0)] + \int_0^\iota \acute{\mathfrak{C}}(\iota - s) l_1\Big(s, y_s + \widetilde{\nu}_s, \int_0^s a(s,\acute{\tau}, y_{\acute{\tau}} + \widetilde{\nu}_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+ \sum_{\acute{r}=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}+1})\mathcal{D}(y(\iota_{\acute{r}+1}^-) + \widetilde{\nu}(\iota_{\acute{r}+1}^-)) - \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\mathcal{D}(y(\iota_{\acute{r}}^+) + \widetilde{\nu}(\iota_{\acute{r}}^+))\Big]$$
$$- \acute{\mathfrak{S}}(\iota - \iota_j)\mathcal{D}(y(\iota_j^+) + \widetilde{\nu}(\iota_j^+)) + \int_0^\iota \acute{\mathfrak{C}}(\iota - s)\mathcal{D}(y(s) + \widetilde{\nu}(s))ds$$
$$+ \int_0^\iota \acute{\mathfrak{S}}(\iota - s) l_2\Big(s, y_s + \widetilde{\nu}_s, \int_0^s b(s,\acute{\tau}, y_{\acute{\tau}} + \widetilde{\nu}_{\acute{\tau}})d\acute{\tau}\Big)ds + \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}\acute{r}(y_{\iota_{\acute{r}}} + \widetilde{\nu}_{\iota_{\acute{r}}})$$
$$+ \sum_{0<\iota_{\acute{r}}<\iota} \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}\acute{r}(y_{\iota_{\acute{r}}} + \widetilde{\nu}_{\iota_{\acute{r}}}),\ \iota \in [\iota_j, \iota_{j+1}],\ j = 0, \ldots, n,$$

has a fp $\omega(\cdot)$. which is a ms of the system (2.2.1)-(2.2.4). From the hypothesis that, Ψ is cts and well defined .

Next we want to show that ∃ $r>0$ s.t. $\Psi(B_r(0,H(P))) \subseteq B_r(0,H(P))$. Suppose this statement is not true, then for each $r>0$, we can take $\omega^r \in B_r(0,H(P))$, $j=\{0,\ldots,n\}$ and $\iota^r \in [\iota_j,\iota_{j+1}]$ s.t. $\|\Psi y^r(\iota^r)\| > r$. Using the notation $\|y_\iota + \phi e_\iota\|_B \le K_P\|y_\iota\| + \|\phi e_\iota\|_B$, we get

$$\widetilde{\varphi}_s, \int_0^s a(s,\acute{\tau},\widetilde{\varphi}_{\acute{\tau}})d\acute{\tau}\Big)\Big\|ds + \acute{\mathfrak{M}}\int_0^{\iota^r}\Big\|l_1\Big(s,\widetilde{\varphi}_s,\int_0^s a(s,\acute{\tau},\widetilde{\varphi}_{\acute{\tau}})d\acute{\tau}\Big)\Big\|ds$$

$+ (3N' + \mathfrak{P}\acute{\mathfrak{M}})\|\mathcal{D}\|(r+\|\varphi\|_{\mathfrak{P}})$

$\le N'[\|\xi\| + \|l_1(0,\phi,0)\|] + PM' K_P(L_g(1+N'_1))r + M' Z (L_g[(1+N'_1)\|\phi e\|_P + L_2] + L_1)ds$

$0 + (3N' + PM')\|D\|(r + \|\phi\|_P) + N'$

$$\int_0^{\mathfrak{P}} m(s)\Omega\Big((1+\acute{\mathfrak{N}}_1)(K_{\mathfrak{P}}r + \|\widetilde{\varphi}_s\|_{\mathcal{B},\mathfrak{P}}) + L_2\Big)ds$$

$$+\acute{\mathfrak{M}}\sum_{\acute{r}=1}^{n}(I$$

and hence

$$+\acute{\mathfrak{N}}\int_0^{\iota^r} m(s)\Omega\Big(\|y_s^r + \widetilde{\varphi}_s\| + \|\int_0^s b(s,\acute{\tau},y_{\acute{\tau}} + \widetilde{\varphi}_{\acute{\tau}})d\acute{\tau}\|\Big)ds + \acute{\mathfrak{M}}\sum_{\acute{r}=1}^{n}[\|\acute{I}\acute{r}(y_{\iota_{\acute{r}}}^r + \widetilde{\varphi}_{\iota_{\acute{r}}})$$

$$1 \le \Big[K_{\mathfrak{P}}\Big(\mathfrak{P}\acute{\mathfrak{M}}(L_g(1+\acute{\mathfrak{N}}_1)) - \acute{I}\acute{r}(\widetilde{\varphi}_{\iota_{\acute{r}}})\| + \|\acute{I}\acute{r}(\widetilde{\varphi}_{\iota_{\acute{r}}})\|] + \acute{\mathfrak{N}}\sum_{\acute{r}=1}^{n}[\|\acute{J}\acute{r}(y_{\iota_{\acute{r}}}^r + \widetilde{\varphi}_{\iota_{\acute{r}}}) - \acute{J}\acute{r}(\widetilde{\varphi}_{\iota_{\acute{r}}})\| + \|\acute{J}\acute{r}(\widetilde{\varphi}_{\iota_{\acute{r}}})\|]$$

PM′)

$\mathfrak{P}$

$$r < \|\Psi y^r(\iota^r)\|$$

$$\le \acute{\mathfrak{N}}[\|\xi\| + \|l_1(0,\varphi,0)\|] + \acute{\mathfrak{M}}\int_0^{\iota^r}\Big\|l_1\Big(s, y_s^r + \widetilde{\varphi}_s, \int_0^s a(s,\acute{\tau},y_{\acute{\tau}}^r + \widetilde{\varphi}_{\acute{\tau}})d\acute{\tau}\Big)$$

which is a contradiction.

$$-l_1\Big(s,$$

Let $r>0$ be s.t. $\Psi(B_r(0,H(P))) \subset B_r(0,H(P))$. To prove that Ψ is a condensing map on $B_r(0,H(P))$ into $B_r(0,H(P))$. We study the decomposition $\Psi = \Psi_1 + \Psi_2$ where

$$\|\mathcal{D}\| + \acute{\mathfrak{N}}\Lambda\int_0^{\mathfrak{P}} m(s)ds + \sum_{i=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\Big)\Big]$$

$$\Psi_1\omega(\iota) = \acute{\mathfrak{S}}(\iota)[\xi - l_1(0,\varphi,0)] + \int_0^{\iota}\acute{\mathfrak{C}}(\iota - s)l_1\Big(s, y_s + \widetilde{\nu}_s, \int_0^s a(s,\acute{\tau},y_{\acute{\tau}} + \widetilde{\nu}_{\acute{\tau}})d\acute{\tau}\Big)ds$$
$$+\sum_{\acute{r}=0}^{j-1}\Big[\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}+1})\mathcal{D}(y(\iota_{\acute{r}+1}^-) + \widetilde{\nu}(\iota_{\acute{r}+1}^-)) - \acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\mathcal{D}(y(\iota_{\acute{r}}^+) + \widetilde{\nu}(\iota_{\acute{r}}^+))\Big]$$
$$-\acute{\mathfrak{S}}(\iota - \iota_j)\mathcal{D}(y(\iota_j^+) + \widetilde{\nu}(\iota_j^+)) + \int_0^{\iota}\acute{\mathfrak{C}}(\iota - s)\mathcal{D}(y(s) + \widetilde{\nu}(s))ds$$
$$+\sum_{0<\iota_i<\iota}\acute{\mathfrak{C}}(\iota - \iota_{\acute{r}})\acute{I}\acute{r}(y_{\iota_{\acute{r}}} + \widetilde{\nu}_{\iota_{\acute{r}}}) + \sum_{0<\iota_{\acute{r}}<\iota}\acute{\mathfrak{S}}(\iota - \iota_{\acute{r}})\acute{J}\acute{r}(y_{\iota_{\acute{r}}} + \widetilde{\nu}_{\iota_{\acute{r}}})$$
$$\Psi_2\omega(\iota) = \int_0^{\iota}\acute{\mathfrak{S}}(\iota - s)l_2\Big(s, y_s + \widetilde{\nu}_s, \int_0^s b(s,\acute{\tau},y_{\acute{\tau}} + \widetilde{\nu}_{\acute{\tau}})d\acute{\tau}\Big)ds.$$

,

From [[14], Lemma 3.1], we infer that Ψ_2 is comp cts. This fact and the estimate

$$\Big(\mathfrak{P}\acute{\mathfrak{M}}(L_g(1+\acute{\mathfrak{N}}_1)) + \frac{1}{K}(3\acute{\mathfrak{N}} + \quad \|\mathcal{D}\| + \sum_{\acute{r}=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\Big)\|u - v\|_{\mathfrak{P}}$$

$\|\Psi_1 v - \Psi_1 w\| = K_P PM')$,

together imply that Ψ is condensing on $B_r(0,H(P))$.

Finally, from Sadovskii's fp theorem we obtain a fp y of Ψ. Clearly, $\omega = y + v$ is a ms of the problem (2.2.1)-(2.2.4). This completes the proof.

Corollary 2.6

If all condns of Theorem 2.5 hold except that (H2) replaced by the following one,

(C1) : ∃ constants $a_{r'} > 0$, $b_{r'} > 0$, $c_{r'} > 0$, $d_{r'} > 0$ and constants $\theta_{r'}$, $\delta_{r'} \in (0,1)$, $r' = 1,2,...,n$ s.t. for each $\nu \in X$,

$$\|I_{r'}(\nu)\| \leq a_{r'} + b_{r'}(\|\nu\|)^{\theta_{r'}}, \quad r' = 1,2,...,n.$$

and

$$\|J_{r'}(\nu)\| \leq c_{r'} + d_{r'}(\|\nu\|)^{\delta_{r'}}, \quad r' = 1,2,...,n.$$

$$\left[K_{\mathfrak{P}}\left(\mathfrak{P}\acute{\mathfrak{M}}(L_g(1+\acute{\mathfrak{N}}_1)) + \frac{1}{K}(3\acute{\mathfrak{N}} + \quad \|\mathcal{D}\| + \acute{\mathfrak{N}}\Lambda\int_0^{\mathfrak{P}} m(s)ds + \sum_{\acute{r}=1}^{n}(\acute{\mathfrak{M}}K_1 + \acute{\mathfrak{N}}K_2)\right)\right] < 1$$

then the system (2.2.1)-(2.2.4) is a ms on J provided th*at* PM´).

3. Conclusion

We used fp theorem of Sadovskii's with a non compact condn on the cosine family of operators. Finally we conclude that the mild solution(ms) exists for impulsive damped second order neutral integrodifferential equations with infinite delay in a Bach spaces.

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