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CARTESIAN PRODUCT OF BIPOLAR L-FUZZY SUB ℓ-HX GROUPS

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Abstract: The aim of the paper is to introduce the Cartesian product between bipolar L-fuzzy sub ℓ-HX groups, and we tend to present the Cartesian product of bipolar L-fuzzy sub ℓ-HX groups under homomorphism, and anti-homomorphism.
Keywords: bipolar L-fuzzy sub ℓ-group, bipolar L-fuzzy sub ℓ-HX group, Cartesian Product of bipolar L-fuzzy subsets, homomorphism and anti-homomorphism, image, pre-image of bipolar L-fuzzy subsets.

1. Introduction

The fuzzy set was initiated by L.A.Zadeh [15]. The membership degree of fuzzy set is defined in the interval of [0,1]. In continue, J.A. Goguen [3] introduced L-Fuzzy set. In L-fuzzy set, the valuation set [0,1] replaced through a complete lattice. Complete lattice may be a posset within which all subsets have each a supremum (join) associated an infimum (meet).The membership degree[−1,1] of bipolar-valued fuzzy set contains two parts. That is, positive membership degree(0,1] and negative membership degree[−1,0). The membership degree (0,1] indicates that components somewhat satisfy the property and also the membership degree [−1,0) indicates that components somewhat satisfy the implicit counter-property. Li Hongxing [4] introduced the idea of HX group and also the authors Luo Chengzhong, Mi Honghai, Li Hongxing [5] introduced the idea of the fuzzy HX group. G.S.V. Satya Saibaba [14] introduced the idea of Fuzzy lattice ordered groups. Muthuraj.R, Sridharan.M, [8] introduced the Cartesian product of bipolar fuzzy HX subgroup,
In this paper, we discuss the Cartesian product of bipolar L-Fuzzy sub ℓ- HX group and present some properties using the concept of homomorphism and anti-homomorphism.

2. Preliminaries

In this section, we provide some basic definitions. Throughout this paper G=(G,*,≤) could be a lattice ordered group or a ℓ-group, e is that the identity of G and mn we tend to mean m*n.

Definition 2.1[16]

A bipolar L-fuzzy subset α of G is said to be bipolar L-fuzzy sub ℓ-group of G if for any m,n∈G

i) $\alpha^+(mn) \geq \alpha^+(m) \wedge \alpha^+(n)$

ii) $\alpha^-(mn) \leq \alpha^-(m) \vee \alpha^-(n)$

iii) $\alpha^+(m^{-1}) = \alpha^+(m)$

iv) $\alpha^-(m^{-1}) = \alpha^-(m)$

v) $\alpha^+(m \vee n) \geq \alpha^+(m) \wedge \alpha^+(n)$

vi) $\alpha^-(m \vee n) \leq \alpha^-(m) \vee \alpha^-(n)$

vii) $\alpha^+(m \wedge n) \geq \alpha^+(m) \wedge \alpha^+(n)$

viii) $\alpha^-(m \wedge n) \leq \alpha^-(m) \vee \alpha^-(n)$

Definition 2.2 [16]

Let α be a bipolar L-fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a ℓ-HX group on G. A bipolar L-fuzzy set ρ^α defined on ϑ is said to be a bipolar L-fuzzy sub ℓ-HX group on ϑ if for all P,Q∈ϑ.

i) $(\rho^\alpha)^+(PQ) \geq (\rho^\alpha)^+(P) \wedge (\rho^\alpha)^+(Q)$

ii) $(\rho^\alpha)^-(PQ) \leq (\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q)$

iii) $(\rho^\alpha)^+(P) = (\rho^\alpha)^+(P^{-1})$

iv) $(\rho^\alpha)^-(P) = (\rho^\alpha)^-(P^{-1})$

v) $(\rho^\alpha)^+(P \vee Q) \geq (\rho^\alpha)^+(P) \wedge (\rho^\alpha)^+(Q)$

vi) $(\rho^\alpha)^-(P \vee Q) \leq (\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q)$

vii) $(\rho^\alpha)^+(P \wedge Q) \geq (\rho^\alpha)^+(P) \wedge (\rho^\alpha)^+(Q)$

viii) $(\rho^\alpha)^-(P \wedge Q) \leq (\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q)$

Where $(\rho^\alpha)^+(P) = \vee\{\alpha^+(m) / \text{ for all } m \in P \subseteq G\}$ and $(\rho^\alpha)^-(P) = \wedge\{\alpha^-(m) / \text{for all } m \in P \subseteq G\}$.

Definition2.3[13]

Let G_1 and G_2 be any two ℓ-groups. Let $\vartheta_1 \subset 2^{G1} - \{\phi\}$ and $\vartheta_2 \subset 2^{G2} - \{\phi\}$ be any two ℓ - HX groups defined on G_1 and G_2 respectively. Let ρ^α be a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group in ϑ_1. Let $\varphi: \vartheta_1 \rightarrow \vartheta_2$ be a function. Then ρ^α is called φ-invariant if the following conditions are satisfied.

i) $((\varphi(P))^+ = ((\varphi(Q))^+$ implies that $(\rho^\alpha)^+(P) = (\rho^\alpha)^+(Q)$

ii) $((\varphi(P))^- = ((\varphi(Q))^-$ implies that $(\rho^\alpha)^-(P) = (\rho^\alpha)^-(Q)$.

3. Properties and example of bipolar L-Fuzzy sub ℓ-HX group using Cartesian product.

In this section, we discuss some of the properties and example of bipolar L-Fuzzy sub ℓ-HX group using Cartesian product.

Definition3.1

Let $\rho^\alpha = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ and $\omega^\beta = ((\omega^\beta)^+, (\omega^\beta)^-)$ be bipolar L-fuzzy subsets of ϑ_1 and ϑ_2 respectively. Then a product $\rho^\alpha \times \omega^\beta = ((\rho^\alpha \times \omega^\beta)^+, (\rho^\alpha \times \omega^\beta)^-)$, Where $(\rho^\alpha \times \omega^\beta)^+ : \vartheta_1 \times \vartheta_2 \rightarrow L$ and $(\rho^\alpha \times \omega^\beta)^- : \vartheta_1 \times \vartheta_2 \rightarrow L$ are mappings defined by

i) $(\rho^\alpha \times \omega^\beta)^+(P,Q) = (\rho^\alpha)^+(P) \wedge (\omega^\beta)^+(Q)$

ii) $(\rho^\alpha \times \omega^\beta)^-(P,Q) = (\rho^\alpha)^-(P) \vee (\omega^\beta)^-(Q)$ for all $P \in \vartheta_1, Q \in \vartheta_2$.

Theorem3.2

Let $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ and $\omega^{\beta}=((\omega^{\beta})^{+},(\omega^{\beta})^{-})$ be bipolar L-fuzzy sub ℓ-HX groups on ℓ-HX groups ϑ_1 and ϑ_2 respectively. Then $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX groups on ℓ-HX groups $\vartheta_1 x \vartheta_2$.

Proof. Assume $(P,Q),(R,S)\in\vartheta_1\times\vartheta_2$

i)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{+}(PR,QS)\\
&= (\rho^{\alpha})^{+}(PR)\wedge(\omega^{\beta})^{+}(QS)\\
&\geq ((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(R))\wedge((\omega^{\beta})^{+}(Q)\wedge(\omega^{\beta})^{+}(S))\\
&= ((\rho^{\alpha})^{+}(P)\wedge(\omega^{\beta})^{+}(Q))\wedge((\rho^{\alpha})^{+}(R)\wedge(\omega^{\beta})^{+}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)(R,S)) &\geq (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)
\end{aligned}$$

ii)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{-}(PR,QS)\\
&= (\rho^{\alpha})^{-}(PR)\vee(\omega^{\beta})^{-}(QS)\\
&\leq ((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(R))\vee((\omega^{\beta})^{-}(Q)\vee(\omega^{\beta})^{-}(S))\\
&= ((\rho^{\alpha})^{-}(P)\vee(\omega^{\beta})^{-}(Q))\vee((\rho^{\alpha})^{-}(R)\vee(\omega^{\beta})^{-}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)(R,S)) &\leq (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)
\end{aligned}$$

iii)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)^{-1}) &= (\rho^{\alpha}\times\omega^{\beta})^{+}(P^{-1},Q^{-1})\\
&= (\rho^{\alpha})^{+}(P^{-1})\wedge(\omega^{\beta})^{+}(Q^{-1})\\
&= (\rho^{\alpha})^{+}(P)\wedge(\omega^{\beta})^{+}(Q)\\
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)^{-1}) &= (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)
\end{aligned}$$

iv)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)^{-1}) &= (\rho^{\alpha}\times\omega^{\beta})^{-}(P^{-1},Q^{-1})\\
&= (\rho^{\alpha})^{-}(P^{-1})\vee(\omega^{\beta})^{-}(Q^{-1})\\
&= (\rho^{\alpha})^{-}(P)\vee(\omega^{\beta})^{-}(Q)\\
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)^{-1}) &= (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)
\end{aligned}$$

v)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)\vee(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{+}(P\vee R,Q\vee S)\\
&= (\rho^{\alpha})^{+}(P\vee R)\wedge(\omega^{\beta})^{+}(Q\vee S)\\
&\geq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(R)\wedge(\omega^{\beta})^{+}(Q)\wedge(\omega^{\beta})^{+}(S)\\
&= ((\rho^{\alpha})^{+}(P)\wedge(\omega^{\beta})^{+}(Q))\wedge((\rho^{\alpha})^{+}(R)\wedge(\omega^{\beta})^{+}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)\vee(R,S)) &\geq (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)
\end{aligned}$$

vi)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)\vee(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{-}(P\vee R,Q\vee S)\\
&= (\rho^{\alpha})^{-}(P\vee R)\vee(\omega^{\beta})^{-}(Q\vee S)\\
&\leq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(R)\vee(\omega^{\beta})^{-}(Q)\vee(\omega^{\beta})^{-}(S)\\
&= ((\rho^{\alpha})^{-}(P)\vee(\omega^{\beta})^{-}(Q))\vee((\rho^{\alpha})^{-}(R)\vee(\omega^{\beta})^{-}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)\vee(R,S)) &\leq (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)
\end{aligned}$$

vii)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)\wedge(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{+}(P\wedge R,Q\wedge S)\\
&= (\rho^{\alpha})^{+}(P\wedge R)\wedge(\omega^{\beta})^{+}(Q\wedge S)\\
&\geq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(R)\wedge(\omega^{\beta})^{+}(Q)\wedge(\omega^{\beta})^{+}(S)\\
&= ((\rho^{\alpha})^{+}(P)\wedge(\omega^{\beta})^{+}(Q))\wedge((\rho^{\alpha})^{+}(R)\wedge(\omega^{\beta})^{+}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{+}((P,Q)\wedge(R,S)) &\geq (\rho^{\alpha}\times\omega^{\beta})^{+}(P,Q)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(R,S)
\end{aligned}$$

viii)
$$\begin{aligned}
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)\wedge(R,S)) &= (\rho^{\alpha}\times\omega^{\beta})^{-}(P\wedge R,Q\wedge S)\\
&= (\rho^{\alpha})^{-}(P\wedge R)\vee\omega^{\beta})^{-}(Q\wedge S)\\
&\leq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(R))\vee((\omega^{\beta})^{-}(Q)\vee(\omega^{\beta})^{-}(S))\\
&= ((\rho^{\alpha})^{-}(P)\vee(\omega^{\beta})^{-}(Q))\vee((\rho^{\alpha})^{-}(R)\vee(\omega^{\beta})^{-}(S))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)\\
(\rho^{\alpha}\times\omega^{\beta})^{-}((P,Q)\wedge(R,S)) &\leq (\rho^{\alpha}\times\omega^{\beta})^{-}(P,Q)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(R,S)
\end{aligned}$$

Hence, a product $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group on$\vartheta_1\times\vartheta_2$.

Theorem 3.3

Let $\alpha\times\beta$ be a bipolar L-fuzzy sub ℓ-group on $G_1\times G_2$ then the bipolar L-fuzzy set $\Omega^{\alpha\times\beta}$ is a bipolar L-fuzzy sub ℓ-HX group on$\vartheta_1\times\vartheta_2$.

Theorem 3.4

If $\rho^{\alpha},\omega^{\beta},\Omega^{\mu\times\alpha}$ are bipolar L-fuzzy sub ℓ-HX groups of ϑ_1,ϑ_2 ,$\vartheta_1\times\vartheta_2$ induced by Bipolar L-fuzzy sub ℓ-groupsα,β and $\alpha\times\beta$ of G_1, G_2 and$G_1\times G_2$ respectively. Then $\Omega^{\alpha\times\beta} = \rho^{\alpha}\times\omega^{\beta}$.

Proof: Let $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ and $\omega^{\beta}=((\omega^{\beta})^{+},(\omega^{\beta})^{-})$ be bipolar L-fuzzy sub ℓ-HX groups of ϑ_1 and ϑ_2 . Then $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of ℓ-HX group $\vartheta_1\times\vartheta_2$ and $\eta^{\alpha\times\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$ induced by $\mu\times\alpha$ of $G_1\times G_2$.

Now

i) $(\Omega^{\alpha\times\beta})^{+}(M,N) = \vee\{ (\alpha\times\beta)^{+}(m,n)/m\in M\subseteq G_1,n\in N\subseteq G_2\}$

$= \vee\{(\alpha^{+}(m)\wedge(\alpha)^{+}(n))/m\in M\subseteq G_1,n\in N\subseteq G_2\}$

$= (\vee\{\alpha^{+}(m)/m\in M\subseteq G_1\})\wedge(\vee\{\alpha)^{+}(n)/n\in N\subseteq G_2\})$

$= (\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(N)$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}(M,N)$

$(\Omega^{\alpha\times\beta})^{+}(M,N) = (\rho^{\alpha}\times\omega^{\beta})^{+}(M,N)$

ii) $(\Omega^{\alpha\times\beta})^{-}(M,N) = \wedge\{ (\alpha\times\beta)^{-}(m,n)/m\in M\subseteq G_1,n\in N\subseteq G_2\}$

$= \wedge\{(\mu^{-}(m)\vee(\alpha)^{-}(n))/m\in M\subseteq G_1,n\in N\subseteq G_2\}$

$= (\wedge\{\mu^{-}(m)/m\in M\subseteq G_1\})\vee(\wedge\{\alpha)^{-}(n)/n\in N\subseteq G_2\})$

$= (\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(N)$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}(M,N)$

$(\Omega^{\alpha\times\beta})^{-}(M,N) = (\rho^{\alpha}\times\omega^{\beta})^{-}(M,N)$

Hence, $\Omega^{\alpha\times\beta} = \rho^{\alpha}\times\omega^{\beta}$.

Example 3.5 The above result illustrated clearly in this example.

Let $(G_1,\cdot_8,\leq)=(\{1,3,5,7\},\cdot_8,\leq)$ be a ℓ-group where G_1 is the non-negative integer relatively prime to 8and $(G_2,\cdot_{12},\leq)=(\{1,5,7,11\},\cdot_{12},\leq)$ be a ℓ-group where G_2 is the non-negative integer relatively prime to 12.Then:

$G_1\times G_2=\{(1,1),(1,5),(1,7),(1,11),(3,1),(3,5),(3,7),(3,11),(5,1),(5,5),(5,7),(5,11),(7,1),(7,5),(7,7),(7,11)\}$.

For all $m\in G$, we define the bipolar L-fuzzy subsets μ and α is on G_1 and G_2 respectively as,

$$\alpha^{+}(m)=\begin{cases} 0.5, \text{ for } m=1 \\ 0.4, \text{ for } m=3 \\ 0.3, \text{ for } m=5 \\ 0.3, \text{ for } m=7 \end{cases}, \quad \alpha^{-}(m)=\begin{cases} -0.4, \text{ for } m=1 \\ -0.3, \text{ for } m=3 \\ -0.2, \text{for } m=50. \\ -0.2, \text{ for } m=70. \end{cases} \quad \beta^{+}(m)=\begin{cases} 0.8, \text{ for } m=1 \\ 0.7, \text{ for } m=5 \\ 6, \text{ for } m=7 \\ 6, \text{ for } m=11 \end{cases}, \quad \beta^{-}(m)=\begin{cases} -0.7, \text{ for } m=1 \\ -0.6, \text{ for } m=5 \\ -0.5, \text{ for } m=7 \\ -0.5, \text{ for } m=11 \end{cases}$$

Clearly,α and $\beta$are bipolar L-fuzzy sub ℓ-group of G_1 and G_2.

$$(\alpha\times\beta)^{+}(m,n)=\begin{cases} 0.5, \text{ for } (m,n) =(1,1),(1,5),(1,7),(1,11) \\ 0.4, \text{ for } (m,n) = (3,1),(3,5),(3,7),(3,11) \\ 0.3, \text{ for } (m,n)=(5,1),(5,5),(5,7),(5,11) \\ 0.3, \text{ for } (m,n)=(7,1),(7,5),(7,7),(7,11) \end{cases}$$

and

$$(\alpha\times\beta)^{-}(m,n)=\begin{cases} -0.4, \text{ for } (m,n) = (1,1),(1,5),(1,7),(1,11) \\ -0.3, \text{ for } (m,n) = (3,1),(3,5),(3,7),(3,11) \\ -0.2, \text{ for } (m,n) = (5,1),(5,5),(5,7),(5,11) \\ -0.2, \text{ for } (m,n) = (7,1),(7,5),(7,7),(7,11) \end{cases}$$

Clearly, $\alpha\times\beta$ is a bipolar L-fuzzy sub ℓ-group of$G_1\times G_2$.

Case(i): if $|M|=1$, for all $M\in\vartheta$

Let$\vartheta_1=\{P,Q,R,S\}$,Where P={1},Q={3},R={5},S={7}and$\vartheta_2=\{I,J,K,Z\}$,Where I={1},J={5},K={7},Z={11} are ℓ-HX groups for all $M\in\vartheta$ with $|M|=1$, Define the bipolar L-fuzzy subsets $\rho^{\alpha}$and ω^{β} on ϑ as,

$$(\rho^{\alpha})^{+}(M)=\begin{cases}0.5, \text{ for } M=P\\0.4, \text{ for } M=Q\\0.3, \text{ for } M=R\\0.3, \text{ for } M=S\end{cases}, \quad (\rho^{\alpha})^{-}(M)=\begin{cases}-0.4, \text{for } M=P\\-0.3, \text{ for } M=Q\\-0.2, \text{for } M=R\\-0.2, \text{ for } M=S\end{cases}$$

$$(\omega^{\beta})^{+}(M)=\begin{cases}0.8, \text{ for } M=I\\0.7, \text{ for } M=J,\\0.6, \text{ for } M=K\\0.6, \text{ for } M=Z\end{cases} \quad (\omega^{\beta})^{-}(M)=\begin{cases}-0.7, \text{ for } M=I\\-0.6, \text{ for } M=J\\-0.5, \text{for } M=K\\-0.5, \text{for } M=Z\end{cases}$$

Clearly, $\rho^{\alpha}$and $\omega^{\beta}$are bipolar L-fuzzy sub ℓ-HX group of ϑ_1 and ϑ_2.

$$(\rho^{\alpha}\times\omega^{\beta})^{+}(M,N)=\begin{cases}0.5, \text{ for } (M,N)=(P,I),(P,J),(P,K),(P,Z)\\0.4, \text{ for } (M,N)=(Q,I),(Q,J),(Q,K),(Q,Z)\\0.3, \text{for } (M,N)=(R,I),(R,J),(R,K),(R,Z)\\0.3, \text{for } (M,N)=(S,I),(S,J),(S,K),(S,Z)\end{cases}$$

and

$$(\rho^{\alpha}\times\omega^{\beta})^{-}(M,N)=\begin{cases}-0.4, \text{ for } (M,N)=(P,I),(P,J),(P,K),(P,Z)\\-0.3, \text{ for } (M,N)=(Q,I),(Q,J),(Q,K),(Q,Z)\\-0.2, \text{ for } (M,N)=(R,I),(R,J),(R,K),(R,Z)\\-0.2, \text{ for } (M,N)=(S,I),(S,J),(S,K),(S,Z)\end{cases}$$

Clearly, $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$.

Let $\Omega^{\alpha\times\beta}=((\Omega^{\alpha\times\beta})^{+},(\Omega^{\alpha\times\beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$ induced by a bipolar L-fuzzy set $\alpha\times\beta$ of $G_1\times G_2$,where

$$(\Omega^{\alpha\times\beta})^{+}(M,N)=\vee\{(\alpha\times\beta)^{+}(m,n)/m\in M\subseteq G,n\in N\subseteq G\}$$
$$(\Omega^{\alpha\times\beta})^{-}(M,N)=\wedge\{(\alpha\times\beta)^{-}(m,n)/m\in M\subseteq G,n\in N\subseteq G\}$$

So,

$$(\Omega^{\alpha\times\beta})^{+}(M,N)=\begin{cases}0.5, \text{ for } (M,N)=(P,I),(P,J),(P,K),(P,Z)\\0.4, \text{ for } (M,N)=(Q,I),(Q,J),(Q,K),(Q,Z)\\0.3, \text{for } (M,N)=(R,I),(R,J),(R,K),(R,Z)\\0.3, \text{for } (M,N)=(S,I),(S,J),(S,K),(S,Z)\end{cases}$$

$$(\Omega^{\alpha\times\beta})^{-}(M,N)=\begin{cases}-0.4, \text{ for } (M,N)=(P,I),(P,J),(P,K),(P,Z)\\-0.3, \text{ for } (M,N)=(Q,I),(Q,J),(Q,K),(Q,Z)\\-0.2, \text{ for } (M,N)=(R,I),(R,J),(R,K),(R,Z)\\-0.2, \text{ for } (M,N)=(S,I),(S,J),(S,K),(S,Z)\end{cases}$$

Clearly, $\Omega^{\alpha\times\beta}=\rho^{\alpha}\times\omega^{\beta}$

Case (ii): if $|M|\geq 2$, for all $M\in\vartheta$

Let $\vartheta_1=\{P,Q\}$,where $P=\{1,3\}$,$Q=\{5,7\}$and $\vartheta_2=\{I,J\}$,where $I=\{1,5\}$,$J=\{7,11\}$are ℓ-HX groups, for all $M\in\vartheta$ with $|M|\geq 2$, Define the bipolar L-fuzzy subsets $\rho^{\alpha},\omega^{\beta}$ on ϑ as,

$$(\rho^{\alpha})^{+}(M)=\begin{cases}0.4, \text{ for } M=P,\\0.3, \text{ for } M=Q\end{cases} \quad (\rho^{\alpha})^{-}(M)=\begin{cases}-0.3, \text{ for } M=P\\-0.2, \text{for } M=Q\end{cases}$$

$$(\omega^{\beta})^{+}(M)=\begin{cases}0.7, \text{ for } M=I,\\0.6, \text{ for } M=J\end{cases} \quad (\omega^{\beta})^{-}(M)=\begin{cases}-0.6, \text{for } M=I\\-0.5, \text{ for } M=J\end{cases}$$

Clearly, $\rho^{\alpha}$and ω^{β} are bipolar L-fuzzy sub ℓ-HX group of ϑ_1 and ϑ_2.

$$(\rho^{\alpha}\times\omega^{\beta})^{+}(M,N)=\begin{cases}0.4, \text{ for } (M,N)=(P,I),(P,J)\\0.3, \text{for } (M,N)=(Q,I),(Q,J)\end{cases}$$

$$(\rho^{\alpha}\times\omega^{\beta})^{-}(M,N)=\begin{cases}-0.3, \text{ for}(M,N)=(P,I),(P,J)\\-0.2, \text{ for } (M,N)=(Q,I),(Q,J)\end{cases}$$

Clearly, $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$.

$$(\Omega^{\alpha\times\beta})^{+}(M,N)=\begin{cases}0.4, \text{ for } (M,N)=(P,I),(P,J)\\0.3, \text{for } (M,N)=(Q,I),(Q,J)\end{cases}$$

$(\Omega^{\alpha\times\beta})^{-}(M,N) = \begin{cases} -0.3 \text{ for } (M,N)=(P,I),(P,J) \\ -0.2 \text{ for } (M,N)=(Q,I),(Q,J) \end{cases}$

Clearly, $\Omega^{\alpha\times\beta} = \rho^{\alpha}\times\omega^{\beta}$.

Hence, by cases (i) and (ii), $\Omega^{\alpha\times\beta} = \rho^{\alpha}\times\omega^{\beta}$ is proved.

Theorem 3.6

Let $\alpha=(\alpha^{+}, \alpha^{-})$ and $\beta=(\beta^{+},\beta^{-})$ are bipolar L-fuzzy subsets of the sub ℓ-groups of G_1 and G_2 respectively. Let $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ and $\omega^{\beta}=((\omega^{\beta})^{+},(\omega^{\beta})^{-})$ be bipolar L-fuzzy sub ℓ-HX groups of the ℓ-HX groups ϑ_1 and ϑ_2 respectively. Suppose that E_1 and E_2 are the identity elements of ϑ_1 and ϑ_2 respectively. If a product $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$ then at least one of the following two statements must hold.

i) $(\omega^{\beta})^{+}(E_2) \geq (\rho^{\alpha})^{+}(M)$, $(\omega^{\beta})^{-}(E_2) \leq (\rho^{\alpha})^{-}(M) \forall M\in\vartheta_1$

ii) $(\rho^{\alpha})^{+}(E_1) \geq (\omega^{\beta})^{+}(N)$, $(\rho^{\alpha})^{-}(E_1) \leq (\omega^{\beta})^{-}(N) \forall N\in\vartheta_2$.

Proof: Let $\rho^{\alpha}\times\omega^{\beta}=((\rho^{\alpha}\times\omega^{\beta})^{+},(\rho^{\alpha}\times\omega^{\beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find M in ϑ_1 and N in ϑ_2 such that $(\rho^{\alpha})^{+}(M)>(\omega^{\beta})^{+}(E_2)$, $(\rho^{\alpha})^{-}(M)<(\omega^{\beta})^{-}(E_2)$ and $(\omega^{\beta})^{+}(N)>(\rho^{\alpha})^{+}(E_1)$, $(\omega^{\beta})^{-}(N)<(\rho^{\alpha})^{-}(E_1)$. We have:

i)
$$\begin{aligned}(\rho^{\alpha}\times\omega^{\beta})^{+}(M,N) &= (\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(N)\\ &> (\omega^{\beta})^{+}(E_2)\wedge(\rho^{\alpha})^{+}(E_1)\\ &= (\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(E_2)\\ &> (\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,E_2)\end{aligned}$$

and

ii)
$$\begin{aligned}(\rho^{\alpha}\times\omega^{\beta})^{-}(M,N) &= (\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(N)\\ &< (\omega^{\beta})^{-}(E_2)\vee(\rho^{\alpha})^{-}(E_1)\\ &= (\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(E_2)\\ &< (\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,E_2)\end{aligned}$$

Thus the product $\rho^{\alpha}\times\omega^{\beta}$ is not a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$.

Hence, either $(\omega^{\beta})^{+}(E_2)\geq(\rho^{\alpha})^{+}(M)$, $(\omega^{\beta})^{-}(E_2)\leq(\rho^{\alpha})^{-}(M)$ for all $M\in\vartheta_1$

(or)

$(\rho^{\alpha})^{+}(E_1)\geq(\omega^{\beta})^{+}(N)$, $(\rho^{\alpha})^{-}(E_1)\leq(\omega^{\beta})^{-}(N)$ for all $N\in\vartheta_2$.

Theorem 3.7

Let ρ^{α} and ω^{β} be bipolar L-fuzzy subsets of the sub ℓ-HX groups of ϑ_1 and ϑ_2 respectively. Such that $(\rho^{\alpha})^{+}(M)\leq(\omega^{\beta})^{+}(E_2)$, $(\rho^{\alpha})^{-}(M)\geq(\omega^{\beta})^{-}(E_2)$ for all $M\in\vartheta_1$, E_2 be the identity element of ϑ_2. If a product $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$ then ρ^{α} is a bipolar L-fuzzy sub ℓ-HX groups of ϑ_1.

Proof: Let $\rho^{\alpha}\times\omega^{\beta}= ((\rho^{\alpha}\times\omega^{\beta})^{+},(\rho^{\alpha}\times\omega^{\beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$ and $M,N\in\vartheta$. Then $(M,E_2),(N,E_2)\in\vartheta_1\times\vartheta_2$. Given:

i) $(\rho^{\alpha})^{+}(M)\leq(\omega^{\beta})^{+}(E_2)$, ii) $(\rho^{\alpha})^{-}(M)\geq(\omega^{\beta})^{-}(E_2)$ for all $M\in\vartheta_1$

We have:

i)
$$\begin{aligned}(\rho^{\alpha})^{+}(MN) &= (\rho^{\alpha})^{+}(MN)\wedge(\omega^{\beta})^{+}(E_2E_2) \text{.....by (i)}\\ &= (\rho^{\alpha}\times\omega^{\beta})^{+}((M,E_2),(N,E_2))\\ &\geq (\rho^{\alpha}\times\omega^{\beta})^{+}(M,E_2)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(N,E_2)\\ &= ((\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(E_2))\wedge((\rho^{\alpha})^{+}(N)\wedge(\omega^{\beta})^{+}(E_2))\\ &= (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)\\ (\rho^{\alpha})^{+}(MN) &\geq (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)\end{aligned}$$

ii)
$$\begin{aligned}(\rho^{\alpha})^{-}(MN) &= (\rho^{\alpha})^{-}(MN)\vee(\omega^{\beta})^{-}(E_2E_2) \text{...... by (ii)}\\ &= (\rho^{\alpha}\times\omega^{\beta})^{-}((M,E_2),(N,E_2))\\ &\leq (\rho^{\alpha}\times\omega^{\beta})^{-}(M,E_2)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(N,E_2)\\ &= ((\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(E_2))\vee((\rho^{\alpha})^{-}(N)\vee(\omega^{\beta})^{-}(E_2))\\ &= (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)\end{aligned}$$

$(\rho^{\alpha})^{-}(MN) \leq (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)$

iii) $(\rho^{\alpha})^{+}(M^{-1}) = (\rho^{\alpha})^{+}(M^{-1})\wedge(\omega^{\beta})^{+}(E_2^{-1})$……by (i)

$= (\rho^{\alpha}\times\omega^{\beta})^{+}(M^{-1}, E_2^{-1})$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}((M,E_2)^{-1})$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}(M,E_2)$

$= (\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(E_2)$

$(\rho^{\alpha})^{+}(M^{-1}) = (\rho^{\alpha})^{+}(M)$

iv) $(\rho^{\alpha})^{-}(M^{-1}) = (\rho^{\alpha})^{-}(M^{-1})\vee(\omega^{\beta})^{-}(E_2^{-1})$ …… by (ii)

$= (\rho^{\alpha}\times\omega^{\beta})^{-}(M^{-1},E_2^{-1})$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}((M,E_2)^{-1})$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}(M,E_2)$

$= (\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(E_2)$

$(\rho^{\alpha})^{-}(M^{-1}) = (\rho^{\alpha})^{-}(M)$

v) $(\rho^{\alpha})^{+}(M\vee N) = (\rho^{\alpha})^{+}(M\vee N)\wedge(\omega^{\beta})^{+}(E_2\vee E_2)$ ……by (i)

$= (\rho^{\alpha}\times\omega^{\beta})^{+}((M\vee N),(E_2\vee E_2))$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}((M,E_2)\vee(N,E_2))$

$\geq ((\rho^{\alpha}\times\omega^{\beta})^{+}(M,E_2))\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(N,E_2))$

$= ((\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(E_2))\wedge((\rho^{\alpha})^{+}(N)\wedge(\omega^{\beta})^{+}(E_2))$

$= (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)$

$(\rho^{\alpha})^{+}(M\vee N) \geq (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)$

vi) $(\rho^{\alpha})^{-}(M\vee N) = (\rho^{\alpha})^{-}(M\vee N)\vee(\omega^{\beta})^{-}(E_2\vee E_2)$ …… by (ii)

$= (\rho^{\alpha}\times\omega^{\beta})^{-}((M\vee N),(E_2\vee E_2))$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}((M,E_2)\vee(N,E_2))$

$\leq ((\rho^{\alpha}\times\omega^{\beta})^{-}(M,E_2))\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(N,E_2))$

$= ((\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(E_2))\vee((\rho^{\alpha})^{-}(N)\vee(\omega^{\beta})^{-}(E_2))$

$= (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)$

$(\rho^{\alpha})^{-}(M\vee N) \leq (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)$

vii) $(\rho^{\alpha})^{+}(M\wedge N) = (\rho^{\alpha})^{+}(M\wedge N)\wedge(\omega^{\beta})^{+}(E_2\wedge E_2)$ ……by (i)

$= (\rho^{\alpha}\times\omega^{\beta})^{+}((M\wedge N),(E_2\wedge E_2))$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}((M,E_2)\wedge(N,E_2))$

$\geq ((\rho^{\alpha}\times\omega^{\beta})^{+}(M,E_2))\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(N,E_2))$

$= ((\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(E_2))\wedge((\rho^{\alpha})^{+}(N)\wedge(\omega^{\beta})^{+}(E_2))$

$= (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)$

$(\rho^{\alpha})^{+}(M\wedge N) \geq (\rho^{\alpha})^{+}(M)\wedge(\rho^{\alpha})^{+}(N)$

viii) $(\rho^{\alpha})^{-}(M\wedge N) = (\rho^{\alpha})^{-}(M\wedge N)\wedge(\omega^{\beta})^{-}(E_2\wedge E_2)$ …… by (ii)

$= (\rho^{\alpha}\times\omega^{\beta})^{-}((M\wedge N),(E_2\wedge E_2))$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}((M,E_2)\wedge(N,E_2))$

$\leq ((\rho^{\alpha}\times\omega^{\beta})^{-}(M,E_2))\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(N,E_2))$

$= ((\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(E_2))\vee((\rho^{\alpha})^{-}(N)\vee(\omega^{\beta})^{-}(E_2))$

$= (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)$

$(\rho^{\alpha})^{-}(M\wedge N) \leq (\rho^{\alpha})^{-}(M)\vee(\rho^{\alpha})^{-}(N)$

Hence,ρ^{α} is a bipolar L-fuzzy sub ℓ-HX group of ϑ_1.

Theorem 3.8

Let ρ^{α} and ω^{β} be bipolar L- fuzzy subsets of the sub ℓ-HX groups of $\vartheta_1$and ϑ_2 respectively. Such that $(\omega^{\beta})^{+}(M)\leq(\rho^{\alpha})^{+}(E_1)$, $(\omega^{\beta})^{-}(M)\geq(\rho^{\alpha})^{-}(E_1)$ for all $M\in\vartheta_2$,E_1be the identity element of ϑ_1 . If a product $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$ then $\omega^{\beta}$is a bipolar L-fuzzy sub ℓ-HX group of ϑ_2.

Proof: Let $\rho^{\alpha}\times\omega^{\beta}=((\rho^{\alpha}\times\omega^{\beta})^{+},(\rho^{\alpha}\times\omega^{\beta})^{-})$ be a bipolar L-fuzzy sub ℓ-HX groups of $\vartheta_1\times\vartheta_2$ and $M,N\in\vartheta_2$, then $(E_1,M),(E_1,N)\in\vartheta_1\times\vartheta_2$.

Let i) $(\omega^{\beta})^{+}(M)\leq(\rho^{\alpha})^{+}(E_1)$, ii)$(\omega^{\beta})^{-}(M)\geq(\rho^{\alpha})^{-}(E_1)$ for all $M\in\vartheta_2$

We have:

$$\begin{aligned}
\text{i)}\quad (\omega^{\beta})^{+}(MN) &= (\rho^{\alpha})^{+}(E_1E_1)\wedge(\omega^{\beta})^{+}(MN) \text{ .....by (i)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1E_1),(MN))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1,M),(E_1,N))\\
&\geq (\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,M)\wedge(\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,N)\\
&= ((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(M))\wedge((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(N))\\
&= (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)\\
(\omega^{\beta})^{+}(MN) &\geq (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)
\end{aligned}$$

$$\begin{aligned}
\text{ii)}\quad (\omega^{\beta})^{-}(MN) &= (\rho^{\alpha})^{-}(E_1E_1)\vee(\omega^{\beta})^{-}(MN) \text{...... by (ii)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1,M),(E_1,N))\\
&\leq (\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,M)\vee(\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,N)\\
&= ((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(M))\vee((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(N))\\
&= (\omega^{\beta})^{-}(M)\vee(\omega^{\beta})^{-}(N)\\
(\omega^{\beta})^{-}(MN) &\leq (\omega^{\beta})^{-}(M)\vee(\omega^{\beta})^{-}(N)
\end{aligned}$$

$$\begin{aligned}
\text{iii)}\quad (\omega^{\beta})^{+}(M^{-1}) &= (\rho^{\alpha})^{+}(E_1^{-1})\wedge(\omega^{\beta})^{+}(M^{-1}) \text{......by (i)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(E_1^{-1},M^{-1})\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1,M)^{-1})\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,M)\\
&= (\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(M)\\
(\omega^{\beta})^{+}(M^{-1}) &= (\omega^{\beta})^{+}(M)
\end{aligned}$$

$$\begin{aligned}
\text{iv)}\quad (\omega^{\beta})^{-}(M^{-1}) &= ((\rho^{\alpha})^{-}(E_1^{-1})\vee(\omega^{\beta})^{-}(M^{-1}) \text{......by (ii)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}(E_1^{-1},M^{-1})\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1,M)^{-1})\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,M)\\
&= (\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(M)\\
(\omega^{\beta})^{-}(M^{-1}) &= (\omega^{\beta})^{-}(M)
\end{aligned}$$

$$\begin{aligned}
\text{v)}\quad (\omega^{\beta})^{+}(M\vee N) &= (\rho^{\alpha})^{+}(E_1\vee E_1)\wedge(\omega^{\beta})^{+}(M\vee N) \text{ ......by (i)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1\vee E_1),(M\vee N))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1,M)\wedge(E_1,N))\\
&\geq ((\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,M))\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,N))\\
&= ((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(M))\wedge((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(N))\\
&= (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)\\
(\omega^{\beta})^{+}(M\vee N) &\geq (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)
\end{aligned}$$

$$\begin{aligned}
\text{vi)}\quad (\omega^{\beta})^{-}(M\vee N) &= (\rho^{\alpha})^{-}(E_1\vee E_1)\vee(\omega^{\beta})^{-}(M\vee N) \text{ ...... by (ii)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1\vee E_1),(M\vee N))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1,M)\vee(E_1,N))\\
&\leq ((\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,M))\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,N))\\
&= ((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(M))\vee((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(N))\\
&= (\omega^{\beta})^{-}(M)\vee(\omega^{\beta})^{-}(N)\\
(\omega^{\beta})^{-}(M\vee N) &\leq (\omega^{\beta})^{-}(M)\vee(\omega^{\beta})^{-}(N)
\end{aligned}$$

$$\begin{aligned}
\text{vii)}\quad (\omega^{\beta})^{+}(M\wedge N) &= (\rho^{\alpha})^{+}(E_1\wedge E_1)\wedge(\omega^{\beta})^{+}(M\wedge N) \text{ ......by (i)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1\wedge E_1),(M\wedge N))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}((E_1,M)\wedge(E_1,N))\\
&\geq ((\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,M))\wedge((\rho^{\alpha}\times\omega^{\beta})^{+}(E_1,N))\\
&= ((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(M))\wedge((\rho^{\alpha})^{+}(E_1)\wedge(\omega^{\beta})^{+}(N))\\
&= (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)\\
(\omega^{\beta})^{+}(M\wedge N) &\geq (\omega^{\beta})^{+}(M)\wedge(\omega^{\beta})^{+}(N)
\end{aligned}$$

$$\begin{aligned}
\text{viii)}\quad (\omega^{\beta})^{-}(M\wedge N) &= (\rho^{\alpha})^{-}(E_1\wedge E_1)\wedge(\omega^{\beta})^{-}(M\wedge N) \text{ ...... by (ii)}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1\wedge E_1),(M\wedge N))\\
&= (\rho^{\alpha}\times\omega^{\beta})^{-}((E_1,M)\wedge(E_1,N))\\
&\leq ((\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,M))\vee((\rho^{\alpha}\times\omega^{\beta})^{-}(E_1,N))\\
&= ((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(M))\vee((\rho^{\alpha})^{-}(E_1)\vee(\omega^{\beta})^{-}(N))
\end{aligned}$$

$$= \quad (\omega^{\beta})^{-}(M) \vee (\omega^{\beta})^{-}(N)$$

$$(\omega^{\beta})^{-}(M \wedge N) \leq (\omega^{\beta})^{-}(M) \vee (\omega^{\beta})^{-}(N)$$

Hence, ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ_2.

Corollary 3.9. Let ρ^{α} and ω^{β} be bipolar L-fuzzy subsets of the sub ℓ-HX groups ϑ_1 and ϑ_2 respectively. If $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$, then either ρ^{α} is a bipolar L-fuzzy sub ℓ-HX group of ϑ_1 or ω^{β} is a bipolar L-fuzzy sub ℓ-HX group of ϑ_2.

4. Properties of a bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group under homomorphism and anti-homomorphism

In this section, we discuss the properties of the product of bipolar L-fuzzy sub ℓ-HX group of a ℓ-HX group under homomorphism and anti-homomorphism. E_1 and E_2 are the identity elements of the ℓ-HX groups, ϑ_1 and ϑ_2 respectively ,(ϑ_1 and ϑ_2 are not necessarily commutative), and mn we mean m*n. Over this section the finite ℓ-groups G_1, G_2, H_1 and H_2 are not necessarily commutative and $\vartheta_1\subset 2^{G1}-\{\phi\}$, $\vartheta_2\subset 2^{G2}-\{\phi\}$, $\vartheta_3\subset 2^{H1}-\{\phi\}$, $\vartheta_4\subset 2^{H2}-\{\phi\}$ are their ℓ-HX groups respectively.

Theorem 4.1 [10]

Let φ be homomorphism from $\vartheta_1\times\vartheta_2$ onto $\vartheta_3\times\vartheta_4$. If $(\rho^{\alpha}\times\omega^{\beta})$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$, then the image $\varphi(\rho^{\alpha}\times\omega^{\beta})$ of $(\rho^{\alpha}\times\omega^{\beta})$ under φ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_3\times\vartheta_4$, if $\rho^{\alpha},\omega^{\beta}$ have sup property and $\rho^{\alpha},\omega^{\beta}$ are φ-invariant.

Theorem 4.2 [10]

Let f be homomorphism from $\vartheta_1\times\vartheta_2$ to $\vartheta_3\times\vartheta_4$. If $(\delta^{\eta}\times\Upsilon^{\varsigma})$ is a bipolar L-fuzzy sub ℓ- HX group of $\vartheta_3\times\vartheta_4$ then the pre-image $\varphi^{-1}(\delta^{\eta}\times\Upsilon^{\varsigma})$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$, ., if $\lambda^{\mu},\omega^{\beta}$ have sup property and $\rho^{\alpha},\omega^{\beta}$ are φ-invariant.

Theorem 4.3 ***[10]***

Let φ be an anti-homomorphism from $\vartheta_1\times\vartheta_2$ on to $\vartheta_3\times\vartheta_4$. If $(\rho^{\alpha}\times\omega^{\beta})$ is a bipolar L-fuzzy sub ℓ-HX group $\vartheta_1\times\vartheta_2$, then the image $\varphi(\rho^{\alpha}\times\omega^{\beta})$ of $(\rho^{\alpha}\times\omega^{\beta})$ under φ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_3\times\vartheta_4$, if $\rho^{\alpha},\omega^{\beta}$ have sup property and $\rho^{\alpha},\omega^{\beta}$ are φ-invariant.

Theorem 4.4 ***[10]***

Let φ be an anti-homomorphism from $\vartheta_1\times\vartheta_2$ to $\vartheta_3\times\vartheta_4$. If $(\delta^{\eta}\times\Upsilon^{\varsigma})$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_3\times\vartheta_4$ then the pre-image $\varphi^{-1}(\delta^{\eta}\times\Upsilon^{\varsigma})$ is a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$, if $\rho^{\alpha},\omega^{\beta}$ have sup property and $\rho^{\alpha},\omega^{\beta}$ are φ-invariant.

Theorem 4.5

Let ϑ_1 and ϑ_2 be any two ℓ-HX groups of the ℓ-groups G_1 and G_2 respectively. Let $\varphi:\vartheta_1\times\vartheta_2\rightarrow\vartheta_3\times\vartheta_4$ be a homomorphism onto ℓ-HX groups. Let $\rho^{\alpha}\times\omega^{\beta}$ be a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$ then $\varphi(\rho^{\alpha}\times\omega^{\beta})=\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta})$, if $\rho^{\alpha},\omega^{\beta}$ have sup property and $\rho^{\alpha},\omega^{\beta}$ are φ-invariant.

Proof: Let α be a bipolar L-fuzzy subset of G_1. Let β be a bipolar L-fuzzy sub ℓ-group of G_2. Let $\rho^{\alpha}\times\omega^{\beta}$ be a bipolar L-fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$. Let $(I,J)\in\vartheta_1\times\vartheta_2$, as φ is a homomorphism such that $(\varphi(I),\varphi(J))\in\vartheta_3\times\vartheta_4$,

$$\begin{aligned}
(\varphi(\rho^{\alpha}\times\omega^{\beta}))^{+}(\varphi(I),\varphi(J)) &= (\varphi(\rho^{\alpha}\times\omega^{\beta}))^{+}(\varphi(I,J)), \text{ as } \varphi \text{ is homomorphism}\\
&= (\rho^{\alpha}\times\omega^{\beta})^{+}(I,J)\\
&= (\rho^{\alpha})^{+}(I)\wedge(\omega^{\beta})^{+}(J)\\
&= \varphi(\rho^{\alpha})^{+}(\varphi(I))\wedge\varphi(\omega^{\beta})^{+}(\varphi(J))\\
&= (\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta}))^{+}(\varphi(I),\varphi(J))
\end{aligned}$$

Therefore, $(\varphi(\rho^{\alpha}\times\omega^{\beta}))^{+}(\varphi(I),\varphi(J))=(\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta}))^{+}(\varphi(I),\varphi(J))$

and

$(\varphi(\rho^{\alpha}\times\omega^{\beta}))^{-}(\varphi(I),\varphi(J)) = (\varphi(\rho^{\alpha}\times\omega^{\beta}))^{-}(\varphi(I,J))$, as φ is homomorphism

$= (\rho^{\alpha}\times\omega^{\beta})^{-}(I,J)$

$= (\rho^{\alpha})^{-}(I)\vee(\omega^{\beta})^{-}(J)$

$= \varphi(\rho^{\alpha})^{-}(\varphi(I))\vee\varphi(\omega^{\beta})^{-}(\varphi(J))$

$= (\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta}))^{-}(\varphi(I),\varphi(J))$

Therefore, $(\varphi(\rho^{\alpha}\times\omega^{\beta}))^{-}(\varphi(I),\varphi(J))=(\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta}))^{-}(\varphi(I),\varphi(J))$

Hence, $\varphi(\rho^{\alpha}\times\omega^{\beta})=\varphi(\rho^{\alpha})\times\varphi(\omega^{\beta})$.

Theorem 4.6

Let μ and α be any two bipolar L-fuzzy sub ℓ-groups of G_1 and G_2 respectively. Let ϑ_1 and ϑ_2 be any two ℓ-HX groups of the ℓ-groups of G_1 and G_2 respectively. Let ρ^{α} and ω^{β} be any two bipolar L-fuzzy sub ℓ-HX groups of the ℓ-HX groups $\vartheta_1$andϑ_2 respectively. Let $\varphi:\vartheta_1\times\vartheta_2\rightarrow\vartheta_3\times\vartheta_4$ be a homomorphism. If $\rho^{\alpha}\times\omega^{\beta}$ is a bipolar L-fuzzy sub ℓ-HX group of$\vartheta_1\times\vartheta_2$, then:

$\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta})=\varphi^{-1}(\rho^{\alpha})\times\varphi^{-1}(\omega^{\beta})$.

Proof: Let $\rho^{\alpha}\times\omega^{\beta}$ be a bipolar L- fuzzy sub ℓ-HX group of $\vartheta_1\times\vartheta_2$. Let $(I,J)\in\vartheta_1\times\vartheta_2$, since φ is a homomorphism such that $(\varphi(I), \varphi(J))\in\vartheta_3\times\vartheta_4$.

$(\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta}))^{+}(I,J) = (\rho^{\alpha}\times\omega^{\beta})^{+}(\varphi(I,J))$

$= (\rho^{\alpha}\times\omega^{\beta})^{+}(\varphi(I),\varphi(J))$

$= (\rho^{\alpha})^{+}(\varphi(I))\wedge(\omega^{\beta})^{+}(\varphi(J))$

$= \varphi^{-1}(\rho^{\alpha})^{+}(I)\wedge\varphi^{-1}(\omega^{\beta})^{+}(J)$

$= (\varphi^{-1}(\rho^{\alpha})\times\varphi^{-1}(\omega^{\beta}))^{+}(I,J)$.

Therefore, $(\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta}))^{+}(I,J)= (\varphi^{-1}(\rho^{\alpha})\times\varphi^{-1}(\omega^{\beta}))^{+}(I,J)$.

and

$(\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta}))^{-}(I,J) = (\rho^{\alpha}\times\omega^{\beta})^{-}(\varphi(I,J))$

$= (\rho^{\alpha}\times\omega^{\beta})^{-}(\varphi(I),\varphi(J))$

$= (\rho^{\alpha})^{-}(\varphi(I))\vee(\omega^{\beta})^{-}(\varphi(J))$

$= \varphi^{-1}(\rho^{\alpha})^{-}(I)\vee\varphi^{-1}(\omega^{\beta})^{-}(J)$

$= (\varphi^{-1}(\rho^{\alpha})\times\varphi^{-1}(\omega^{\beta}))^{-}(I,J)$.

Therefore, $(\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta}))^{-}(I,J) = (\varphi^{-1}(\rho^{\alpha})\times\varphi^{-1}(\omega^{\beta}))^{-}(I,J)$

Hence, $\varphi^{-1}(\rho^{\alpha}\times\omega^{\beta})=\varphi^{-1}(\rho^{\alpha}) \times \varphi^{-1}(\omega^{\beta})$.

Definition 4.7[11]

Let ρ^{α} be a bipolar L- fuzzy sub ℓ-HX group of a ℓ-HX groupϑ. The set $I(\rho^{\alpha};\tau,\psi)=\{P\in\vartheta/(\rho^{\alpha})^{+}(P) \geq\tau, (\rho^{\alpha})^{+}(P)\leq\psi\}$, for any $<\tau,\psi>\in[0,1]\times[-1,0]$ is called the bipolar level subset of ρ^{α} or $<\tau, \psi>$ - level subset of ρ^{α} or upper level subset of ρ^{α} or level subset of ρ^{α}.

Theorem 4.8

Let ρ^{α} and ω^{β} be any two bipolar L- fuzzy sub ℓ-HX groups of the ℓ-HX groups$\vartheta_1$andϑ_2 respectively. Let $<\tau, \psi>\in[0,1]\times[-1,0]$ then $I(\rho^{\alpha}\times\omega^{\beta};\tau, \psi)=I(\rho^{\alpha};\tau, \psi)\times I(\omega^{\beta};\tau, \psi)$

Proof: For all $(M,N)\in I((\rho^{\alpha}\times\omega^{\beta});\tau,\psi)$

If $(\rho^{\alpha}\times\omega^{\beta})^{+}(M,N)\geq\tau$ and $(\rho^{\alpha}\times\omega^{\beta})^{-}(M,N)\leq\psi$

If $((\rho^{\alpha})^{+}(M)\wedge(\omega^{\beta})^{+}(N))\geq\tau$ and $((\rho^{\alpha})^{-}(M)\vee(\omega^{\beta})^{-}(N))\leq\psi$

If $(\rho^{\alpha})^{+}(M)\geq\tau$ and $(\omega^{\beta})^{+}(N)\geq\tau,(\rho^{\alpha})^{-}(M)\leq\psi$ and $(\omega^{\beta})^{-}(N)\leq\psi$

If $(\rho^{\alpha})^{+}(M)\geq\tau$ and $(\rho^{\alpha})^{-}(M)\leq\psi$, $(\omega^{\beta})^{+}(N)\geq\tau$ and $(\omega^{\beta})^{-}(N)\leq\psi$

If $M\in I(\rho^{\alpha};\tau,\psi)$ and $N\in I(\omega^{\beta}; \tau, \psi)$

If $(M,N)\in I(\rho^{\alpha};\tau,\psi) \times I(\omega^{\beta}; \tau, \psi)$

Hence, $I((\rho^{\alpha}\times\omega^{\beta});\tau, \psi)= I(\rho^{\alpha};\tau, \psi) \times I(\omega^{\beta};\tau, \psi)$.

5. Conclusions

In this paper, we have presented some properties of the Cartesian product of bipolar L-fuzzy subsets of a set and some results of the Cartesian product of bipolar L-fuzzy subℓ-HX groups of a ℓ-HX group under homomorphism and anti-homomorphism.

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