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gη-HOMEOMORPHISM IN TOPOLOGICAL SPACES

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Abstract In this paper a new class of maps namely gη-closed maps, gη-open maps and gη-homeomorphism in topological spaces are introduced. Further some of their characterizations are investigated.

Keywords gη-closed maps, gη-open maps, gη-homeomorphism, gη*-homeomorphism

1. Introduction

In recent years a number of generalizations of open sets have been developed by many mathematicians. In 1963, Levine [9] introduced the notion of semi-open sets in topological spaces. In 1984, Andrijevic [1] introduced some properties of the topology of α-sets. In 2016, Sayed MEL and Mansour FHAL introduced [19] new near open set in Topological Spaces. Motivated by various open and closed sets are discussed in the previous literature, in this paper a new class of maps called gη-closed maps and gη-open maps has been introduced using the concept of gη-closed sets, gη-continuous by Subbulakshmiet al [22, 23]. Further we study the basic properties of gη-closed maps and gη-open maps.

2. Preliminaries

Definition 2.1 A subset A of a topological space (X, τ) is called:
(i) α-open set [1] if $A \subseteq int(cl(int(A)))$, α-closed set if $cl (int (cl(A))) \subseteq A$.
(ii) pre-openset [15] if $A \subseteq int (cl (A))$, pre-closed set if $cl (int(A)) \subseteq A$.
(iii) semi-openset [9] if $A \subseteq cl(int (A))$, semi-closed set if $int (cl(A) \subseteq A$.
(iv) regular-open set [18] if $A = int (cl(A))$, regular-closed set if $A = cl (int (A)))$.
(v) β-open (or semi-pre-open) set [2] if $A \subseteq (cl(int(cl(A)))$, semi-pre-closed set if $int(cl(int(A))) \subseteq A$.
(vi) η-open set [21] if $A \subseteq int (cl(int(A))) \cup cl (int (A))$,η-closed set if $cl (int (cl (A))) \cap int(cl(A)) \subseteq A$.

Definition 2.2 A subset A of a topological space $(X, \boldsymbol{\tau})$ is called:
(i) g-closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in$(X, \boldsymbol{\tau})$.
(ii)g*-closed set [25] if $cl(A) \subseteq U$ wheneverA $\subseteq$ Uand U is g open in$(X, \boldsymbol{\tau})$.
(iii) gα-closed set [13] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α open in$(X, \boldsymbol{\tau})$.
(iv) αg-closed set [12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in$(X, \boldsymbol{\tau})$.
(v) sg-closed set [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in$(X, \boldsymbol{\tau})$.
(vi) gpr-closed set [8] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in(X, τ).
(vii) gαr-closed set [20] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in(X, τ).
(viii) rg-closed set [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in(X, τ).
(ix) gη-closed set [22] if $\eta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in$(X, \boldsymbol{\tau})$.

Definition 2.3 A functionf: $(X, \tau) \rightarrow (Y, \sigma)$is called:
(i) continuous [3] if $f^{-1}(V)$ is a closed in$(X, \boldsymbol{\tau})$for every closed set V of(Y, σ).
(ii) semi-continuous [9] if $f^{-1}(V)$ is a semi-closed in $(X, \boldsymbol{\tau})$ for every closed set V of (Y, σ).
(iii) α-continuous [13] if $f^{-1}(V)$ is aα-closed in$(X, \boldsymbol{\tau})$ for every closed set V of(Y, σ).
(iv) r-continuous [11] if $f^{-1}(V)$ is ar-closed in $(X, \boldsymbol{\tau})$for every closed set V of(Y, σ).
(v) g-continuous [3] if $f^{-1}(V)$ is a g-closed in$(X, \boldsymbol{\tau})$ for every closed set V of(Y, σ).
(vi) g*-continuous [16] if $f^{-1}(V)$ is a g*-closed in$(X, \boldsymbol{\tau})$ for every closed set V of(Y, σ).
(vii) sg-continuous [24] if $f^{-1}(V)$ is a sg-closed in$(X, \boldsymbol{\tau})$for every closed set V of(Y, σ).
(viii) gα-continuous [6] if $f^{-1}(V)$ is a gα-closed in$(X, \boldsymbol{\tau})$ for every closed set V of (Y, σ).
(ix) αg-continuous [12] if $f^{-1}(V)$ is a αg-closed in$(X, \boldsymbol{\tau})$for every closed set V of(Y, σ).
(x) η-continuous [23] if $f^{-1}(V)$ is a η-closed in $(X, \boldsymbol{\tau})$for every closed set V of(Y, σ).
(xi) gαr-continuous [20] if $f^{-1}(V)$ is a gαr-closed in$(X, \boldsymbol{\tau})$ for every regular-closed set V of(Y, σ).
(xii) rg-continuous [17] if $f^{-1}(V)$ is arg-closed in$(X, \boldsymbol{\tau})$ for every regular-closed set V of(Y, σ).
(xiii) gpr-continuous [8] if $f^{-1}(V)$ is a gpr-closed in$(X, \boldsymbol{\tau})$ for every regular-closed set V of(Y, σ).
(xiv) gη-continuous [23] iff $^{-1}(V)$ is a gη-closed in$(X, \boldsymbol{\tau})$ for every closed set V of(Y, σ).
(xv)gη-irresolute[23] if $f^{-1}(V)$ is gη-closed in$(X, \boldsymbol{\tau})$ for every gη-closed V of (Y, σ).

Definition 2.4 A bijective function f: $(X, \tau) \rightarrow (Y, \sigma)$ is called:
(i) homeomorphism [3] if both f and f^{-1} are continuous.
(ii) semi-homeomorphism [5] if both f and f^{-1} are semi-continuous.
(iii) α-homeomorphism [13] if both f and $f^{-1}$are α-continuous.
(iv) r-homeomorphism [11] if both f and f^{-1} are r-continuous.
(v) g-homeomorphism [14] if both f and f^{-1} are g-continuous.
(vi) g *-homeomorphism [16] if both f and f^{-1} are g*-continuous.
(vii) sg-homeomorphism [7] if both f and f^{-1} are sg-continuous.
(viii) gα-homeomorphism [6] if both f and $f^{-1}$are gα-continuous.
(ix) αg-homeomorphism [12] if both f and f^{-1} are αg-continuous.
(x) rg-homeomorphism [17] if both f and f^{-1} are rg-continuous.

(xi) gαr-homeomorphism [20] if both f and f^{-1} are gαr-continuous.
(xii) gpr-homeomorphism [8] if both f and f^{-1} are gαr-continuous.

3. gη-closed maps

Definition 3.1 A map f : (X, τ) → (Y, σ) is said to be a gη-closed map if the image of every closed set in (X, τ) is gη-closed in (Y, σ).

Example 3.2 Let X = Y = {a, b, c, d}, τ = {X, φ, {c}, {a, b}, {a, b, c}} and σ = {Y, φ, {b},{c, d},{b, c, d}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = d, f (d) = c. Then f ({d}) = {c}, f ({c, d}) = {c, d}, f ({a, b, d}) = {a, b, c}. Therefore f is gη-closed map. Since the image of every closed set in X is gη-closed in Y.

Theorem 3.3 Let (X, τ) and (Y, σ) be topological spaces. Then for a mapping f : (X, τ) → (Y, σ). The following results are true.

(i) Every closed map is gη-closed map.
(ii) Every semi-closed map is gη-closed map.
(iii) Every α-closed map is gη-closed map.
(iv) Every r-closed map is gη-closed map.
(v) Every η-closed map is gη-closed map.
(vi) Every g-closed map is gη-closed map.
(vii) Every g*-closed map is gη-closed map.
(viii) Every sg-closed map is gη-closed map.
(ix) Every αg-closed map is gη-closed map.
(x) Every gα-closed map is gη-closed map.

Proof. (i) Let f : (X, τ) → (Y, σ) be a closed map and V be a closed set in (X, τ), then f (V) is closed in (Y, σ) and hence gη-closed in (Y, σ). Thus f is gη-closed.
Proof of (ii) to (x) are similar to (i).

Remark. The converse of the above theorem need not be true as seen from the following example.

Example 3.4 (i). Let X = Y = {a, b, c, d}, τ = {X, φ, {a}, {b}, {a, b}, {a, b, c}} and σ = {Y, φ, {a},{b, d},{a, b, d}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = c, f (d) = d. Then the function is gη-closed but not closed, semi-closed, r-closed, α-closed, g-closed, g* closed, gα-closed, αg-closed as the image of closed set{d} in X is {d} which is not closed, semi-closed, r-closed, α-closed, g-closed, g* closed, gα-closed, αg-closed in Y.
(ii). Let X = Y = {a, b, c, d}, τ = {X, φ, {a}, {b}, {a, b}, {a, b, c}} and σ = {Y, φ, {b},{c, d},{b, c, d}}.Define f: X →Y as f (a) = a, f (b) = c, f (c) = b, f (d) = d. Then the function is gη-closed but not sg-closed, η-closed as the image of closed set {a, c, d} in X is {a, b, d} which is not sg-closed, η-closed Y.

Remark. The concept of rg-closed map, gαr-closed map, gpr-closed map and gη-closed map are independent.

Example 3.5 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}, {b}, {a, b}}. Define f : X →Y as f (a) = b, f (b) = c, f (c) = a. Here f is gη-closed map. But f is not rg-closed map and gαr-closed map, gpr closed map. Since for closed set {a, b} in X, f ({a, b}) = {b, c}is not rg-closed and gαr-closed, gpr closed in Y.

Example 3.6 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}}. Define f : X →Y as f (a) = c, f (b) = a, f (c) = b. Here f is rg-closed map, gpr-closed map, gαr-closed map. But f is not gη-closed map. Since for the closed set {c} in X, f ({c}) = {c}is not gη-closed in Y.

Theorem 3.7 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a gη-closed map then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is gη-closed.

Proof. Let V be a closed set in (X, τ). Then f (V) is a closed set in (Y, σ). Hence $g(f(V)) = (g \circ f)(V)$ is gη-closed set in (Z, η). Therefore, $g \circ f$ is a gη-closed map.

Remark. The composition of two gη-closed maps need not be gη-closed map as seen from the following example.

Example 3.8 Let X = Y = Z = {a,b,c, d} with $\tau = \{X, \varphi, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\sigma = \{Y, \varphi, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\mu = \{Z, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ as f(a) = b, f(b) = a, f(c) = d, f(d) = c and $g : (Y, \sigma) \rightarrow (Z, \mu)$ be defined by g(a) = b, g(b) = c, g(c) = a, g(d) = d. Then the functions f and g are gη-closed maps but their composition $g \circ f : (X, \tau) \rightarrow (Z, \mu)$ is not gη-closed map, since for the closed set {a, b, d} in (X, τ), $(g \circ f)\{a, b, d\} = \{a, b, c\}$ is not gη-closed in (Z, μ).

Theorem 3.9 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \mu)$ be two mappings such that their composition $g \circ f : (X, \tau) \rightarrow (Z, \mu)$ be a gη-closed mapping. Then the following statements are true:

(i). if f is continuous and surjective then g is gη-closed.

(ii). If g is gη-irresolute, injective then f is gη-closed.

Proof. (i). Let A be a closed set in (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is gη-closed, $(g \circ f)(f^{-1}(A)) = g(A)$ is a gη-closed in (Z, η), since f is surjective. Therefore, g is a gη-closed map.
(ii). Let A be a closed set in (X, τ). Since $g \circ f$ is gη-closed, then $(g \circ f)(A)$ is gη-closed in (Z, η). Since g is gη-irresolute, then $g^{-1}[(g \circ f)(A)] = f(A)$ is gη-closed in (Y, σ), since g is injective. Thus, f is a gη-closed map.

Theorem 3.10 Let (X, τ) and (Y, σ) be any topological spaces, Then if :

(i). $f : (X, \tau) \rightarrow (Y, \sigma)$ is gη-closed and A is a closed subset of (X, τ) then $f_A: (A, \tau_A) \rightarrow (Y, \sigma)$ is gη-closed.

(ii). $f : (X, \tau) \rightarrow (Y, \sigma)$ is gη-closed and $A = f^{-1}(B)$, for some closed set B of (Y, σ), then $f_A: (A, \tau_A) \rightarrow (Y, \sigma)$ is gη-closed.

Proof. (i). Let B be a closed set of (A, τ_A). Then $B = A \cap F$ for some closed set F of (X, τ) and so B is closed in (X, τ). Since f is gη-closed, then f (B) is gη-closed in (Y, σ). But $f(B) = f_A(B)$ and therefore f_A is a gη-closed map.
(ii). Let F be a closed set of (A, τ_A). Then $F = A \cap H$ for some closed set H of (X, τ). Now $f_A(F) = f(F) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since f is gη-closed, f (H) is gη-closed in (Y, σ) and so $B \cap f(H)$ is gη-closed in (Y, σ). Therefore, f_A is a gη-closed map.

Theorem 3.11 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is gη-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there is a gη-open set V of (Y, σ) such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Suppose f is gη-closed. Let $S \subseteq Y$ and U be an open set of (X, τ) such that $f^{-1}(S) \subseteq U$. Now X – U is closed set in (X, τ). Since f is gη-closed, f(X-U) is an gη-closed set in (Y,σ). Then V = Y – f(X – U) is gη-open set in (Y, σ). $f^{-1}(S) \subseteq U$ implies $S \subseteq V$ and $f^{-1}(V) = X - f^{-1}(f(X - U)) \subset X - (X - U) = U$, ie, $f^{-1}(V) \subseteq U$.
Conversely, let F be a closed set of (X,τ). Then $f^{-1}(f(F)^c) \subset F^c$ is an open set in (X,τ). By hypothesis, there exists an gη-open set V in (Y,σ) such that $f(F)^c \subseteq V$ and $f^{-1}(V) \subseteq F^c$ and so $F \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(F) \subseteq f((f^{-1}(V))^c) \subseteq V^c$ which implies $f(F) \subseteq V^c$. Since V^c is gη-closed, f(F) is gη-closed. That is f(F) is gη-closed in (Y,σ). Therefore, f is gη-closed map.

4. gη open maps

Definition 4.1 A map f : (X, τ) → (Y, σ) is said to be a gη-open map if the image of every open set in (X, τ) is gη-open in (Y, σ).

Example 4.2 Let X = Y = {a, b, c, d}, τ = {X, φ, {b}, {c, d}, {b, c, d}} and σ = {Y, φ, {a}, {b}, {a, b},{a, b, c}}. Define f : X →Y as f (a) = a, f (b) = c, f (c) = b, f (d) = d. Then f ({b}) = {c}, f ({c, d}) = {b, d}, f ({b, c, d}) = {b, c, d}. Therefore, f is gη-open map. Since the image of every open set in X is gη-open in Y.

Theorem4.3 Let (X ,τ) and (Y, σ) be a topological spaces. Then for a mapping f : (X , τ) → (Y, σ). The following results are true.

(i) Every open map is gη-open map.
(ii) Every semi-open map is gη-open map.
(iii) Every α-open map is gη-open map.
(iv) Every r-open map is gη-open map.
(v) Every η-open map is gη-open map.
(vi) Every g-open map is gη-open map.
(vii) Every g*- open map is gη-open map.
(viii) Every sg-open map is gη-open map.
(ix) Every αg-open map is gη-open map.
(x) Every gα-open map is gη-open map.

Proof. (i). Let f : (X, τ) → (Y, σ) be a open map and V be an open set in (X, τ), then f (V) is open in (Y, σ) and hence gη-open in (Y, σ). Thus f is gη-open.
Proof of (ii) to (x) are similar to (i).

Remark. The converse of the above theorem need not be true as may be seen by the following example.

Example 4.4 (i). Let X = Y = {a, b, c}, τ = {X, φ, {a}, {b}, {a, b}} and σ = {Y, φ,{a}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c. Then the function is gη-open but notsemi-open, sg-open, η-open as the image of open set {a} in X is {b} which is not semi-open, sg-open, η-open in Y.
(ii). Let X = Y = {a, b, c, d}, τ = {X, φ, {b}, {c, d}, {b, c, d}} and σ = {Y, φ, {a},{b},{a, b}, {a, b, c}}. Define f : X →Y as f (a) = a, f (b) = c, f (c) = b, f (d) = d. Then the function is gη-open but not open, α-open, r-open,g-open, g*-open, gα -open, αg-open as the image of open set {b, c, d} in X is {b, c, d} which is not open, α-open, r-open,g-open, g*-open, gα-open, αg-open in Y.

Remark The concept of rg-open map, gαr-open map, gpr-open map and gη-open map are independent.

Example 4.5 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c. Here f is rg-open map, gαr-open map, gpr-open map. But f is not gη-open map. Since for the open set {a} in X, f({a}) = {c}is not gη-open in Y.

Example 4.6 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {b, c}} and σ = {Y, φ, {a}, {c}, {a, c}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = c. Here f is gη-open map. But f is not rg-open map, gαr-open map, gpr-open map. Since for the open set {b, c} in X,f({b, c}) = {b, c}is not rg-open, gαr-open, gpr-open in Y.

Remark. The composition of two gη-open map need not be gη-open map as seen from the following example.

Example 4.7 Let X = Y = Z = {a,b,c} with τ = {X, φ, {a}, {c}, {a, c}}, σ = {Y, φ,{b, c}} and μ= {Z, φ, {a}, {b}, {a, b}}. Define f : (X, τ) → (Y, σ) as f(a) = b, f(b) = a, f(c) = c and g : (Y, σ)

$\rightarrow$ (Z,μ) as g(a) = b, g(b) = c, g(c) = a. Then the functions f and g are gη-open map but their composition $g \circ f$: (X, τ) → (Z, μ) is not gη-open map, since the open set {a} in (X, τ), $(g \circ f)\{a\}$ = {c} is not gη-open in (Z, μ).

Theorem 4.8 For any bijection f : (X, τ) → (Y, σ), the following statements are equivalent.
(i). f^{-1} : (Y, σ) → (X, τ) is gη-continuous
(ii). f is a gη-open map and
(iii). f is a gη-closed map.

Proof. (i)⇒ (ii). Let U be an open set of (X, τ). By assumption, $(f^{-1})^{-1}$ (U) = f (U) is gη-open in (Y, σ) and so f is a gη-open map.
(ii) ⇒ (iii). Let V be a closed set of (X, τ). Then V^c is open in (X, τ). By assumption f (V^c) = $(f(V))^c$ is gη-open in (Y, σ) and therefore f (V) is gη-closed in (Y, σ). Hence f is a gη-closed map.
(iii) ⇒ (i) Let V be a closed set of (X, τ). By assumption f (V) is gη-closed in (Y, σ). But f (V) = $(f^{-1})^{-1}$ (V) and therefore f^{-1} is gη-continuous on (Y, σ).

Theorem 4.9 Let f : (X, τ) → (Y, σ) be mapping. If f is a gη-open mapping, then for each x∊X and for each neighborhood U of x in (X, τ), there exists a gη-neighborhood W of f(x) in (Y, σ) such that W⊂ f (U).

Proof. Let x∊X and U be an arbitrary neighborhood of x. Then there exists an open set V in (X, τ) such that x∊V ⊆ U. By assumption, f (V) is a gη-open set in (Y, σ). Further, f(x)∊f(V) ⊆ f(U), clearly f(U) is a gη-neighborhood of f(x)in (Y, σ) and so the theorem holds, by taking W = f(V).

Theorem 4.10 Let X, Y and Z be topological spaces.
(i). If f : X → Y is an open map and g : Y → Z is a gη-open map, then $g \circ f$: X → Z is a gη-open map.
(ii).If f : X → Y and g : Y → Z are open maps, then $g \circ f$: X → Z is a gη-open map.
(iii). If f : X → Y is an open map and g : Y → Z is anη-open map, then $g \circ f$: X → Z is a gη-open map.

Proof. (i). Let U be an open set in X. Since f is an open map, f(U) is open in Y. Then g(f(U)) = $(g \circ f)$(U) is a gη-open set in Z. Therefore, $g \circ f$ is a gη-open map.
(ii). Let U be an open set in X. Since f is an open map, f(U) is open in Y. Also, since g is an open map, g(f(U)) isopen in Z. That is, $(g \circ f)$(U) is an open set in Z. And every open set is gη-open, $(g \circ f)$(U) is a gη-open set in Z. Therefore, $g \circ f$ is a gη-open map.
(iii). Let U be an open set in X. Since f is an open map, f(U) is open in Y. Then g(f(U)) is a nη-open set in Z. That is,$(g \circ f)$(U) is a nη-open set in Z. As every η-open set is gη-open,$(g \circ f)$(U) is a gη-open set in Z. Hence $g \circ f$ is a gη-open map.

Theorem 4.11 A map f:(X, τ) → (Y, σ) is gη-open if and only if for any subset S of (Y, σ) and any closed set containing f^{-1}(S), there exists a gη-closed set K of (Y, σ) containing S such that f^{-1}(K) ⊆ F.

Proof. Suppose f is a gη-open map. Let S ⊆ Y and F be a closed set of (X, τ), such that f^{-1}(S) ⊆ F. Now X-F is an open set in (X, τ). Since f is gη-open map, f (X - F) is gη-open set in (Y, σ). Then K = Y- f (X - F) is a gη-closed set in (Y, σ). Note that f^{-1}(S) ⊆ F implies S ⊆ K and f^{-1}(K) = X – f(f^{-1}(X – F))⊆ X - (X – F) = F. That is f^{-1}(K) ⊆ F. For the converse let U be an open set of (X, τ), Then f^{-1}($(f(U))^c$) ⊆ U^c and U^c is a closed set in (X, τ). By hypothesis, there exists a gη-closed set K of (Y, σ) such that $(f(U))^c$ ⊆ K and f^{-1}(K) ⊆ U^c and so U⊆ $(f^{-1}(K))^c$. Hence K^c ⊆ f (U) ⊆f $((f^{-1}(K))^c$ which implies f (U) = K^c. Since K^c is a gη-open, f (U) is gη-open in (Y, σ) and therefore f is gη-open map.

5. gη-Homeomorphism

Definition 5.1 A bijection f:(X, τ) → (Y, σ)is called a η-homeomorphism if f is both η-continuous map and η-open map. That is, both f and $f^{-1}$are η-continuous map.

Definition 5.2 A bijection f:(X, τ) → (Y, σ)is called a gη-homeomorphism if f is both gη-continuous map and gη-open map. That is, both f and $f^{-1}$are gη-continuous map.

Example 5.3 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {b, c}} and σ = {Y, φ, {a},{c},{a, c}}. Definef : X →Y as f (a) = c, f (b) = b, f (c) = a. Here the sets {b}, {a, b}, {b, c} are closed in Y. Then f^{-1} ({b}) = {b}, f^{-1} ({a, b}) = {b, c}, f^{-1} ({b, c}) = {a, b} are gη-closed in X. Therefore, f is gη-continuous. And the sets {a}, {b, c} are open in X. Then f (a) = c, f ({b, c}) = {a, b} are gη-open in Y. Therefore f is open map. Hence, f is gη-homeomorphism.

Theorem 5.4

(i) Every homeomorphism is gη-homeomorphism.
(ii) Every semi-homeomorphism is gη-homeomorphism.
(iii) Every α-homeomorphism is gη-homeomorphism.
(iv) Every r-homeomorphism is gη-homeomorphism.
(v) Every η-homeomorphism is gη-homeomorphism map.
(vi)Every g-homeomorphism is gη-homeomorphism map.
(vii) Every g*-homeomorphism is gη-homeomorphism.
(viii) Every sg-homeomorphism is gη-homeomorphism.
(ix) Every αg-homeomorphism is gη-homeomorphism.
(x) Every gα-homeomorphism is gη-homeomorphism.

Proof. (i). Let f : (X, τ) → (Y, σ) be a homeomorphism. Then f and f^{-1} are continuous and f is bijection. Since every continuous function is gη-continuous, f and f^{-1} are gη-continuous. Hence f is gη-homeomorphism.
Proof of (ii) to (x) are similar to (i).

Remark. The converse of the above theorem need not be true as may be seen by the following example.

Example 5.4. (i). Let X = Y = {a, b, c}, τ = {X, φ, {a}} and σ = {Y, φ, {b, c}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c. Then this function is gη-homeomorphism. But f^{-1} ({a}) = {b} is not closed X. Here the set {a} is closed in Y. Therefore, f is not continuous. Hence f is not homeomorphism.
(ii). Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}, {b}, {a, b}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = c. Then this function is gη-homeomorphism. But f({a, c}) = {a, c} is not r-open Y. Here the set {a, c} is open in X. Therefore, f is not r-open map. Hence f is not r-homeomorphism.
(iii). Let X = Y = {a, b, c}, τ = {X, φ, {b, c}} and σ = {Y, φ, {a}, {c}, {a, c}}. Define f : X →Y as f (a) = c, f (b) = b, f (c) = a. Then this function is gη-homeomorphism. Here the set f({b, c}) = {a, b} is not g-open, g*-open, α-open, αg-open, gα-open in Y. Here the set {b, c} is open in X. Therefore, f is not g-open, g*-open, α-open, αg-open, gα-open map. Hence f in notg-homeomorphism, g*-homeomorphism,α-homeomorphism, αg-homeomorphism, gα-homeomorphism.
(iv). Let X = Y = {a, b, c, d}, τ = {X, φ, {c}, {a, b}, {a, b, c}} and σ = {Y, φ, {a}, {b}, {a, b}, {a, b, c}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c, f (d) = d. Then the function is gη-homeomorphism. Here the set f({c}) = {c} is not semi-open, η-open, sg-open in Y. Here the set {c} is open in X. Therefore, f is not semi-homeomorphism, η-homeomorphism, sg-homeomorphism.

Remark. The concept of rg-homeomorphism, gαr-homeomorphism and gη-homeomorphism are independent.

Example 5.5 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}\}$ and $\sigma = \{Y, \varphi, \{b, c\}\}$. Define $f : X \to Y$ as $f(a) = a$, $f(b) = c$, $f(c) = b$. Clearly f is rg-continuous, gpr-continuous, gαr-continuous. Then $f^{-1}(\{a\}) = \{a\}$ is not gη-closed in X. Therefore, f is not gη-continuous. Hence f is not gη-homeomorphism.

Example 5.6 Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\sigma = \{Y, \varphi, \{c\}, \{a, b\}, \{a, b, c\}\}$. Define $f : X \to Y$ as $f(a) = a$, $f(b) = b$, $f(c) = c$. Clearly f is gη-continuous. Then $f^{-1}(\{c, d\}) = \{c, d\}$ is not rg-closed, gpr-closed, gαr-closed in X. Therefore f is not rg-continuous, gpr-continuous, gαr-continuous. Hence f is not rg-homeomorphism, gpr-homeomorphism, gαr-homeomorphism.

Remark. The composition of two gη-homeomorphism need not be gη-homeomorphism as seen from the following example.

Example 5.7 Let $X = Y = Z = \{a,b,c\}$ with $\tau = \{X, \varphi, \{b, c\}\}$, $\sigma = \{Y, \varphi, \{a\}\}$ and $\mu = \{Z,\varphi,\{a\}, \{b, c\}\}$.Define $f : (X, \tau) \to (Y, \sigma)$ as $f(a) = b$, $f(b) = c$, $f(c) = a$ and $g : (Y, \sigma) \to (Z,\mu)$ as $g(a) = b$, $g(b) = a$, $g(c) = c$. Then the functions f and g are gη-continuous but their composition $g \circ f : (X, \tau) \to (Z, \mu)$ is not gη-continuous, since for the closed set $\{b, c\}$ in (Z, μ), $(g \circ f)^{-1}\{b, c\} = \{b, c\}$ is not gη-closed in (X, τ).

Theorem 5.8 Let $f : (X, \tau) \to (Y, \sigma)$be a bijective and gη-continuous map. Then the following statements are equivalent:

(i) f is gη-open map
(ii) f is gη-homeomorphism
(iii) f is gη-closed map

Proof. (i)⇒(ii) Let F be a closed set in (X, τ). Then $\{X – F\}$ is open in (X, τ). Since f is gη-open, then $f(X - F)$ is gη-open in (Y, σ). This implies $Y - f(F)$ is gη-open in (Y, σ). That is, $f(F)$ is gη-closed in (Y, σ). Thus f is gη-closed. Further $(f^{-1})^{-1}(F) = f(F)$ is gη-closed in (Y, σ). Thus $f^{-1}$is gη-continuous. By assumption f is gη-continuous and bijective. Hence f is gη-homeomorphism.
(ii)⇒(iii) Suppose f is an gη-homeomorphism. Then f is bijective, f and f^{-1} are gη-continuous. Let f be a closed set in (X, τ). Since f^{-1} is gη-continuous. Then $(f^{-1})^{-1}(F) = f(F)$ is gη-closed in (Y, σ). Thus f is gη-closed.
(iii)⇒(i) Let f be a gη-closed map. Let V be an open in X. Then $X - V$ is a closed in (X, τ). Since f is gη-closed, $f(X - V)$ is gη-closed in (Y, σ). This implies $Y-f(V)$ is gη-closed in (Y, σ). Therefore, $f(V)$ is gη-open in (Y, σ).

Definition 5.9 A bijection $f:(X, \tau) \to (Y, \sigma)$is called a gη*-homeomorphism if both f and $f^{-1}$are gη-irresoulte.

Theorem 5.10 Every gη*-homeomorphism is gη-homeomorphism but not conversely.

Proof. Let $f :(X,\tau)\to(Y,\sigma)$ be an gη*-homeomorphism. Then f and f^{-1} are gη-irresolute and f is bijection. Also f and f^{-1} are gη-continuous. Therefore f is gη-homeomorphism.
The converse of the above theorem need not be true as seen from the following example.

Example 5.11 Let $X = Y = \{a,b,c\}$, $\tau = \{X, \varphi,\{a\}\}$ and $\sigma = \{Y,\varphi, \{b, c\}\}$. Define $f : X \to Y$ as $f(a) = b$, $f(b) = a$, $f(c) = c$. Here the sets $\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ are gη-closed in Y. Then $f^{-1}(\{c\}) = \{c\}$,$f^{-1}(\{a, c\}) = \{b, c\}$ are not gη-closed in X. Therefore, f is not gη-irresolute. But $f^{-1}(\{b, c\}) = \{a, c\}$ is gη-closed in X. Hence f is gη-continuous

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