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PRIME CORDIAL AND SIGNED PRODUCT CORDIAL LABELING ON IDENTITY GRAPH

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Abstract The following scientific paper deals with the exploration of the prime cordial labeling. Further, we investigate the signed product cordial, square difference and analytic mean labeling on identity graph

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1. Introduction

Graph labelling is nothing but allocating integers to the vertices or edges or both subject to specific conditions. It was first promulgated in the later days of 1960's. Though much has been said and done on the graph labelling in the paramount five decades, every one base their findings to the theory articulated by Rosa [7] in 1967. In the meantime something noteworthy sprang up in 1980 with the advent of I. Cahit [2] and he pioneered the cordial labelling of graphs. However, over the past five decades in excess of two thousand papers have spawned a bewildering array of graph labelling methods. A chain of survey on graph labelling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatorics. A vast amount of literature is made accessible on varied types graph labelling and more than thousand research papers have been published during the past three decades. Indeed, all these papers focus on particular classes of graphs and methods, and feature ad hoc arguments.

Labelled graphs opened up new vistas for a wide array of applications like coding theory, rudiments of radar functioning, study of celestial bodies, designing complex circuits, network communication, handling of countless data base, clandestine sharing schemes, constraint programming designs for the finite domains and X-ray crystallography. As H. Wang, B. Yao and M. Yao [11] expounded “Graph labelling is used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards and configuring resistor networks”, reinstate the previous claim and its broad use in the modern scientific advancements.

2. Preliminaries

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be $(\boldsymbol{p}, \boldsymbol{q})$ graph. In this paper we considered only simple and undirected graphs. The number of vertices and the number of edges in a graph $\boldsymbol{G}$ denoted by $|\boldsymbol{V}|$ and $|\boldsymbol{E}|$ respectively. In 1960, Ian C.Ross and Frank Harary [4] have first introduced by the notion of Square of a tree.

Definition 2.1 [4] For any graph $\boldsymbol{G}$, its square graph $\boldsymbol{G}^{\mathbf{2}}$ has the same vertex set $\boldsymbol{G}$, with two vertices adjacent in $\boldsymbol{G}^{\mathbf{2}}$ whenever they are at distance 1 or 2 in $\boldsymbol{G}$.

Definition 2.2 [2] Let $\boldsymbol{f}$ be a function from the vertices of $\boldsymbol{G}$ to $\{\mathbf{0}, \mathbf{1}\}$ and for each edge $\boldsymbol{xy}$ assign the label $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})|$. Call $\boldsymbol{f}$ a *cordial labeling* of $\boldsymbol{G}$ if the number of vertices labeled 0 and the number of vertices labeled 1differ by atmost 1, and the number of edges labeled 0 and the number of edges labeled 1differ by atmost 1.

In 2005, Sundaram, Ponraj and Somasundaram [8] have first introduced by the notion of prime cordial labeling.

Definition 2.3 [8] A *prime cordial labeling* of a graph $\boldsymbol{G}$ with vertex set $\boldsymbol{V}(\boldsymbol{G})$ is a bijection $\boldsymbol{f}$ from $\boldsymbol{V}(\boldsymbol{G})$ to $\{\mathbf{1}, \mathbf{2}, \dots, |\boldsymbol{v}|\}$ such that if each $\boldsymbol{uv}$ is assigned the label $\mathbf{1}$ if $\boldsymbol{gcd}(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{f}(\boldsymbol{v})) = \mathbf{1}$ and $\mathbf{0}$ if $\boldsymbol{gcd}(\boldsymbol{f}(\boldsymbol{u}), \boldsymbol{f}(\boldsymbol{v})) > \mathbf{1}$, the number of edges labeled number with $\mathbf{0}$ and the number of edges labeled with $\mathbf{1}$ differ by atmost $\mathbf{1}$.

In 2009 W.B. Vasantha Kandasamy and Florentin Smarandache [10] introduced the new graph namely Identity graph. We motivated by the construction of identity graph from the algebraic structure and also proved some varieties of cordiality for this graph.

Definition 2.4 [10] Let $\boldsymbol{\Gamma}$ be a group and a graph $\boldsymbol{G}$ is said to be $\boldsymbol{identity\ graph}$ if the vertex set $\boldsymbol{V}(\boldsymbol{G}) = \boldsymbol{\Gamma}$ and the edge set $\boldsymbol{E}(\boldsymbol{G}) = \{(\boldsymbol{x}, \boldsymbol{y}) \cup (\boldsymbol{x}, \boldsymbol{e}) | \boldsymbol{x} * \boldsymbol{y} = \boldsymbol{e}, \boldsymbol{x} \neq \boldsymbol{e}, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Gamma}\}$ where $\boldsymbol{e}$ is the identity element.

In 2011, Jayapal Baskar Babujee and Shobana Loganathan[5] introduced signed product cordial labeling.

Definition 2.5 [5] A vertex labeling of a graph $\boldsymbol{G}$, define a function $\boldsymbol{f}: \boldsymbol{V}(\boldsymbol{G}) \rightarrow \{-\mathbf{1}, \mathbf{1}\}$ with induced edge labeling $\boldsymbol{f}^*: \boldsymbol{E}(\boldsymbol{G}) \rightarrow \{-\mathbf{1}, \mathbf{1}\}$ defined by $\boldsymbol{f}^*(\boldsymbol{uv}) = \boldsymbol{f}(\boldsymbol{u})\boldsymbol{f}(\boldsymbol{v})$ is called a signed product cordial labeling if $|\boldsymbol{v_f}(-\mathbf{1}) - \boldsymbol{v_f}(\mathbf{1})| \leq \mathbf{1}$ and $|\boldsymbol{e_f}(-\mathbf{1}) - \boldsymbol{e_f}(\mathbf{1})| \leq \mathbf{1}$, where $\boldsymbol{v_f}(-\mathbf{1})$is the number of vertices labeled with $-\mathbf{1}$, $\boldsymbol{v_f}(\mathbf{1})$ is the number of vertices labeled with $\mathbf{1}$, $\boldsymbol{e_f}(-\mathbf{1})$is the number of edges labeled with $-\mathbf{1}$ and $\boldsymbol{e_f}(\mathbf{1})$ is the number of edges labeled with $\mathbf{1}$. A graph $\boldsymbol{G}$ is *signed product cordial* if it admits signed product cordial labeling.

In 2012, V.Ajitha and et al.[1] introduced on square difference graphs.

Definition 2.6 [1] A graph $\boldsymbol{G}(\boldsymbol{p}, \boldsymbol{q})$ to be a square difference graph if there exist a bijection $\boldsymbol{f}$ from $\boldsymbol{V}(\boldsymbol{G})$ to $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{p} - \mathbf{1}\}$ such that the induced function $\boldsymbol{f}^*$ from $\boldsymbol{E}(\boldsymbol{G})$to the natural numbers given by $\boldsymbol{f}^*(\mathbf{uv}) = |\mathbf{f}(\mathbf{u})^{\mathbf{2}} - \mathbf{f}(\mathbf{v})^{\mathbf{2}}|$ for every edge $\boldsymbol{uv}$ of $\boldsymbol{G}$ is a bijection. Such a function is called a *square difference* labeling of graph $\boldsymbol{G}$.

In 2014, T.Thangaraj and P.B.Sarasija[9] introduced analytic mean labeled graphs.

Definition 2.7 [9] A (p, q) graph $G(V, E)$ is said to be analytic mean graph if it is possible to label the vertices v in V with distinct from $1, 2, \dots, p-1$ in such a way that when each edge $e = uv$ is labeled with $f^*(e = uv) = \frac{|[f(u)]^2 - [f(v)]^2|}{2}$ if $|[f(u)]^2 - [f(v)]^2|$ is even and $\frac{|[f(u)]^2 - [f(v)]^2|+1}{2}$ if $|[f(u)]^2 - [f(v)]^2|$ is odd and the edge labels are distinct. In this case f is called an analytic mean labeling of G.

We already observe the following properties of identity graphs in the research paper "Some cordial labeling on identity Graphs" $Id(D_{2n})$ Contains $k-1$ triangles when $2n \equiv 0$(mod 4) and k triangles when $2n \not\equiv 0$ (mod 4) where k quotient. It contains $2k+3$ lines when $n \equiv 0, 2$ (mod 4) and $2k+1$ lines when $n \equiv 1, 3$ (mod 4). $Id(Z_n)$ Contains $\left\lfloor\frac{n}{2}\right\rfloor$ triangles when n is odd, and $\frac{n}{2} - 1$ triangles and a line when n is even [6].

3. Main Results

In this section we find the existence of signed product cordial, prime cordial, square difference and analytic mean labeling for identity graphs.

Theorem 3.1 Let Γ be a finite group of order n and the identity graph $Id(\Gamma)$ where $\Gamma = D_{2n}$, $n \geq 3$ admits Signed product cordial labeling.

Proof. Note that the star $K_{1,2n-1}$ is the spanning subgraph of the identity graph $Id(\Gamma)$ where $\Gamma = D_{2n}$, $n \geq 3$ and it contains 2n vertices. Now we can partition the vertex set $V(Id(\Gamma)) = \{v_1, v_2, v_3, \dots, v_{2n}\}$ into the four sets $V_1 = \{v_1, v_5, \dots, v_{2n-3}\}$, $V_2 = \{v_2, v_6, \dots, v_{2n-2}\}$, $V_3 = \{v_3, v_7, \dots, v_{2n-1}\}$, and $V_4 = \{v_4, v_8, \dots, v_{2n}\}$. Observe that $|V_i| = \lfloor\frac{n}{2}\rfloor$, where $1 \leq i \leq 4$. Fix v_{2n} as apex vertex in the graph $K_{1,2n-1}$, define a vertex labeling $l : V \rightarrow \{1, -1\}$ as $l(v_i) = (-1)^{i+1}$, $i = 1, 2, 3, \cdots, 2n$. This will lead an induced edge labeling for the star $K_{1,2n-1}$ such that the number of edges with label -1 namely $|l(e_{-1})| = n$ for all $n \geq 1$ and the number of edges with label 1 namely $|l(e_1)| = \begin{cases} n-1 & \textit{if n is odd} \\ n & \textit{if n is even.} \end{cases}$ Thus in either case we get $|l(e_1) - l(e_{-1})| \leq 1$. In other words, the signed product cordial labeling for $K_{1,2n-1}$ is obtained.

Observe that, all the vertices belong to V_1 and V_3 are labeled with 1 and those belong to V_2 and V_4 are labeled with -1. The identity graph $Id(\Gamma)$ together with a signed product cordial labeling is obtained by introducing edges between the vertex sets in the following manner. Introduce an edge between the vertices: v_2 and v_4(this will increase the edge label count $l(e_1)$ by one), v_1 and v_6 (this will increase the edge label count $l(e_{-1})$ by one), v_8 and v_{10} (this will increase the edge label count $l(e_1)$ again by one), v_5 **and** v_{14} (this will increase the edge label count $l(e_{-1})$ again by one), etc., This process will end if the number of triangles required to construct $Id(\Gamma)$ is then obtained. Hence, we may either end up with $|l(e_1) - l(e_{-1})| = 0$ if number of triangles is odd and $|l(e_1) - l(e_{-1})| = 1$ if number of triangles is even in $Id(\Gamma)$. Thus, the required edge conditions exist.

Theorem 3.2 Let Γ be a finite group of order n and the identity graph $Id(\Gamma)$ where $\Gamma = D_{2n}$, $n \geq 3$ admits Square difference labeling.

Proof. Define a labeling $l: V(Id(\Gamma)) \rightarrow \{0, 1, 2, \dots, p-1\}$. Consider the identity graph $Id(\Gamma)$, where $\Gamma = D_{2n}(n \geq 3)$ and k as the number of triangle in $Id(\Gamma)$. Assign label $l(v_p) = p-1$ for the apex vertex v_p and give the labeling of pendent vertices successively starting from 0 to $(2k-1)+i$ where $(i = 1 \; or \; 3)$. Then assign the label $(2k+i)$ and $(2k+i+1)$ for the degree 2 vertices respectively in a triangle and $(2k+i+2)$ and $(2k+i+3)$ respectively for the degree 2 vertices in the next triangle and so on.

Let $\alpha_n = (p-1)^2 - n^2$, where $n < (p-1)$ and $\alpha_n's$ are distinct for every n. Further $m^2 - (m-1)^2$ are distinct and they are different from α_n's. Since $Id(\Gamma)$ contains $k-1$ triangles when $2n \equiv 0\ (mod\ 4)$, k triangles when $2n \not\equiv 0\ (mod\ 4)$ and $i + 2k$ where $i = 1$ or 3, number of lines (pendant vertices), the above said labeling of $Id(\Gamma)$ will turn out to be the square labeling and hence the proof.

Theorem 3.3 Let Γ be a finite group of order n and the identity graph $Id(\Gamma)$ where $\Gamma = Z_n$, $n \geq 3$ admits square difference labeling.

Proof. We prove this theorem by two cases. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of $Id(Z_n)$ and v_n be the apex vertex.
Case I. If n is odd, $Id(Z_n)$ Contains $\left\lfloor\frac{n}{2}\right\rfloor$ triangles. Define a labeling $l: V\big(Id(Z_n)\big) \to \{0, 1, \dots, n-1\}$. Label the vertices consecutively such that $l(v_i) = n - 1$ where $1 \leq i \leq n$.
Applying the definition, the induced edge labels are distinct.
Case II. If n is even, $Id(Z_n)$ Contains $\frac{n}{2} - 1$ triangles and a line. Define a labeling $l: V\big(Id(Z_n)\big) \to \{0, 1, \dots, n-1\}$. Starting from the pendent vertex, Label all the vertices consecutively such that $l(v_i) = n - 1$ where $1 \leq i \leq n$. Observe that the induced edge labels are distinct. In both the cases the graph the edge labels of $Id(Z_n)$ are distinct. Hence $Id(Z_n)$ admits Square difference labeling.

Result. Let $G = Id(\Gamma)$ also Γ be a group then $G^2 \cong K_n$.Since, $d(u, v) \leq 2$ for all $u, v \in Id(D_{2n})$, the above relation follows.

Theorem 3.4 Let Γ be a finite group of order n and the identity graph $Id(\Gamma)$ where $\Gamma = D_{2n}$, $n \geq 3$ admits Prime cordial labeling.

Proof. Consider the star graph $k_{1,2n-1}$, with $2n$ vertices $\{v_1, v_2, \cdots, v_{2n}\}$ with v_{2n} as apex vertex. Partition the vertex set $V = \{v_1, v_2, v_3, \dots, v_{2n}\}$ into the sets $V_1 = \{v_1, v_3, \dots, v_{2n-1}\}$ and $V_2 = \{v_2, v_4, \dots, v_{2n-2}\}$. Define a vertex labeling $l : V \to \{1, 2, .., |V|\}$ as follows: $l(v_{2n}) = 2$. Assign labels for all pendent vertices in V_1 as $1, 3, 5 \dots, 2n-1$ and give labels for all pendant vertices in V_2 as $4, 6, 8, \dots, 2n$. Due to this vertex labeling, the induced edge labels counts are $l(e_1) = n$ and $l(e_0) = n - 1$. The identity graph $Id(\Gamma)$ together with a prime cordial labeling is obtained in the following way: First introduce an edge between the vertices v_2 and v_4 which belongs to V_2(this will increase the edge label count $l(e_0)$ by 1), introduce an edge between v_1 and v_3 in the set V_1(this will increase the edge label $l(e_1)$ by 1), again introduce an edge betweenv_6 and v_8 (this will increase the edge label count $l(e_0)$ by 1) and so on. By continuing this process we may either have$|l(e_1) - l(e_0)| = 0$, if number of triangles is odd in $Id(\Gamma)$ and $|l(e_1) - l(e_0)| = 1$ if number of triangles is even in $Id(\Gamma)$, sincethe number of triangles appeared in $Id(\Gamma)$ is either $\frac{n-1}{2}$ where $(n = 2k+1)$ or $\frac{n-2}{2}$ where$(n = 2k+2)$ and the choice of label for apex vertex as 2.

Theorem 3.5 Let Γ be a group. Then identity graph $Id(\Gamma)$ where $\Gamma = D_{2n}$ admits analytic mean labeling for $n \geq 3$.

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of $Id(D_{2n})$. Let v_n be the apex vertex.
Define a labeling $l: V\big(Id(D_{2n})\big) \to \{0, 1, \dots, n-1\}$. Starting from the pendent vertex, label all the vertices consecutively such that $l(v_i) = n - 1$ where $1 \leq i \leq n$. Observe that the induced edge labels are distinct. Hence $Id(D_{2n})$ admits analytic mean labeling.

References

1) V. Ajitha, K.L. Princy, V. Lokesha, and P.S. Ranjini, On square difference graphs, Math. Combin. Internat. Book Ser., 1(2012) 31-40.
2) I. Cahit, “Cordial graphs: A weaker version of graceful and harmonious Graphs”, *Ars Combinatoria*, Vol. 23(1987) 201-207.

3) J.A. Gallian, A Dynamic survey of graph labeling, *The Electronic journal of Combinatorics*, Vol 29 (2017) 1-298.
4) F. Harary and I.C. Ross, The square of a tree, *Bell. System. Tech.*, J.39 (1960) 641-647.
5) Jayapal Baskar Babujee and Shobana Loganathan, On signed product cordial labeling, *Journal of Applied Mathematics*, 2(2011) 1525-1530.
6) N. Mohamed Rilwan and A.Syed Hussain, Some cordial labeling on identity Graphs, Communicated to *Tamkang Journal of Mathematics*.
7) A. Rosa, On Certain Valuations of the vertices of a graph, Theory of graphs (*Internat. Symposium*, *Rome*, July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967) 349-355.
8) M. Sundaram, R. Ponraj, and S. Somasundaram , Prime cordial labeling of graphs, *J.Indian Acad*. Math., 27(2005) 373-390.
9) T. Tharmaraj and P.B. Sarasija, Analytic mean labeled graphs, International Journal of Mathematical Archive. (2014) 136-146.
10) W.B.Vasantha Kandasamy, Florentin Smarandache, *Groups as graphs*, (2009).
11) H. Wang, B. Yao, and M. Yao, Generalized edge-magic total labeling of models from researching networks, *Inform. Sci*., (2014) 460-467.