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STEPWISE EDGE IRREGULAR GRAPHS

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Abstract A simple graph $G(V,E)$ is stepwise edge irregular graph(SEI) if the difference between the edge degrees of any two adjacent edges in G is either zero or one and for some edge e in $E(G)$, the difference between the edge degrees of e and at least one of its neighbour edges is 1. In this work, we have given the proof for the existence of such SEI graphs and SEI trees. We have also examined some methods of construction of SEI graphs from some standard graphs and also a given SEI graph. We have got some results on SEI graphs and explored some of its basic properties. It is also exhibited that any graph can be imbedded as an induced subgraph in a SEI graph.

Keywords . edge degree, highly irregular graphs, stepwise irregular graphs, stepwise edge irregular graphs.

AMS subject classification : Primary: 05C12, Secondary: 03E72, 05C72.

1. Introduction

Throughout this paper we consider finite, simple connected graphs. Let G be a graph with n vertices and m edges. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v and is denoted by $d_G(v)$ or simply $d(v)$.

The concept of stepwise irregular graphs was introduced and studied by Ivan Gutman [2] in 2018. The purpose of this paper is to introduce a new class of irregular graphs based on distance property in edge sense. The concept of Stepwise edge irregular graphs is analogous to Stepwise irregular graph but considering the distance between the edges instead of vertices.

2. Preliminaries

We post some definitions for reference to get through the work exposed in this paper.

Definition 2.1 The degree of an edge e = (u,v) as the number of edges which have a common vertex with the edge e, (i.e) deg(e) = deg(u) + deg(v) −2[5].

Definition 2.2. The distance between two edges $e_1 = (u_1,v_1)$ and $e_2 = (u_2,v_2)$ is defined as $ed(e_1,e_2)= \min\{d(u_1,u_2),d(u_1,v_2),d(v_1,u_2),d(v_1,v_2)\}$. If $ed(e_1,e_2) = 0$, these edges are neighbour edges[4].

Definition 2.3 A graph G is said to be Stepwise irregular graph if the difference between the degrees of any two adjacent vertices is exactly one [3].

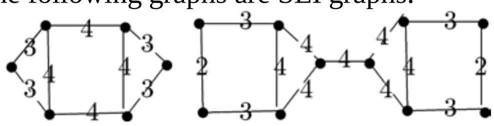
Definition 2.4 A graph G is said to be highly irregular graph if each of its vertices is adjacent only to the vertices with distinct degree [1].

3. Stepwise edge irregular graphs(SEI)

In this section, we put forward the definition of Stepwise edge irregular graphs and study some properties of these graphs.

Definition 3.1 A simple graph G(V,E) is Stepwise edge irregular graph(SEI) if (i)The difference between the edge degrees of any two adjacent edges in G is either zero or one (ii) For some edge e in E(G), the difference between the edge degrees of e and at least one of its neighbour is 1. That is (i) For any two edges e_i and e_j in E(G), $|ed(e_i) - ed(e_j)| = 0$ or 1, (ii)For some edge e in E(G), $|ed(e) - ed(e_i)| = 1$ for al least one edge $e_i \in N(e)$.

Example 3.2 The following graphs are SEI graphs.



G_1 G_2

Result 3.3 Any highly irregular graph cannot be a SEI.

Result 3.4 In any complete graph K_n , if we remove one edge from K_n the obtained graph is a SEI with edge degree sequence {2n −4,...,2n −4,2n −5,...,2n −5}, if we introduce one edge between any one of the vertex of two copies of K_n the obtained graph is also a SEI graph with edge degree sequence {2d −2,...2d −2,2d −3,...,2d −3,2d −4,...,2d −4}.

Result 3.5 We can attach an r-regular graph with K_{r+1} by an edge to get a SEI graph with edge degree sequence {2r−2,...,2r−2,2r−1,...,2r−1,2r,...,2r}.

Result 3.6 For any positive integer r ≥ 1, we can construct a SEI graph of order 2r and edge degree sequence {2r −1,...,2r −1,2r,...,2r} by joining 2 r-regular graphs at any one of its vertices.

Result 3.7 For any positive integer d ≥ 2, by joining the roots 2 stars' S_d and S_{d-1} with a common vertex v , we will get a SEI graph of order 2d + 2, maximum degree d+1 and edge degree sequence {d−1,...d−1,d,...d,d+1,...,d+ 1} .

Result 3.8 Let G be a SEI graph.We cannot find P_3 (say uvw) s.t d(u) = 1, d(v) = 2 and d(w) >4 in G as an induced subgraph of G.

Result 3.9 Let G be a SEI graph.We cannot find P_n with extreme vertices of degree ≥ 2 as an induced subgraph of G.

Result 3.10 We can make 3 copies of P_n to coincide at one of its end vertices, the obtained graph is a SEI graph with edge degree sequence {3,3,..,3,2,2,...2,1, 1,...,1}.

Theorem 3.11 A graph G is a SEI graph if and only if for any two adjacent edges uv and vw , either d(u) = d(w) or |d(u) −d(w)| = 1.

Proof. Let *G* be a SEI graph. Let *uv* and *vw* be any two adjacent edges. If possible $d(u) \neq d(w)$ and $d(u)-d(w) \neq 1 \implies d(u)+d(v) \neq d(v)+d(w)$ and $d \quad d \;) \neq d(v)+d(w)+1 \implies ed(uv) \neq ed(vw)$ and $ed(uv) \neq ed(vw)+1$,(*u*)+ (*v* which is a contradiction.

Conversely suppose that , *d*(*u*) = *d*(*w*) or |*d*(*u*) −*d*(*w*)| = 1, for any two adjacent edges *uv* and *vw*, then *ed*(*uv*) −*ed*(*vw*) = 0 or 1. Hence *G* is SEI graph. □

Theorem 3.12 For any given positive integer n, there exists a SEI graph of order n.

Proof. For *n* = 3, C_3 and for *n* = 4 and 5, P_4 and P_5 are SEI graphs respectively. Suppose *n* = 2*k*, *k* ≥ 3. The *n* vertices are partitioned into two partites sets say $\{u_1,u_2,...,u_k\}$ and $\{v_1,v_2,...,v_k\}$ which will be the vertex set for our desired graph *H* . For the edge set $E(H)=\{u_iv_i$ and u_iv_j , $1 \leq i \leq k-1$, $i+1 \leq j \leq k\}$. From the construction, the resulting graph is SEI graph of order *n* with edge degree sequence {2*k* −2,2*k* −3,2*k* −4}.

For *n* = 2*K* + 1, by joining the vertices of any one partite with a newly introduced vertex *w*, we will get a SEI graph of order *n* = 2*k* + 1, with edge degree sequence {2*k* −1,2*k* −2,2*k* −3}. □

Illustration: The following figure elucidates theorem 3.12 for n = 6 and n = 7

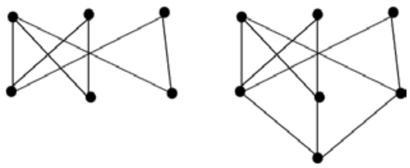


Figure 2

Theorem 3.13 For every positive integer d ≥ 2, there exists a SEI graph of order 2d and size $\frac{d^2+2}{2}$ with maximum degree d and α(G) = d and girth 4.

Proof. The require SEI graph *G* is constructed as follows. The 2*d* vertices are partitioned into 3 sets say *U*,*V* and *W* . The first set *U* consist of 2 vertices say *u* and *v*, the second set *V* consists of *d* vertices say $u_{11}, u_{12}, ..., u_{1\lceil \frac{d}{2} \rceil}$, $u_{21}, u_{22}, ..., u_{2\lceil \frac{d}{2} \rceil}$ and third set *W* consist of the vertices say $v_{11}, v_{12}, ..., v_{1\lfloor \frac{d}{2} \rfloor}$, $v_{21}, v_{22}, ..., v_{2\lfloor \frac{d}{2} \rfloor}$. The vertices in the sets *U* , *V* and *W* constitute the vertex set for the desired graph *G*. For the edge set $E(G)=\{uv, uu_{1j}, u_{2k}$, $1 \leq j \leq \lceil \frac{d}{2} \rceil,\ 1 \leq k \leq \lfloor \frac{d}{2} \rfloor,\ u_{1j}v_{1k}$ and $u_{2l}v_{1m},\ 1 \leq j \leq \lceil \frac{d}{2} \rceil,\ 1 \leq k \leq \lceil \frac{d}{2} \rceil - 1,\ 1 \leq l \leq \lfloor \frac{d}{2} \rfloor,\ 1 \leq m \leq \lfloor \frac{d}{2} \rfloor - 1\}$. From the construction the resulting graph is SEI graph. Moreover, *ed*(*uv*) = *d*,$ed(uu_{1j}) = d$, $ed(vu_{2k}) = d-1$ for$1 \leq j \leq \lceil \frac{d}{2} \rceil,\ 1 \leq k \leq \lfloor \frac{d}{2} \rfloor$ and $ed(u_{1j}v_{1k}) = d-1$, $ed(u_{2l}v_{2m}) = d-2$

for $1 \leq j \leq \lceil \frac{d}{2} \rceil,\ 1 \leq k \leq \lceil \frac{d}{2} \rceil - 1,\ 1 \leq l \leq \lfloor \frac{d}{2} \rfloor,\ 1 \leq m \leq \lfloor \frac{d}{2} \rfloor - 1\}$

$1 + \frac{d}{2} + \frac{d}{2} + 2\frac{d}{2}(\frac{d}{2} - 1) = \frac{d^2+2}{2}$. , □

Illustration: The following graph elucidates the theorem 3.13 for d = 4.

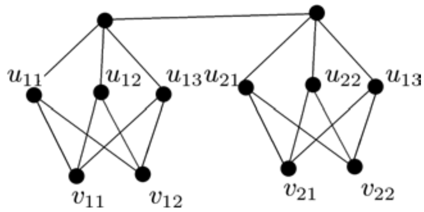


Figure 3

Theorem 3.14 For any integers $m,n \geq 1$, $K_{m,n}$ is an induced subgraph of SEI of order m + n + 2. For, if the vertices u_i and v_j , $1 \leq i \leq m$ and $1 \leq j \leq n$ are two partitions of $K_{m,n}$, then join u_i and v_j with u and v respectively. The resulting graph is a SEI graph of order m + n + 2.

Result 3.15 For any integer $r \geq 3$, $K_{r-2,r}$ is an induced subgraph of SEI graph of order 4r −4. For, take two copies G_1 and G_2 of $K_{r-2,r}$, join the vertices of one partite in G_1 which contains r vertices with the vertices of one partite in G_2 which contains r vertices of G_1 . The resulting graph is a SEI graph of order 4r −4.

Result 3.16 Let G be a SEI graph. Let u be a vertex of degree ≥ 4 which is adjacent to one(or more) vertices of degree 1, then N(u) is independent. For, if there are two vertices v and w in N(u) which are adjacent, then ed(uv) = ed(uw) = d(u), ed(vw) = 2 and ed(ux) = d(u) −1 which implies d(u) ≤ 3 which is a contradiction.

Result 3.17 Suppose G is a SEI graph. Let u be a vertex which is adjacent to a vertex w of degree 1 and if $ed(uu_i) = ed(u_iu_j)$ or $ed(u_iu_j) + 1$, $u_i \in N(u)$ and $u_j \in N(u)$, then we can introduce one more pendant vertex at u. The resulting graph is also a SEI graph.

Result 3.18 In a SEI graph, if uv is an edge with d(v) = 1, then we can attach one leaf of the star $K_{1,n-1}$, $K_{1,n}$ or $K_{1,n+1}$ at v , the resulting graph is also a SEI graph.

Result 3.19 In a SEI graph, let u be a vertex of degree d and $d(u) = d(u_i)$ or $d(u_i)-1$ where $u_i \in N(u)$. We can attach v by an edge with any one of the vertex of K_{d+1} or k_d .

Result 3.20 In a SEI G, for any two adjacent vertices u and v and if ed(uv) = d, then d(N(u)) = d(v) or d(v)±1 and d(N(v)) = d(u) or d(u)±1.

Result 3.21 In a SEI, let u be a pendant vertex at v , we can join the vertex u to the root of $K_{1,d(v)-1}$ or $K_{1,d(v)}$, we will get a SEI graph.

Result 3.22 In a SEI, let u be any vertex of degree m s.t $ed(uu_i) = ed(u_iu_j)+ 1$ and if $ed(uu_i)$ = d or d−1, we can join u to the root of $K_{1,d-m}$, the obtained graph is also a SEI graph.

Result 3.23 For any given positive integer d, there exists unicyclic, 2-cyclic and (n −2)-cyclic SEI graph of order 2d, d + 2 and 2d + 4 respectively.

Result 3.24 In a SEI, if there is a triangle (uvw), then not all the three edges of uvw is of distinct edge degree.

Theorem 3.25 In a SEI graph G, if there is a triangle (uvw), then the edge degrees of the three edges of uvw are even or 2 edges are of odd edge degree and 1 edge is of even edge degree.

Proof. Clearly at least 2 edges of the triangle *uvw* (say *uv* and *vw*) are of same edge degree say *d*. Then $d(u) = d(w)$ and $ed(uw) = d(u)+d(w)-2=2(d(u)-1)$ which is even.
$ed(uw) = 2(d(u)-1) = d \pm 1$ or d. Then $d(u) = \frac{d\pm1}{2} + 1$ or $\frac{d}{2}$.

If d is even, $d(u) = \frac{d}{2}$ and $ed(uw) = d$, which is even, hence all the three edges of uvw will have even edge degree. If d is odd, $d(u) = \frac{d\pm1}{2}$ and $ed(uw) = d\pm1$ which is even, then 2 edges are of odd degree and 1 is of even degree. □

Theorem 3.26 Let G be a SEI graph. Let e = uv be any pendant edge s.t d(u) = 1and $ed(e_i) = ed(e_j)$ or $ed(e_j)+1$, $\forall e_i \in N(e)$ and $e_j \in N(e_i)$. Then $G - \{u\}$ is also a SEI graph.

Proof. Let $e = uv$ be any edge in a SEI graph G such that $d(v) = m$ and $d(u) = 1$. Then $ed(e_i) = m-1$ or m where $\forall e_i \in N(e)$. The edges with edge degree $m-1$ cannot be adjacent to any new vertices other than $N(v)$ and $ed(e_i) = ed(e_j)$ or $ed(e_j) + 1$ which implies $ed(e_i) = m$ or $m-1$ in $G - \{v\}$, then $|ed(e_i) - ed(e_j)| = 0$ or 1.
□

Theorem 3.27 Let G be a SEI graph. The line graph of G is also SEI graph iff for any edge $e \in E(G)$, $|ed(e_i) - ed(e_j)| = 0$ or 1 $\forall e_i, e_j \in N(e)$.

Proof. Suppose G is a SEI graph and $L(G)$ is also a SEI graph. Let $e = uv$ be any edge in G. The edge e is a vertex in $L(G)$. Then $ed(e_ie) - ed(ee_j) = 0$ or 1 in $L(G) \Longrightarrow d(e_i) - d(e_j) = 0$ or 1 in $L(G) \Longrightarrow ed(e_i) - ed(e_j) = 0$ or 1 in G $\forall e_i, e_j \in N(e)$.

Conversely Suppose for any edge $e \in E(G)$, $|ed(e_i) - ed(e_j)| = 0$ or 1 $\forall e_i, e_j \in N(e)$. Let e_1e_2 and e_2e_3 be any two adjacent edges in $L(G)$. $ed(e_1e_2$ ed d d $) - (e_2e_3) = (e_1) - (e_3)$ in L G $( ) =$ ed $(e_1) -$ $ed(e_3)$ in G=0 or 1. Hence $L(G)$ is a SEI graph.

Theorem 3.28 Let G be a SEI graph. Then G^C is also a SEI graph if (i)d(u) = d(v) or d(v) + 1 $\forall uv \in E(G)$ and (ii)diam(G) = 2 $\forall uv \notin E(G)$.

Proof. Let G be a SEI graph. In G^c, let uv and vw be any two adjacent edges. Then $e^*(uv) - ed^*(vw)$ in $G^c = d^*(u) - d^*(w)$ in $G^c = d(u) - d(w)$ in
G. If u and w are adjacent in G^c, then u and w are not adjacent in G and at distance 2 which implies $d(u) - d(w) = 0$ or 1. If u and w are not adjacent in G^c, then u and w are adjacent in G, by (*i*) $ed^*(uv) - ed^*(vw) = 0$ or 1, for any two adjacent edges in G^c. Then G^c is also a SEI graph. □

Theorem 3.29 *Any path $P_n, n \geq 3$ is an induced subgraph of a proper SEI graph of order* $n + \lfloor \frac{n}{4} \rfloor$ *or* $n + \lfloor \frac{n}{4} \rfloor - 1$.

Proof. Let P_n be a path of order n and $n = 4k + i$, $1 \leq i \leq 3$. Let $V(G) = \{v_1, v_2, ..., v_n\}$. If $n = 4k + 2$ and $4k + 3$, the new pendant vertices are to be introduced at $4i + 1$, $1 \leq i \leq k-1$ and if $n = 4k$ or $4k + 1$, the new vertices are to be introduced at $4i$, $1 \leq i \leq k$. Thus we will get a SEI graph of order $n + k$ or $n + k - 1$ which has P_n as induced subgraph. □

Theorem 3.30 Every graph of order n is an induced subgraph of a proper SEI graph of order 4n and edge degree sequence $\{2n, 2n-1, 2n-1\}$.

Proof. Let G be a given graph of order n. Consider two copies of G as G and G^*. Let $V(G) = v_i$ and $V(G^*) = v_i^*$, $1 \leq i \leq n$. We construct a graph H with vertex set $V(H) = V(G) \cup V(G^*)$ and for the edge set, join $v_iv_j^*$if $v_iv_j \notin E(G)$. Take two copies of H say H_1 and H_2, join any one of the vertex v of H_1 with any one of the vertex w of H_2. The obtained graph is a proper SEI of order $4n$ and the edge sequence will be $\{2n, 2n-1, 2n-2\}$. Moreover $ed(vw) = 2n$, $ed(N(vw)) = 2n-1$ and $ed(e) = 2n-2$ $\forall e \in E(H) \backslash N([vw])$. □

Theorem 3.31 For any positive integer d ≥ 3, there exists a proper SEI tree T with maximum degree d and edge degree sequence {1,2,3,...,d}. Also in T , edge degree difference between the edge with maximum edge degree(say e) and any edge at odd(even) edge distance from e is even(odd).

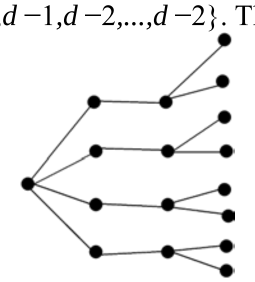
Proof. For any integer $d \geq 3$, we will prove by induction method. If $d = 3$, in P_3, by attaching one pendant vertex at one end vertex and two at the other end vertex and if $d = 4$, joining 2 copies of P_7 at its middle vertices, we will get the required trees. Assume that the result is true for d, that is there exist a SEI tree T with maximum degree d and edge degree sequence $\{1,2,...,d\}$. It is enough to prove the result is true for $d + 2$. For, let $e = uv$ be the edge with maximum edge degree.

If d is even, T_1 is a minimal proper SEI subtree of T which the edge e and the component $T_1 - uv$ contains u. we will get T_2 by attaching $\frac{d}{2}$ copies of T_1 at their v, and next we will get T_3 by attaching $\frac{d}{2}+1$ copies of T_2 at a common vertex x. Again we will get T_3 by attaching two copies of T_2 at their x by an edge e. By construction the resulting graph is a proper SEI tree with maximum degree $d + 2$ and edge degree sequence $\{1,2,3,..,d + 2\}$. Moreover, the edge degree difference between the edge with maximum edge degree and any edge at odd(even) edge distance is even(odd).

If d is odd, T_1 and T_2 are minimal proper SEI subtree of T with an edge e s.t $T_1 - uv$ and $T_2 - uv$ contains u and v respectively. We will get T_3, by attaching d$^{d}_{2}$e copies of T_1 at their u. Next we will get T_4, by attaching u of each $\lceil \frac{d}{2} \rceil$ copies of T_3 with a common vertex x. Similarly we will get T_5, by attaching $\lceil \frac{d}{2} \rceil - 1$ copies of T_2 at their v. Next we will get T_6, by attaching v of each $\lceil \frac{d}{2} \rceil$ copies of T_5 with a common vertex y. We will get a required graph by attaching x and y of T_4 and T_6. By our construction T is a proper SEI tree with maximum degree d and edge sequence $\{1,2,3,...,d,d+1,d+2\}$. Moreover, the edge degree difference between the edge with maximum edge degree and any edge at odd(even) edge distance is even(odd). □

Theorem 3.32 For any positive integer d, there exists a SEI tree of order $d^2 + 1$. Moreover maximum degree=maximum edge degree=d and edge degree sequence is $\{d,d,...,d,d-1,...,d-1,d-2,...,d-2\}$.

Proof. Suppose a positive integer d is given. Let u be a vertex of degree d which will be the root of the required SEI tree. Suppose u has d children say $\{u_1,u_2,...,u_d\}$, the new d vertices say $\{v_1,v_2,...,v_d\}$ are to be introduced and join the edges u_iv_i, $1 \leq i \leq d$. For each v_i, $1 \leq i \leq d$, new $d-2$ children w_{ij}, $1 \leq j \leq d-2$ are to be introduced and join the edges v_iw_{ij} for $1 \leq i \leq d$, $1 \leq j \leq d-2$. The obtained graph is a SEI tree of order d^2+1. Moreover maximum degree=maximum edge degree=d and edge degree sequence say $\{d,d,...,d,d-1,...,d-1,d-2,...,d-2\}$. The following graph is the example for $d = 4$.



□

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