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Fixed Point and Common Fixed Point Theorems in Complex Valued *b*-Metric Spaces

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Abstract. In this paper, we discuss the existence and uniqueness of fixed point and common fixed point theorems in complex valued *b*-metric space for a pair of mappings satisfying some rational contraction conditions which generalize and unify some well-known results in the literature.

Keywords. Complex valued b-metric space, Rational contraction, Common fixed point.

1. Introduction

The Banach contraction mapping principle [1] plays a vital role in fixed point theory. Bakhtin [2] introduced the concept of *b*-metric space which is the generalization of metric space. Czerwik [5] extended the Banach principle in *b*-metric space. Many researchers proved fixed point theorem on single valued and multi valued mapping in *b*-metric space [8]. Azam et al. [7] initiated a new space described as complex valued metric space. Several researchers studied various common fixed point theorems in complex valued metric space. Rao et al. [3] introduced complex valued *b*-metric space, continuously Mukheimer [6] and A. K. Dubey [4] verified the existence of some common fixed point theorems in complex valued *b*-mertic space. In this paper, we continue the study of fixed point theorems in complex valued *b*-mertic space.

2. Preliminaries

[4] Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:
$z_1 \preceq z_2$ if and only if $\operatorname{Re}(z_1) \le \operatorname{Re}(z_2), \operatorname{Im}(z_1) \le \operatorname{Im}(z_2)$. Thus $z_1 \preceq z_2$ if one of the following holds:
(1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;

(2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$;
(3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$;
(4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$;

We will write $z_1 \underset{\neq}{\prec} z_2$ if $z_1 \neq z_2$ and one of (2), (3), and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied. It follows that

(i) $0 \preceq z_1 \underset{\neq}{\prec} z_2$ implies $|z_1| < |z_2|$;
(ii) $z_1 \preceq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$;
(iii) $0 \preceq z_1 \preceq z_2$ implies $|z_1| \leq |z_2|$;
(iv) if $a,b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \preceq z_2$, then $az_1 \preceq bz_2$ for all $z_1, z_2 \in \mathbb{C}$. Recently Rao et.al[3] introduced the following definition.

Definition 2.1 [8] Let W be a non-empty set and let $s \geq 1$ be a given real number. A function $d : W \times W \to \mathbb{C}$ is called a complex valued b-metric on W if for all $l,m,n \in W$ the following conditions are satisfied:

i. $0 \preceq d(l,m)$ and $d(l,m) = 0$ if and only if $l = m$;
ii. $d(l,m) = d(m,l)$;
iii. $d(l,m) \preceq s[d(l,n) + d(n,m)]$.

Then the pair (W,d) is called a complex valued b‒metric space.

Example 2.2 [3] If $W = [0,1]$, define the mapping $d : W \times W \to \mathbb{C}$ by

$$d(l,m) = |l-m|^2 + i|l-m|^2$$

for all $l,m \in W$. Then (W,d) is complex valued b-metric space with s = 2.

Definition 2.3 [3] Let (W,d) be a complex valued b-metric space

(i) A point $l \in W$ is called interior point of a set $L \subseteq W$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(l,r) = \{m \in W : d(l,m) \prec r\} \subseteq L$

(ii) A point $l \in W$ is called limit point of a set W, whenever for every $0 \prec r \in \mathbb{C}$, $B(l,r) \cap (L - \{l\}) \neq \phi$

(iii) A subset $L \subseteq W$ is called closed whenever each element of L belongs to L.

(iv) A subbasis for a Hausdorff topology τ on W is a family $F = \{B(l,r) : l \in W \text{ and } 0 \prec \text{r}\}$

Definition 2.4 [3] Let (W,d) be a complex valued b-metric space and $\{l_n\}$ be a sequence in W and $l \in W$

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(a_n, l) \prec c$, then $\{l_n\}$ is said to be convergent, $\{l_n\}$ converges to *l*, and *l* is the limit point of $\{l_n\}$. It is denoted by $\lim_{n\to\infty} l_n = l$ or $\{l_n\} \to l$ as $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(a_n, a_{n+p}) \prec c$, where $p \in \mathbb{N}$ then $\{l_n\}$ is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in W is convergent, then (W,d) is said to be a complete complex valued *b*-metric space.

Lemma 2.5 [3] Let (W,d) be a complex valued *b*-metric space and let $\{l_n\}$ be a sequence in *W*. Then $\{l_n\}$ converges to *l* if and only if $|d(l_n,l)| \to 0$ as $n \to \infty$.

Lemma 2.6 [3] Let (W,d) be a complex valued b-metric space and let $\{l_n\}$ be a sequence in *W*. Then $\{l_n\}$ is a Cauchy sequence if and only if $\left|d\left(l_n,l_{n+p}\right)\right| \to 0$ as $n \to \infty$, where $p \in \mathbb{N}$.

3. Main Results

Theorem 3.1 Let (W,d) be a complete complex valued b-metric space with $s \ge 1$ and let U,V be self-mappings from W into itself satisfy the following inequality,

$$d(Ul,Vm) \preceq \mu_1 d(l,m) + \mu_2 \frac{d(l,m)}{1+d(m,Ul)} + \mu_3 \frac{d(l,Ul)d(m,Vm)}{d(l,Vm)+d(m,Ul)+d(l,m)} \tag{3.1}$$

for all $l,m \in W$, *such that* $l \ne m$, $d(l,Vm)+d(m,Ul)+d(l,m) \ne 0$ where μ_1,μ_2 and μ_3 are non-negative reals with $\mu_1 + s\mu_2 + \mu_3 < 1$ or $d(Ul,Vm)=0$ if $d(l,Vm)+d(m,Ul)+d(l,m)=0$.
Then *U* and *V* have a unique common fixed point.

Proof. For any arbitrary point $l_0 \in W$, define sequence $\{l_n\}$ in *W* such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = Vl_{2n+1} \quad \forall n \ge 0 \tag{3.2}$$

Now, we prove that $\{l_n\}$ is a Cauchy sequence.
Let $l = l_{2n}$, $m = l_{2n+1}$.

$$\begin{aligned} d(l_{2n+1},l_{2n+2}) &= d(Ul_{2n},Vl_{2n+1}) \\ &\preceq \mu_1 d(l_{2n},l_{2n+1}) + \frac{\mu_2 d(l_{2n},l_{2n+1})}{1+d(l_{2n+1},Ul_{2n})} + \mu_3 \frac{d(l_{2n},Ul_{2n})d(l_{2n+1},Vl_{2n+1})}{d(l_{2n},Vl_{2n+1})+d(l_{2n+1},Ul_{2n})+d(l_{2n},l_{2n+1})} \quad (3.3) \\ &= \mu_1 d(l_{2n},l_{2n+1}) + \mu_2 \frac{d(l_{2n},l_{2n+1})}{1+d(l_{2n+1},l_{2n+1})} + \mu_3 \frac{d(l_{2n},l_{2n+1})d(l_{2n+1},l_{2n+2})}{d(l_{2n},l_{2n+2})+d(l_{2n+1},l_{2n+1})+d(l_{2n},l_{2n+1})} \\ &= \mu_1 d(l_{2n},l_{2n+1}) + \mu_2 d(l_{2n},l_{2n+1}) + \mu_3 \frac{d(l_{2n},l_{2n+1})d(l_{2n+1},l_{2n+2})}{d(l_{2n},l_{2n+2})+d(l_{2n},l_{2n+1})} \end{aligned}$$

then

$$|d(l_{2n+1},l_{2n+2})| \le \mu_1 |d(l_{2n},l_{2n+1})| + \mu_2 |d(l_{2n},l_{2n+1})| + \mu_3 \frac{|d(l_{2n},l_{2n+1})||d(l_{2n+1},l_{2n+2})|}{|d(l_{2n},l_{2n+2})|+|d(l_{2n},l_{2n+1})|} \tag{3.4}$$

$$\begin{aligned} &\le \mu_1 |d(l_{2n},l_{2n+1})| + \mu_2 |d(l_{2n},l_{2n+1})| + \mu_3 \frac{|d(l_{2n},l_{2n+1})||d(l_{2n+1},l_{2n+2})|}{|d(l_{2n+1},l_{2n+2})|} \\ &= (\mu_1 + \mu_2 + \mu_3)|d(l_{2n},l_{2n+1})| \end{aligned}$$

$$|d(l_{2n+1},l_{2n+2})| \le (\mu_1 + \mu_2 + \mu_3)|d(l_{2n},l_{2n+1})| \tag{3.5}$$

Similarly, we can get

$$|d(l_{2n+2},l_{2n+3})| \le (\mu_1 + \mu_2 + \mu_3)|d(l_{2n+1},l_{2n+2})| \tag{3.6}$$

Since $\mu_1 + s\mu_2 + \mu_3 < 1$ and $s \geq 1$, we get $\mu_1 + \mu_2 + \mu_3 < 1$, therefore with $\zeta = \mu_1 + \mu_2 + \mu_3 < 1$ and for all $n \geq 0$, and consequently, we have

$$\begin{aligned} \left|d\left(l_{2n+1}, l_{2n+2}\right)\right| &\leq \zeta\left|d\left(l_{2n}, l_{2n+1}\right)\right| \leq \zeta\zeta\left|d\left(l_{2n-1}, l_{2n}\right)\right| = \zeta^2\left|d\left(l_{2n-1}, l_{2n}\right)\right| \\ &\leq \zeta^3\left|d\left(l_{2n-2}, l_{2n-1}\right)\right| \leq \ldots \\ &\leq \zeta^{2n+1}\left|d\left(l_0, l_1\right)\right| \end{aligned} \tag{3.7}$$

That is,

$$\left|d\left(l_{n+1}, l_{n+2}\right)\right| \leq \zeta\left|d\left(l_n, l_{n+1}\right)\right| \leq \zeta^2\left|d\left(l_{n-1}, l_n\right)\right| \leq \ldots \leq \zeta^{n+1}\left|d\left(l_0, l_1\right)\right|. \tag{3.8}$$

Thus, for any $m > n,\ m, n \in \mathbb{N}$ we have

$$\begin{aligned} \left|d\left(l_n, l_m\right)\right| &\leq s\left|d\left(l_n, l_{n+1}\right)\right| + s\left|d\left(l_{n+1}, l_m\right)\right| \\ &\leq s\left|d\left(l_n, l_{n+1}\right)\right| + s^2\left|d\left(l_{n+1}, l_{n+2}\right)\right| + s^2\left|d\left(l_{n+2}, l_m\right)\right| \\ &\leq \ldots \\ &\leq s\left|d\left(l_n, l_{n+1}\right)\right| + s^2\left|d\left(l_{n+1}, l_{n+2}\right)\right| + \ldots + s^{m-n}\left|d\left(l_{m-1}, l_m\right)\right| \end{aligned} \tag{3.9}$$

$$\begin{aligned} \left|d\left(l_n, l_m\right)\right| &\leq s\zeta^n\left|d\left(l_0, l_1\right)\right| + s^2\zeta^2\left|d\left(l_0, l_1\right)\right| + \ldots + s^{m-n}\zeta^{m-1}\left|d\left(l_0, l_1\right)\right| \\ &= s\zeta^n\left(1 + s\zeta + \left(s\zeta\right)^2 + \ldots + \left(s\zeta\right)^{m-n-1}\right)\left|d\left(l_0, l_1\right)\right| \\ &\leq s\zeta^n\left(1 + s\zeta + \left(s\zeta\right)^2 + \ldots + \left(s\zeta\right)^{m-n-1} + \ldots\right)\left|d\left(l_0, l_1\right)\right| \\ &= s\zeta^n\left(1 - s\zeta\right)^{-1}\left|d\left(l_0, l_1\right)\right| \end{aligned} \tag{3.10}$$

$$\left|d\left(l_n, l_m\right)\right| \leq \frac{s\zeta^n}{\left(1 - s\zeta\right)}\left|d\left(l_0, l_1\right)\right| \to 0 \quad \text{as } m, n \to \infty \tag{3.11}$$

Thus, $\{l_n\}$ is a Cauchy sequence in *W*. Since *W* is complete there exists some $t \in W$ such that $l_n \to t$ as $n \to \infty$.
Assume not, then there exits $z \in W$ such that

$$\left|d\left(t, Ut\right)\right| = |z| > 0. \tag{3.12}$$

So by using the triangular inequality and (1), we get

$$\begin{aligned} z = d\left(t, Ut\right) &\preceq sd\left(t, l_{2n+2}\right) + sd\left(l_{2n+2}, Ut\right) \\ &= sd\left(t, l_{2n+2}\right) + sd\left(Vl_{2n+1}, Ut\right) \\ &\preceq sd\left(t, l_{2n+2}\right) + s\mu_1 d\left(t, l_{2n+1}\right) + s\mu_2\frac{d\left(t, l_{2n+1}\right)}{1 + d\left(l_{2n+1}, Ut\right)} \\ &\quad + s\mu_3\frac{d\left(t, Ut\right)d\left(l_{2n+1}, Vl_{2n+1}\right)}{d\left(t, Vl_{2n+1}\right) + d\left(l_{2n+1}, Ut\right) + d\left(t, l_{2n+1}\right)} \\ &= sd\left(t, l_{2n+2}\right) + s\mu_1 d\left(t, l_{2n+1}\right) + s\mu_2\frac{d\left(t, l_{2n+1}\right)}{1 + d\left(l_{2n+1}, Ut\right)} + s\mu_3\frac{d\left(t, Ut\right)d\left(l_{2n+1}, l_{2n+2}\right)}{d\left(t, l_{2n+2}\right) + d\left(l_{2n+1}, Ut\right) + d\left(t, l_{2n+1}\right)} \end{aligned} \tag{3.13}$$

$$\begin{aligned} |z| = \left|d\left(t, Ut\right)\right| &\leq s\left|d\left(t, l_{2n+2}\right)\right| + s\mu_1\left|d\left(t, l_{2n+1}\right)\right| + s\mu_2\frac{\left|d\left(t, l_{2n+1}\right)\right|}{1 + \left|d\left(l_{2n+1}, Ut\right)\right|} \\ &\quad + s\mu_3\frac{\left|d\left(t, Ut\right)\right|\left|d\left(l_{2n+1}, l_{2n+2}\right)\right|}{\left|d\left(t, l_{2n+2}\right)\right| + \left|d\left(l_{2n+1}, Ut\right)\right| + \left|d\left(t, l_{2n+1}\right)\right|} \end{aligned} \tag{3.14}$$

Taking the limit (3.14) as $n \to \infty$,we obtain that $|z| = |d(t,Ut)| \le 0$, a contradiction with (12). So $|z| = 0$ Hence $Ut = t.$ Similarly we obtain $Vt = t.$

Now, we show that U and V have unique common fixed point of U and V . To prove this assume t' is another common fixed point of U and V . Then

$$d(t,t') = d(Ut,Vt') \preceq \mu_1 d(t,t') + \mu_2 \frac{d(t,t')}{1 + d(t',Ut)} + \mu_3 \frac{d(t,Ut)d(t',Vt')}{d(t,Vt') + d(t',Ut) + d(t,t')} \quad (3.15)$$

So that

$$|d(t,t')| \le \mu_1 |d(t,t')| + \mu_2 \frac{|d(t,t')|}{1 + |d(t',Ut)|} + \mu_3 \frac{|d(t,Ut)||d(t',Vt')|}{|d(t,Vt')| + |d(t',Ut)| + |d(t,t')|}$$

$$|d(t,t')| \le \mu_1 |d(t,t')| \quad (3.16)$$

Which is contradiction. Hence $t = t'$ which shows the uniqueness of common fixed point.

Now we consider the second case.

$$d(l,Vm) + d(m,Ul) + d(l,m) = 0$$

$$l = l_{2n} \ m = l_{2n+1}$$

$$d(l_{2n},Vl_{2n+1}) + d(l_{2n+1},Ul_{2n}) + d(l_{2n},l_{2n+1}) = 0$$

$$d(Ul_{2n},Vl_{2n+1}) = 0 \text{ so that } l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}.$$

Thus we have $l_{2n+1} = Ul_{2n} = l_{2n}$ so there exists E_1 and f_1 such that $E_1 = Uf_1 = f_1$ where $E_1 = l_{2n+1}$ & $f_1 = l_{2n}$ using the foregoing arguments, we show that there exists E_2 and f_2 such that

$$E_2 = Vf_2 = f_2 \text{ where } E_2 = l_{2n+2} \ \& \ f_2 = l_{2n+1}.$$

As $d(f_1,Vf_2) + d(f_2,Uf_1) + d(f_1,f_2) = 0$ which implies $d(Uf_1,Vf_2) = 0$. $E_1 = Uf_1 = Vf_2 = E_2$.
Thus we obtain that $E_1 = Uf_1 = UE_1$ similarly one can also have $E_2 = VE_2$. As $E_1 = E_2$ implies $UE_1 = VE_1 = E_1$, therefore $E_1 = E_2$ is the common fixed point of U and V. For uniqueness of common fixed point, assume that, assume that E_1' in W is another common fixed point of U and V. Then we have $UE_1' = VE_1' = E_1'$.

As $d(E_1,VE_1') + d(E_1',UE_1) + d(E_1,E_2') = 0$, therefore $d(E_1,E_1') = d(UE_1,VE_1') = 0$.

This implies that $E_1 = E_1'$. This completes the proof of theorem.

Corollary 3.2 Let (W,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $V : W \to W$ be a mapping satisfying

$$d(Vl,Vm) \preceq \mu_1 (l,m) + \mu_2 \frac{d(l,m)}{1 + d(m,Vl)} + \mu_3 \frac{d(l,Vl)d(m,Vm)}{d(l,Vm) + d(m,Vl) + d(l,m)} \quad (3.17)$$

for all $l,m \in W$, *such that* $l \ne m$, $d(l,Vm) + d(m,Vl) + d(l,m) \ne 0$ where μ_1, μ_2 and μ_3 are non-negative reals with $\mu_1 + s\mu_2 + \mu_3 < 1$ or $d(Vl,Vm) = 0$ if $d(l,Vm) + d(m,Vl) + d(l,m) = 0$.
Then V has a unique common fixed point in W.

Proof. By using the theorem 3.1 with $U = V$, we can prove this result.

Corollary 3.3 Let (W,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $V : W \to W$ be a mapping satisfying (for some fixed n)

$$d\left(V^n l, V^n m\right) \preceq \mu_1 d(l,m) + \mu_2 \frac{d(l,m)}{1+d\left(m,V^n l\right)} + \mu_3 \frac{d\left(l,V^n l\right) d\left(m,V^n m\right)}{d\left(l,V^n m\right)+d\left(m,V^n l\right)+d(l,m)} \tag{3.18}$$

for all $l,m \in W$, *such that* $l \neq m$, $d\left(l,V^n m\right)+d\left(m,V^n l\right)+d(l,m) \neq 0$ where μ_1, μ_2 and μ_3 are non-negative reals with $\mu_1 + s\mu_2 + \mu_3 < 1$ or $d\left(V^n l, V^n m\right)=0$ if $d\left(l,V^n m\right)+d\left(m,V^n l\right)+d(l,m)=0$. Then V has a unique common fixed point in W.

Proof. By using the corollary 3.2 with $V = V^n$, we can prove this result.

Theorem 3.4 Let (W,d) be a complete complex valued b-metric space with $s \geq 1$ and let U,V be self-mappings from W into itself satisfy the following inequality,

$$d(Ul,Vm) \preceq \mu_1 (l,m) + \mu_2 \left[d(l,m)+d(l,Vm)\right] + \mu_3 \left[d(l,Ul)+d(m,Vm)\right] + \mu_4 \frac{\left[d^2(l,Vm)+d^2(m,Ul)\right]}{d(l,Vm)+d(m,Ul)} \tag{3.19}$$

for all $l,m \in W$, *such that* $l \neq m$, $d(l,Vm)+d(m,Ul) \neq 0$ where μ_1, μ_2, μ_3 and μ_4 are non-negative reals with $\mu_1 + (2s+1)\mu_2 + 2\mu_3 + 2s\mu_4 < 1$ or $d(Ul,Vm)=0$ if $d(l,Vm)+d(m,Ul)=0$. Then U and V have a unique common fixed point.

Proof. For any arbitrary point $l_0 \in W$, define sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = Vl_{2n+1} \quad \forall n \geq 0 \tag{3.20}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence.

Let $l = l_{2n}$, $m = l_{2n+1}$.

$$\begin{aligned} d(l_{2n+1}, l_{2n+2}) &= d(Ul_{2n}, Vl_{2n+1}) \\ &\preceq \mu_1 d(l_{2n}, l_{2n+1}) + \mu_2 \left[d(l_{2n}, l_{2n+1}) + d(l_{2n}, Vl_{2n+1})\right] + \mu_3 \left[d(l_{2n}, Ul_{2n}) + d(l_{2n+1}, Vl_{2n+1})\right] \\ &\quad + \mu_4 \frac{\left[d^2(l_{2n}, Vl_{2n+1}) + d^2(l_{2n+1}, Ul_{2n})\right]}{d(l_{2n}, Vl_{2n+1}) + d(l_{2n+1}, Ul_{2n})} \\ &= \mu_1 d(l_{2n}, l_{2n+1}) + \mu_2 \left[d(l_{2n}, l_{2n+1}) + d(l_{2n}, l_{2n+2})\right] + \mu_3 \left[d(l_{2n}, l_{2n+1}) + d(l_{2n+1}, l_{2n+2})\right] \\ &\quad + \mu_4 \frac{\left[d^2(l_{2n}, l_{2n+2}) + d^2(l_{2n+1}, l_{2n+1})\right]}{d(l_{2n}, l_{2n+2}) + d(l_{2n+1}, l_{2n+1})} \\ &= \mu_1 d(l_{2n}, l_{2n+1}) + \mu_2 \left[d(l_{2n}, l_{2n+1}) + d(l_{2n}, l_{2n+2})\right] + \mu_3 \left[d(l_{2n}, l_{2n+1}) + d(l_{2n+1}, l_{2n+2})\right] \\ &\quad + \mu_4 \frac{\left[d^2(l_{2n}, l_{2n+2})\right]}{d(l_{2n}, l_{2n+2})} \end{aligned}$$

taking modulus

$$\begin{aligned} \left|d(l_{2n+1}, l_{2n+2})\right| &\leq \mu_1 \left|d(l_{2n}, l_{2n+1})\right| + \mu_2 \left[\left|d(l_{2n}, l_{2n+1})\right| + \left|d(l_{2n}, l_{2n+2})\right|\right] + \mu_3 \left[\left|d(l_{2n}, l_{2n+1})\right| + \left|d(l_{2n+1}, l_{2n+2})\right|\right] \\ &\quad + \mu_4 \frac{\left[\left|d^2(l_{2n}, l_{2n+2})\right|\right]}{\left|d(l_{2n}, l_{2n+2})\right|} \end{aligned}$$

$$\begin{aligned} \left|d(l_{2n+1}, l_{2n+2})\right| &\leq \mu_1 \left|d(l_{2n}, l_{2n+1})\right| + \mu_2 \left[\left|d(l_{2n}, l_{2n+1})\right| + \left|d(l_{2n}, l_{2n+2})\right|\right] + \mu_3 \left[\left|d(l_{2n}, l_{2n+1})\right| + \left|d(l_{2n+1}, l_{2n+2})\right|\right] \\ &\quad + \mu_4 \left|d(l_{2n}, l_{2n+2})\right| \end{aligned}$$

As
$\left|d\left(l_{2n},l_{2n+2}\right)\right| \le s\left[\left|d\left(l_{2n},l_{2n+1}\right)\right| + \left|d\left(l_{2n+1},l_{2n+2}\right)\right|\right]$,
Therefore

$$\begin{aligned}\left|d\left(l_{2n+1},l_{2n+2}\right)\right| &\le \mu_1\left|d\left(l_{2n},l_{2n+1}\right)\right| + \mu_2\left|d\left(l_{2n},l_{2n+1}\right)\right| + s\mu_2\left[\left|d\left(l_{2n},l_{2n+1}\right)\right| + \left|d\left(l_{2n+1},l_{2n+2}\right)\right|\right]\\ &\quad + \mu_3\left[\left|d\left(l_{2n},l_{2n+1}\right)\right| + \left|d\left(l_{2n+1},l_{2n+2}\right)\right|\right] + s\mu_4\left[\left|d\left(l_{2n},l_{2n+1}\right)\right| + \left|d\left(l_{2n+1},l_{2n+2}\right)\right|\right]\\ &\le \left(\mu_1 + (s+1)\mu_2 + \mu_3 + s\mu_4\right)\left|d\left(l_{2n},l_{2n+1}\right)\right| + \left(s\mu_2 + \mu_3 + s\mu_4\right)\left|d\left(l_{2n+1},l_{2n+2}\right)\right|\end{aligned}$$

$$\left|d\left(l_{2n+1},l_{2n+2}\right)\right| \le \left(\frac{\mu_1 + (s+1)\mu_2 + \mu_3 + s\mu_4}{1 - s\mu_2 - \mu_3 - s\mu_4}\right)\left|d\left(l_{2n},l_{2n+1}\right)\right|$$

Similarly, we can get

$$\left|d\left(l_{2n+2},l_{2n+3}\right)\right| \le \left(\frac{\mu_1 + (s+1)\mu_2 + \mu_3 + s\mu_4}{1 - s\mu_2 - \mu_3 - s\mu_4}\right)\left|d\left(l_{2n+1},l_{2n+2}\right)\right|$$

since $\mu_1 + (2s+1)\mu_2 + 2\mu_3 + 2s\mu_4 < 1$ and $s \ge 1$, we get $\left(\frac{\mu_1 + (s+1)\mu_2 + \mu_3 + s\mu_4}{1 - s\mu_2 - \mu_3 - s\mu_4}\right) < 1$, therefore wit

$\zeta = \left(\frac{\mu_1 + (s+1)\mu_2 + \mu_3 + s\mu_4}{1 - s\mu_2 - \mu_3 - s\mu_4}\right) < 1$ and for all $n \ge 0$, and consequently, we have

$$\begin{aligned}\left|d\left(l_{2n+1},l_{2n+2}\right)\right| &\le \zeta\left|d\left(l_{2n},l_{2n+1}\right)\right| \le \zeta\zeta\left|d\left(l_{2n-1},l_{2n}\right)\right| = \zeta^2\left|d\left(l_{2n-1},l_{2n}\right)\right|\\ &\le \zeta^3\left|d\left(l_{2n-2},l_{2n-1}\right)\right| \le \dots\\ &\le \zeta^{2n+1}\left|d\left(l_0,l_1\right)\right|\end{aligned}$$

That is

$$\left|d\left(l_{n+1},l_{n+2}\right)\right| \le \zeta\left|d\left(l_n,l_{n+1}\right)\right| \le \zeta^2\left|d\left(l_{n-1},l_n\right)\right| \le \dots \le \zeta^{n+1}\left|d\left(l_0,l_1\right)\right|.$$

Thus, for any $m > n,\ m,n \in \mathbb{N}$, we have

$$\begin{aligned}\left|d\left(l_n,l_m\right)\right| &\le s\left|d\left(l_n,l_{n+1}\right)\right| + s\left|d\left(l_{n+1},l_m\right)\right|\\ &\le s\left|d\left(l_n,l_{n+1}\right)\right| + s^2\left|d\left(l_{n+1},l_{n+2}\right)\right| + s^2\left|d\left(l_{n+2},l_m\right)\right|\\ &\le \dots\\ &\le s\left|d\left(l_n,l_{n+1}\right)\right| + s^2\left|d\left(l_{n+1},l_{n+2}\right)\right| + \dots + s^{m-n}\left|d\left(l_{m-1},l_m\right)\right|\end{aligned}$$

$$\begin{aligned}\left|d\left(l_n,l_m\right)\right| &\le s\zeta^n\left|d\left(l_0,l_1\right)\right| + s^2\zeta^2\left|d\left(l_0,l_1\right)\right| + \dots + s^{m-n}\zeta^{m-1}\left|d\left(l_0,l_1\right)\right|\\ &= s\zeta^n\left(1 + s\zeta + (s\zeta)^2 + \dots + (s\zeta)^{m-n-1}\right)\left|d\left(l_0,l_1\right)\right|\\ &\le s\zeta^n\left(1 + s\zeta + (s\zeta)^2 + \dots + (s\zeta)^{m-n-1} + \dots\right)\left|d\left(l_0,l_1\right)\right|\\ &= s\zeta^n\left(1 - s\zeta\right)^{-1}\left|d\left(l_0,l_1\right)\right|\end{aligned}$$

$$\left|d\left(l_n,l_m\right)\right| \le \frac{s\zeta^n}{\left(1 - s\zeta\right)}\left|d\left(l_0,l_1\right)\right| \to 0 \quad \text{as } m,n \to \infty$$

Thus $\{l_n\}$ is a Cauchy sequence in *W*. since *W* is complete there exists some $t \in W$ such that $l_n \to t$ as $n \to \infty$.
Assume not, then there exits $z \in W$ such that

$$\left|d\left(t,Ut\right)\right| = |z| > 0$$

so by using the triangular inequality and (3.1), we get

$$z = d(t,Ut) \preceq sd(t,l_{2n+2}) + sd(l_{2n+2},Ut)$$
$$= sd(t,l_{2n+2}) + sd(Vl_{2n+1},Ut)$$
$$\preceq sd(t,l_{2n+2}) + s\mu_1 d(t,l_{2n+1}) + s\mu_2\left[d(t,l_{2n+1}) + d(t,Vl_{2n+1})\right] + s\mu_3\left[d(t,Ut) + d(l_{2n+1},Vl_{2n+1})\right]$$
$$+ s\mu_4 \frac{\left[d^2(t,Vl_{2n+1}) + d^2(l_{2n+1},Ut)\right]}{d(t,Vl_{2n+1}) + d(l_{2n+1},Ut)}$$
$$= sd(t,l_{2n+2}) + s\mu_1 d(t,l_{2n+1}) + s\mu_2\left[d(t,l_{2n+1}) + d(t,l_{2n+2})\right] + s\mu_3\left[z + d(l_{2n+1},l_{2n+2})\right]$$
$$+ s\mu_4 \frac{\left[d^2(t,l_{2n+2}) + d^2(l_{2n+1},Ut)\right]}{d(t,l_{2n+2}) + d(l_{2n+1},Ut)}$$
$$|z| = |d(t,Ut)| \le s|d(t,l_{2n+2})| + s\mu_1|d(t,l_{2n+1})| + s\mu_2\left[|d(t,l_{2n+1})| + |d(t,l_{2n+2})|\right] + s\mu_3\left[|z| + |d(l_{2n+1},l_{2n+2})|\right]$$
$$+ s\mu_4 \frac{\left[|d^2(t,l_{2n+2})| + |d^2(l_{2n+1},Ut)|\right]}{|d(t,l_{2n+2})| + |d(l_{2n+1},Ut)|}$$
taking the limit as $n \to \infty$ we obtain that $|z| = |d(t,Ut)| \le 0$ a contradiction, so $|z| = 0$

Hence $Ut = t.$ similarly we obtain $Vt = t.$

Now, we show that U and V have unique common fixed point of U and V. To prove this assume t' is another common fixed point of U and V. Then

$$d(t,t') = d(Ut,Vt') \preceq \mu_1 d(t,t') + \mu_2\left[d(t,t') + d(t,Vt')\right] + \mu_3\left[d(t,Ut) + d(t',Vt')\right] + \mu_4 \frac{\left[d^2(t,Vt') + d^2(t',Ut)\right]}{d(t,Vt') + d(t',Ut)}$$

So that

$$|d(t,t')| \le \mu_1|d(t,t')| + \mu_2\left[|d(t,t')| + |d(t,Vt')|\right] + \mu_3\left[|d(t,Ut)| + |d(t',Vt')|\right] + \mu_4 \frac{\left[|d^2(t,Vt')| + |d^2(t',Ut)|\right]}{|d(t,Vt')| + |d(t',Ut)|}$$
$$|d(t,t')| \le \mu_1 + 2\mu_2 + \mu_4|d(t,t')|$$

Which is contradiction. Hence $t = t'$ which shows the uniqueness of common fixed point.

For the second case, $d(Ul,Vm) = 0$ if $d(l,Vm) + d(m,Ul) = 0$ .the proof of unique common fixed point can be completed in the line of Theorem 3.1.This completes the proof of the theorem.

Corollary 3.5 Let (W,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $V : W \to W$ be a mapping satisfying

$$d(Vl,Vm) \preceq \mu_1(l,m) + \mu_2\left[d(l,m) + d(l,Vm)\right] + \mu_3\left[d(l,Vl) + d(m,Vm)\right] + \mu_4 \frac{\left[d^2(l,Vm) + d^2(m,Vl)\right]}{d(l,Vm) + d(m,Vl)}$$

for all $l,m \in W$, *such that* $l \ne m$, $d(l,Vm) + d(m,Vl) \ne 0$ where μ_1,μ_2,μ_3 and μ_4 are non-negative reals with $\mu_1 + (2s+1)\mu_2 + 2\mu_3 + 2s\mu_4 < 1$ or $d(Vl,Vm) = 0$ if $d(l,Vm) + d(m,Vl) = 0$.

Then *V* has a unique common fixed point in *W*..

Proof. By using the theorem 3.4 with $U = V$, we can prove this result.

Corollary 3.6 Let (W,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $V : W \to W$ be a mapping satisfying (for some fixed n)

$$d(V^n l,V^n m) \preceq \mu_1(l,m) + \mu_2\left[d(l,m) + d(l,V^n m)\right] + \mu_3\left[d(l,V^n l) + d(m,V^n m)\right] + \mu_4 \frac{\left[d^2(l,V^n m) + d^2(m,V^n l)\right]}{d(l,V^n m) + d(m,V^n l)}$$

for all $l,m \in W$, *such that* $l \neq m$, $d\left(l,V^n m\right)+d\left(m,V^n l\right) \neq 0$ where μ_1,μ_2,μ_3 and μ_4 are non-negative reals with $\mu_1+(2s+1)\mu_2+2\mu_3+2s\mu_4<1$ or $d(Vl,Vm)=0$ if $d\left(l,V^n m\right)+d\left(m,V^n l\right)=0$.
Then *V* has a unique common fixed point in *W*.

Proof. By using the corollary 3.5 with $V=V^n$, we can prove this result.

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