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# NATURAL QUINTIC SPLINE

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**Abstract**. In this paper we seek to describe the derivation of natural quintic spline interpolation.

**Keywords** Interpolation, Spline, quintic spline

## 1. Introduction

Spline Interpolation plays an important role in Numerical Analysis, Computation, Integration, Differentiation etc. In Interpolating problems, Spline Interpolation is often preferred to Polynomial Interpolation. In 1946, I.J.Schoenberg [5] has introduced spline functions. A spline is a function defined by piecewise polynomials, each piece is a function which is a polynomial on each of its subintervals, but possibly a different one on each of its interval.

A Spline is also a flexible curve which consists of a long strip of metal or other material, which may be bent into a curve and fixed in position at a number of predefined points called as knots. These predefined points allows to draw a smooth curve for the purpose of transferring the curve to another material. Consider a quintic spline function which is a polynomial of degree 5, whose first derivative is a quartic function. In $16^{th}$ century solving a quintic function is a major problem in Algebra, whereas Cubic spline and quartic spline has been derived [2,4]. In a quintic spline function continuity conditions are applied to the function itself, and also to the first, second, third and fourth order derivatives of the spline functions, considering the known data values as the spline knots and these points are the polynomial pieces which are joined together. The continuity conditions are also applied at the knots.

## 2. Natural quintic spline

Consider the data values $(x_i, y_i), i = 0,1,2 \ldots\ldots n$ of the function $y = f(x)$. Let $I = [x_i, x_{i+1}]$be a subinterval of $[x_0, x_n]$. Let $S_i(x)$ be the spline function of n functions and each of the spline function is defined in the interval $[x_i, x_{i+1}]$. In the case of quintic spline, the functions $S_i(x)$ are the polynomials of fifth degree whose coefficients has to be determined.

***Definition 2.1*** A quintic spline is a polynomial of degree 5 and it has continuous derivative up to order 4,where each subintervals are polynomials of degree 5.

$$S(x) = \begin{cases} S_0(x), & x_0 < x < x_1 \\ S_1(x), & x_1 < x < x_2 \\ & \cdot \\ & \cdot \\ & \cdot \\ S_n(x), & x_{n-1} < x < x_n \end{cases}$$

**Boundary conditions:** First let us define the set of variables $M_i$ for , $i = 0,1,2 \ldots\ldots n$ as the values of the fourth order derivative of the spline function $S(x)$ at the points ie) $M_i = S_i'^{v}(x)$.The boundary conditions for the quintic splines are,

$$S_i(x_i) = y_i, = M_i, \quad i = 0,1,2 \ldots n$$
$$S_i(x_{i+1}) = y_{i+1} = M_{i+1}, \quad i = 0,1,2 \ldots n-2$$
$$S_i(x_i), S_i'(x_i), S_i''(x_i), S_i'''(x_i) \text{ and } S_i'^{v}(x_i)$$

are continuous.

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), i = 0,1,2 \ldots n-2$$
$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}), i = 0,1,2 \ldots n-2$$

(iii) $$S_i'''(x_{i+1}) = S_{i+1}'''(x_{i+1}), i = 0,1,2 \ldots n-2$$

(iv) $$S_i'^{v}(x_{i+1}) = S_{i+1}'^{v}(x_{i+1}), i = 0,1,2 \ldots n-2$$

$$S_i'^{v}(x_0) = S_{i+1}'^{v}(x_n) = 0 \;\; ie)\; M_0 = M_n = 0$$
$$S_{n-1}''(x_n) = 0$$
$$S_{n-1}'''(x_n) = 0$$

On applying these conditions, we get a set of equation with coefficients. On solving the coefficients of the functions which on substituting gives the derivation of the natural quintic spline functions $S_i(x)$.

**Derivation:** Since $S_i(x)$ is a quintic spline, $S_i'^{v}(x)$is linear. Let us define the fourth derivative of the function as follows:

Let $h_i = x_{i+1} - x_i$

$$S_i'^{v}(x) = \frac{1}{h_i}[(x_{i+1} - x)M_i + (x - x_i)M_{i+1}], \; i = 0,1,2 \ldots n-1 \qquad (1)$$

The functions $S_i(x)$can be obtained by integrating (1) four times, On gathering the coefficients and by selecting the integration constants, the following expression is obtained:

$$S_i(x) = \frac{1}{h_i}\left[\frac{(x_{i+1}-x)^5}{120}M_i + \frac{(x-x_i)^5}{120}M_{i+1}\right] + C_i(x_{i+1} - x)(x - x_i)^2 + D_i(x_{i+1} - x)(x - x_i) + E_i(x_{i+1} - x) + F_i(x - x_i) \qquad (2)$$

Where $C_i, D_i, E_i and\ F_i's$ are the coefficients to be determined in terms of $M_i's$

Using the boundary conditions (a) and (b) in (1),

$$E_i = \frac{y_i}{h_i} - \frac{h_i^3}{120}M_i \text{ for } i = 0,1,2 \ldots n-1 \text{ and} \qquad (3)$$

$$F_i = \frac{y_{i+1}}{h_i} - \frac{h_i^3}{120}M_{i+1}, \text{for } i = 0,1,2 \ldots n-1 \qquad (4)$$

Now on differentiating (2), the first derivative of $S_i(x)$ can be obtained:

$$S_i'(x) = \frac{1}{h_i}\left[-\frac{(x_{i+1}-x)^4}{24}M_i + \frac{(x-x_i)^4}{24}M_{i+1}\right] + C_i(2xx_{i+1} - 2x_ix_{i+1} - 3x^2 + 4xx_i - x_i^2) + D_i(x_{i+1} + x_i - 2x) + E_i(-1) + F_i \qquad (5)$$

On replacing the equation (3) and applying the continuity condition defined in (c) (i), then the following equation is obtained,:

$$\left[\frac{h_i^{\ 3}}{24} - \frac{h_{i+1}^{\ 3}}{24}\right] M_{i+1} + \frac{h_{i+1}^{\ 3}}{120}[M_{i+2} - M_{i+1}] - \frac{h_i^{\ 3}}{120}[M_{i+1} - M_i] + [C_{i+1}h_{i+1}^{\ 2} - C_i h_i^{\ 2}] + [D_{i+1}h_{i+1} - D_i h_i] = \Delta_{i+1} - \Delta_i \quad (6)$$

Where $\Delta_i = \frac{y_{i+1} - y_i}{h_i}$

Again on Differentiating (5), the second derivative is obtained:

$$S_i''(x) = \frac{1}{h_i}\left[\frac{(x_{i+1}-x)^3}{6} M_i + \frac{(x-x_i)^3}{6} M_{i+1}\right] + C_i[2x_{i+1} - 6x + 4x_i] + D_i[-2] \quad (7)$$

On applying the continuity condition defined in (c) (ii), we get the next expression follows:

$$D_i = D_{i+1} - \left[\frac{Z_i}{12}\right] M_{i+1} + 2[C_{i+1}h_{i+1} - C_i h_i] \text{for } i = 0, 1, 2 \ldots\ldots n-1 \quad (8)$$

Once again differentiating (7), the third derivative is obtained:

$$S_i'''(x) = \frac{1}{h_i}\left[-\frac{(x_{i+1}-x)^2}{2} M_i + \frac{(x-x_i)^2}{2} M_{i+1}\right] - 6C_i \quad (9)$$

Again applying the continuity condition defined in (c), (iii) the new expression is:

$$C_i = C_{i+1} + \left[\frac{h_i - h_{i+1}}{12}\right] M_{i+1} \text{for } i = 0, 1, 2 \ldots\ldots n-1 \quad (10)$$

Replacing equation (8) in (6) then the following expression is obtained:

$$h_i^3[M_i - 6M_{i+1}] + h_{i+1}^3[M_{i+2} - 6M_{i+1}] + 120C_{i+1}Z_i + 10h_i^2 h_{i+1} M_{i+1} + 120[D_{i+1}h_{i+1} - D_i h_i] = 120[\Delta_{i+1} - \Delta_i]$$

where $Z_i = h_{i+1}^2 - h_i^2 \text{for } i = 0, 1, 2 \ldots\ldots n-1$ (11)

Using the conditions (d), (e) and (f) in (8), (10) and (11) gives

$$C_{n-1} = 0 \text{ and } D_{n-1} = 0 \quad (12)$$

Finally on solving (6), (8) and (9) and replacing it into (2) along with (3), (4) and (12), the spline function $S_i(x)$ is obtained.

## 3. Illustration

Consider a set of data points (5,0.0872), (15,0.2588), (30,0.5),(35, .5736), (45, 0.7071), (60, 0.8660), (65, 0.9063)

$h_0 = 10, h_1 = 15, h_2 = 5, \ h_3 = 10, \ h_4 = 15, h_5 = 5$ ,

$\Delta_0 = 0.01716$ $\quad Z_0 = 125$

$\Delta_1 = 0.01608$ $\quad Z_1 = -200$

$\Delta_2 = 0.01472$ $\quad Z_2 = 75$

$\Delta_3 = 0.01335$ $\quad Z_3 = 125$

$\Delta_4 = 0.01059$ $\quad Z_4 = -200$ $\quad \Delta_5 = 0.00806$

$C_0 = -1.314532 * 10^{-4}$ $\quad D_0 = -1.5533 * 10^{-5}$

$C_1 = -1.2215 * 10^{-4}$ $\quad D_1 = 0.001112664 * 10^{-5}$

$C_2 = -6.404315 * 10^{-5}$ $\quad D_2 = -7.4923218 * 10^{-5}$

$C_3 = -7.435589 * 10^{-5}$ $\quad D_3 = -5.72369 * 10^{-5}$

$C_4 = -6.29085 * 10^{-5}$ $\quad D_4 = 6.29085 * 10^{-4}$

$C_5 = 0$ $\quad D_5 = 0$

$E_0 = 0.00872$ $\quad F_0 = 0.025694$

$E_1 = 0.016625$ $\quad F_1 = 0.035294$

$E_2 = 0.10007263$ $\quad F_2 = 0.114746$

$E_3 = 0.05755663$ $\quad F_3 = 0.070481$

$E_4 = 0.0463673$ $\quad F_4 = 0.059856$

$E_5 = 0.1732786$ $\quad F_5 = 0.18126$

The Natural Quintic Spline $S_i(x)$ are

$S_0(x) = \ 1.861338 * 10^{-8}x^5 - 4.6534 * 10^{-7}x^4 + 12.679985 * \ 10^{-5}x^3 + 0.0034184x^2 - 0.009079x + 0.0632178, \quad x \in [5, 15]$

$$S_1(x) = -5.114452 * 10^{-8}x^5 + 4.76651 * 10^{-6}x^4 - 7.66853 * 10^{-5}x^3 - 0.0037839x^2 + 0.1460965x - 1.024919, \quad x \in [15, 30]$$

$$S_2(x) = 7.496 * 10^{-8}x^5 - 1.414852 * 10^{-5}x^4 + 111.631582 * 10^{-5}x^3 - 0.0447624x^2 + 0.91116219x - 7.050316, \quad x \in [30, 35]$$

$$S_3(x) = 4.352 * 10^{-8}x^5 - 0.864733 * 10^{-5}x^4 + 77.248347 * 10^{-5}x^3 - 0.0371621x^2 + 0.932891x - 8.983685, \quad x \in [35, 45]$$

$$S_4(x) = -5.72 * 10^{-8}x^5 + 1.401523 * 10^{-5}x^4 - 133.58313 * 10^{-5}x^3 + 0.06174287x^2 - 1.367691x + 12.035735, \quad x \in [45, 60]$$

$$S_5(x) = 1.2582 * 10^{-8}x^5 - 4.089052 * 10^{-5}x^4 + 531.57683 * 10^{-5}x^3 - 0.345525x^2 + 11.237542x - 145.596776, \quad x \in [60, 65]$$

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