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ULAM STABILITY RESULTS FOR BOUNDARY VALUE PROBLEM OF FRACTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. Boundary value problems have wide applications in science and technology. This paper is concerned with various kinds of Ulam stability analysis for the nonlinear discrete boundary value problem of fractional order $\sigma \in (2,3]$ with Riemann-Liouville fractional difference operator. Finally, some examples are presented to illustrate the main results.

1. INTRODUCTION

Ability to possess long memory effects by fractional order differential equations (FODEs) is the main reason for the renewed interest in the study of properties of solutions to FODEs in different research areas of applied sciences. Also fractional calculus is a natural tool to model and study the non local phenomena. The solutions arising from fractional calculus models can provide more accurate insights in to the models. But there are several forms of definitions for fractional integral and fractional derivative. During the past few decades FODEs are found in numerous applications in different types of complex systems from diverse disciplines for example diffusion, viscoelasticity, turbulence, oscillation and statistical distribution theory [11], [13]. Moreover they have many applications in various branches of sciences and engineering.

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Stability analysis of FODEs is more complicated than the classical differential equations and it has been developed in a slow pace. In many cases, an integer order differential equation when we compared with the great attention to a fractional order equation the solution is feasible in terms of a Hyers-Ulam (HU) stable and Hyers-Ulam-Rassias (HUR) stable and Mittag-Leffler function. Among all these concepts, HU stability analysis for FODEs is studied in a simple way. There are many papers on FODEs involving Caputo derivative and Riemann - Liouville derivative [1, 2, 10].

The fractional boundary value problems (FBVPs) are used in various physical process of stochastic transport with different boundary conditions (BCs). The study of 2, 3, 4, multipoint and nonlocal BVPs have been examined by many researchers by using different methods which include fixed point theorems in cones, Kronoselskii theorem, degree theory and Leray Schauder 2 type. Integral BCs appear in semiconductors, thermal conduction and hydrodynamic problems [5]. Qualitative properties of fractional difference equations (FDEs) with fractional order sum and difference operators as basic notions have been studied by several authors. Ever since, Kuttner [12] mentioned for the first time the FDEs in 1956, the theory of difference equations of fractional order has systematically evolved over the past decades [3, 9].

Qualitative analysis of these equations has gained momentum in recent years with numerous publications investigating the qualitative properties of the solutions to these equations [4, 6]. However, only two were papers published on the HU stability for discrete FBVPs of different orders with Caputo derivative [7, 8]. Inspired by the above discussions, in this paper, our objective is to discuss the HU stability of solutions for discrete FBVP of the form

(1.1)
$$\begin{cases} {}_{0}^{RL}\Delta_{\kappa}^{\sigma}\omega(\kappa) = \phi\left(\kappa+\sigma-1, \omega(\kappa+\sigma-1)\right), \\ \omega(\sigma-3) = 0,\ \Delta\omega(\sigma-3) = -\Delta^2\omega(\sigma-3), \Delta\omega(\sigma+B) = -\Delta^2\omega(\sigma+B), \end{cases}$$

for $\kappa \in [0, B+2]_{\mathbb{N}_0} = \{0, 1, ..., B+2\}$, where $\phi : [\sigma-2, \sigma+B]_{\mathbb{N}_{\sigma-2}} \times \mathbb{R} \to \mathbb{R}$ is continuous, ${}_{0}^{RL}\Delta_{\kappa}^{\sigma}$ is the Riemann - Liouville fractional difference operator with order $2 < \sigma \leq 3$ and $B \in \mathbb{N}_0$.

The paper is organized with basic concepts introduced in section 2. Ulam stability of solution of FBVP is established in section 3 and in section 4, we develop conditions for HU and HUR stability of solutions of (1.1). In section 5, suitable examples are presented as the applications of our results.

2. Background Materials

We recollect some well known useful definitions and lemmas needed for this study.

Definition 2.1. *(see [4,9]). Let $\sigma > 0$. The σ^{th} fractional sum of a function Ψ, is defined by*

$$\Delta^{-\sigma}\Psi(\kappa) = \frac{1}{\Gamma(\sigma)}\sum_{\ell=a}^{\kappa-\sigma}(\kappa-\ell-1)^{(\sigma-1)}\Psi(\ell),$$

for all $\kappa \in \mathbb{N}_{a+\sigma}$ and $\kappa^{(\sigma)} := \dfrac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\sigma)}$.

Lemma 2.1. *(see [4,9]). Let n and ρ be any number for which $n^{(\rho)}$ and $n^{(\rho-1)}$ are defined. Then $\Delta n^{(\rho)} = \rho n^{(\rho-1)}$.*

Lemma 2.2. *(see [4,9]). Let $0 \le L-1 < \rho \le L$. Then*

$${}_{0}^{RL}\Delta_n^{\rho}\Delta^{-\rho}\omega(n) = \omega(n) + B_1 n^{(\rho-1)} + B_2 n^{(\rho-2)} + \cdots + B_L n^{(\rho-L)},$$

for some $B_i \in \mathbb{R}$, where $i = 1, 2, ..., L$.

Lemma 2.3. *(see [6]) Let $\Psi : \mathbb{N}_{a+\sigma} \times \mathbb{N}_a \to \mathbb{R}$ be given. Then*

$$\Delta\left(\sum_{\ell=a}^{\kappa-\sigma}\Psi(\kappa,\ell)\right) = \sum_{\ell=a}^{\kappa-\sigma}\Delta_\kappa\Psi(\kappa,\ell) + \Psi(\kappa+1, \kappa+1-\sigma),$$

for all $\kappa \in \mathbb{N}_{a+\sigma}$.

Lemma 2.4. *(see [3]) Let $\mu \in \mathbb{R}\backslash\{..., -2, -1\}$. Then*

$$\Delta^{-\sigma}\kappa^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)}\kappa^{(\mu+\sigma)}.$$

3. SOLUTION TO DISCRETE FBVP

The Ulam stability of solution to a discrete FBVP (1.1) is established, if the solution exists.

Theorem 3.1. *Let* $2 < \sigma \leq 3$ *and* $\hbar : [\sigma - 2, \sigma + B]_{\mathbb{N}_{\sigma-2}} \times \mathbb{R} \to \mathbb{R}$ *be given. A function* $\omega(\kappa)$ *is a solution to a discrete FBVP*

(3.1)
$$\begin{cases} {}^{RL}_{0}\Delta^{\sigma}_{\kappa}\omega(\kappa) = \hbar(\kappa + \sigma - 1), \\ \omega(\sigma - 3) = 0,\ \Delta\omega(\sigma - 3) = -\Delta^2\omega(\sigma - 3),\ \Delta\omega(\sigma + B) = -\Delta^2\omega(\sigma + B), \end{cases}$$

for $\kappa \in [0, B + 2]_{\mathbb{N}_0}$*, if and only if* $\omega(\kappa)$*, for* $\kappa \in [\sigma - 3, \sigma + B]_{\mathbb{N}_{\sigma-3}}$ *has the form*

$$\begin{aligned} \omega(\kappa) = {} & \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa - g(\ell))^{(\sigma-1)}\hbar(\ell + \sigma - 1) \\ & + \frac{\beta A(\kappa)}{\Gamma(\sigma - 1)}\sum_{\ell=0}^{B+1}(\sigma + B - g(\ell))^{(\sigma-2)}\hbar(\ell + \sigma - 1) \\ & + \frac{\beta A(\kappa)}{\Gamma(\sigma - 2)}\sum_{\ell=0}^{B+2}(\sigma + B - g(\ell))^{(\sigma-3)}\hbar(\ell + \sigma - 1), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} & g(\ell) = \ell + 1,\ \beta = \frac{\Gamma(B + 5)}{\Gamma(\sigma + B + 1)\left[B(B + 4) + \sigma(B + 3) + 3\right]} \text{ and} \\ & A(\kappa) = \left[\frac{\kappa^{(\sigma-2)}}{(\sigma - 2)} - \frac{\kappa^{(\sigma-1)}}{(\sigma - 1)}\right]. \end{aligned} \tag{3.3}$$

Proof. Suppose that $\omega(\kappa)$ is a solution to (3.1). Applying Lemma 2.2, we obtain a general solution to (3.1) in the form

(3.4)
$$\omega(\kappa) = \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa - g(\ell))^{(\sigma-1)}\hbar(\ell + \sigma - 1) + D_1\kappa^{(\sigma-1)} + D_2\kappa^{(\sigma-2)} + D_3\kappa^{(\sigma-3)},$$

for some D_1, D_2, $D_3 \in \mathbb{R}$. By $\omega(\sigma - 3) = 0$ and from definition 2.1, we obtain $D_3 = 0$. Equation (3.4) becomes

$$\omega(\kappa) = \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa - g(\ell))^{(\sigma-1)}\hbar(\ell + \sigma - 1) + D_1\kappa^{(\sigma-1)} + D_2\kappa^{(\sigma-2)}. \tag{3.5}$$

Using Lemma 2.3 in (3.5), we obtain

$$\Delta\omega(\kappa) = \frac{1}{\Gamma(\sigma-1)} \sum_{\ell=0}^{\kappa-\sigma+1} (\kappa - g(\ell))^{(\sigma-2)} \hbar(\ell+\sigma-1) + D_1(\sigma-1)\kappa^{(\sigma-2)} + D_2(\sigma-2)\kappa^{(\sigma-3)},$$

$$\Delta^2\omega(\kappa) = \frac{1}{\Gamma(\sigma-2)} \sum_{\ell=0}^{\kappa-\sigma+2} (\kappa - g(\ell))^{(\sigma-3)} \hbar(\ell+\sigma-1) + D_1(\sigma-1)(\sigma-2)\kappa^{(\sigma-3)} + D_2(\sigma-2)(\sigma-3)\kappa^{(\sigma-4)}.$$

In view of boundary conditions $\Delta\omega(\sigma-3) = -\Delta^2\omega(\sigma-3)$ and $\Delta\omega(\sigma+B) = -\Delta^2\omega(\sigma+B)$, the values of D_1 and D_2 are as follows

$$D_1 = -\frac{\beta}{(\sigma-1)} \left(\frac{1}{\Gamma(\sigma-1)} \sum_{\ell=0}^{B+1} (\sigma+B-g(\ell))^{(\sigma-2)} + \frac{1}{\Gamma(\sigma-2)} \sum_{\ell=0}^{B+2} (\sigma+B-g(\ell))^{(\sigma-3)} \right) \hbar(\ell+\sigma-1), \text{ and}$$

$$D_2 = \frac{\beta}{(\sigma-2)} \left(\frac{1}{\Gamma(\sigma-1)} \sum_{\ell=0}^{B+1} (\sigma+B-g(\ell))^{(\sigma-2)} + \frac{1}{\Gamma(\sigma-2)} \sum_{\ell=0}^{B+2} (\sigma+B-g(\ell))^{(\sigma-3)} \right) \hbar(\ell+\sigma-1).$$

Substituting the values of D_1 and D_2 in (3.5), we obtain (3.2). This shows that if (3.1) is a solution, then it can be represented as (3.2).

Conversely, every function of the form (3.2) is a solution to (3.1). The proof is completed. □

Lemma 3.1. *(see [8]). If σ and κ are any number then*

(1) $\sum_{\ell=0}^{\kappa-\sigma} (\kappa - g(\ell))^{(\sigma-1)} = \frac{\Gamma(\kappa+1)}{\sigma\Gamma(\kappa-\sigma+1)}$.

(2) $\sum_{\ell=0}^{B+1} (\sigma+B-g(\ell))^{(\sigma-2)} = \frac{1}{\sigma-1}\frac{\Gamma(\sigma+B+1)}{\Gamma(B+2)}$.

(3) $\sum_{\ell=0}^{B+2} (\sigma+B-g(\ell))^{(\sigma-3)} = \frac{1}{\sigma-2}\frac{\Gamma(\sigma+B+1)}{\Gamma(B+3)}$.

4. Stability Analysis

In this section, a stability analysis is presented for the discrete FBVP (1.1). The following definitions for FDE are given on the basis of [7,8].

For computational convenience, let us consider the Banach space $\mathbb{E}$ with the norm $\|\omega\| = \max|\omega(\kappa)|$ for $\kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}$.

Definition 4.1. *If for every function $v(\kappa) \in \mathbb{E}$ of*

$$\left|{}_0^{RL}\Delta_\kappa^\sigma v(\kappa) - \phi(\kappa+\sigma-1, v(\kappa+\sigma-1))\right| \leq \delta, \tag{4.1}$$

for $\kappa \in [0, B+2]_{\mathbb{N}_0}$, where $\delta > 0$, there is a solution $\omega(\kappa) \in \mathbb{E}$ of (1.1) and positive constant $K > 0$ such that

$$|v(\kappa) - \omega(\kappa)| \leq K\delta, \quad \kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}.$$

Then the BVP (1.1) is HU stable.

Definition 4.2. *If for every function $v(\kappa) \in \mathbb{E}$ of*

$$\left|{}_0^{RL}\Delta_\kappa^\sigma v(\kappa) - \phi(\kappa+\sigma-1, v(\kappa+\sigma-1))\right| \leq \delta\Psi(\kappa+\sigma-1), \tag{4.2}$$

for $\kappa \in [0, B+2]_{\mathbb{N}_0}$, where $\delta > 0$, there is a solution $\omega(\kappa) \in \mathbb{E}$ of (1.1) and positive constant $K > 0$ such that

$$|v(\kappa) - \omega(\kappa)| \leq K\delta\Psi(\kappa), \quad \kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}.$$

Then the BVP (1.1) is HUR stable.

We assume the following:

- (H_1) Let us have constant $\eta > 0$, which satisfies $|\phi(\kappa, v) - \phi(\kappa, \omega)| \leq \eta\,|v-\omega|$, for all $v, \omega \in \mathbb{E}$ and κ in $[\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}$.
- (H_2) For a non decreasing function $\Psi \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}} \to \mathbb{R}^+$, there exists $\lambda > 0$ such that

$$\frac{\delta}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa - g(\ell))^{(\sigma-1)}\Psi(\ell+\sigma-1) \leq \lambda\delta\Psi(\kappa+\sigma-1), \quad \kappa \in [0, B+2]_{\mathbb{N}_0}.$$

Theorem 4.1. *Suppose that the hypothesis (H_1) together with the inequality (4.1) are satisfied, then a discrete FBVP (1.1) is HU stable provided that*

$$\eta < \frac{\Gamma(B+1)\Gamma(\sigma-1)}{\Gamma(\sigma+B+1)\left[\frac{1}{\sigma(\sigma-1)} + \frac{Q}{(B+1)}\left(\frac{1}{\sigma-1} + \frac{1}{B+2}\right)\right]}, \tag{4.3}$$

where $Q = \frac{\Gamma(B+5)}{[B(B+4)+\sigma(B+3)+3]}\left[\frac{1}{(\sigma-1)\Gamma(B+2)} + \frac{1}{(\sigma-2)\Gamma(B+3)}\right]$.

Proof. From inequality (4.1), for $\kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}$, we can find a function ${}_0^{RL}\Delta_\kappa^\sigma v(\kappa) = \phi(\kappa+\sigma-1, v(\kappa+\sigma-1)) + f(\kappa+\sigma-1)$ and $|f(\kappa+\sigma-1)| \leq \delta$. It follows that

$$\begin{aligned}
&\Bigg| v(\kappa) - \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)}\phi(\ell+\sigma-1, v(\ell+\sigma-1)) \\
&\quad - \frac{\beta A(\kappa)}{\Gamma(\sigma-1)}\sum_{\ell=0}^{B+1}(\sigma+B-g(\ell))^{(\sigma-2)}\phi(\ell+\sigma-1, v(\ell+\sigma-1)) \\
&\quad - \frac{\beta A(\kappa)}{\Gamma(\sigma-2)}\sum_{\ell=0}^{B+2}(\sigma+B-g(\ell))^{(\sigma-3)}\phi(\ell+\sigma-1, v(\ell+\sigma-1))\Bigg| \\
&\qquad\qquad \leq \frac{\delta}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)} \\
&\qquad\qquad \leq \frac{\delta\Gamma(\sigma+B+1)}{\Gamma(\sigma+1)\Gamma(B+1)}. \qquad (4.4)
\end{aligned}$$

With the help of solution (3.2) and (4.4), for $\kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}$, we have

$$\begin{aligned}
|v(\kappa) - \omega(\kappa)| \leq &\Bigg| v(\kappa) - \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)}\phi(\ell+\sigma-1, v(\ell+\sigma-1)) \\
&- \frac{\beta A(\kappa)}{\Gamma(\sigma-1)}\sum_{\ell=0}^{B+1}(\sigma+B-g(\ell))^{(\sigma-2)}\phi(\ell+\sigma-1, v(\ell+\sigma-1)) \\
&- \frac{\beta A(\kappa)}{\Gamma(\sigma-2)}\sum_{\ell=0}^{B+2}(\sigma+B-g(\ell))^{(\sigma-3)}\phi(\ell+\sigma-1, v(\ell+\sigma-1))\Bigg| \\
&+ \frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)} \\
&\quad \times |\phi(\ell+\sigma-1, v(\ell+\sigma-1)) - \phi(\ell+\sigma-1, \omega(\ell+\sigma-1))| \\
&+ \frac{\beta\,|A(\kappa)|}{\Gamma(\sigma-1)}\sum_{\ell=0}^{B+1}(\sigma+B-g(\ell))^{(\sigma-2)} \\
&\quad \times |\phi(\ell+\sigma-1, v(\ell+\sigma-1)) - \phi(\ell+\sigma-1, \omega(\ell+\sigma-1))|
\end{aligned}$$

$$+\frac{\beta\,|A(\kappa)|}{\Gamma(\sigma-2)}\sum_{\ell=0}^{B+2}(\sigma+B-g(\ell))^{(\sigma-3)}$$
$$\times\left|\phi(\ell+\sigma-1,v(\ell+\sigma-1))-\phi(\ell+\sigma-1,\omega(\ell+\sigma-1))\right|$$
$$|v(\kappa)-\omega(\kappa)|\leq\frac{\delta\Gamma(\sigma+B+1)}{\Gamma(\sigma+1)\Gamma(B+1)}+\frac{\eta\,\|v-\omega\|}{\Gamma(\sigma)}\left[\frac{\Gamma(\kappa+1)}{\sigma\Gamma(\kappa+1-\sigma)}\right]$$
$$+\frac{\eta\beta\,\|v-\omega\|\,|A(\kappa)|}{\Gamma(\sigma-1)}\left[\frac{\Gamma(\sigma+B+1)}{(\sigma-1)\Gamma(B+2)}\right]+\frac{\eta\beta\,\|v-\omega\|\,|A(\kappa)|}{\Gamma(\sigma-2)}\left[\frac{\Gamma(\sigma+B+1)}{(\sigma-2)\Gamma(B+3)}\right].$$

Then

$$\|v(\kappa)-\omega(\kappa)\|\leq\frac{\delta\Gamma(\sigma+B+1)}{\Gamma(\sigma+1)\Gamma(B+1)}$$
$$+\,\eta\left[\frac{\Gamma(\sigma+B+1)}{\Gamma(\sigma+1)(B+1)}+\frac{Q\Gamma(\sigma+B+1)}{\Gamma(\sigma)\Gamma(B+2)}+\frac{Q\Gamma(\sigma+B+1)}{\Gamma(\sigma-1)\Gamma(B+3)}\right]\|v-\omega\|\,.$$

From the above inequality, we have

$$\|v-\omega\|\leq\frac{\Gamma(\sigma+B+1)}{\left[\Gamma(\sigma+1)\Gamma(B+1)-\eta\Gamma(\sigma+B+1)\left(1+\frac{Q\sigma}{(B+1)}\left[1+\frac{(\sigma-1)}{(B+2)}\right]\right)\right]}\delta;$$

where $\dfrac{\Gamma(\sigma+B+1)}{\left[\Gamma(\sigma+1)\Gamma(B+1)-\eta\Gamma(\sigma+B+1)\left(1+\frac{Q\sigma}{(B+1)}\left[1+\frac{(\sigma-1)}{(B+2)}\right]\right)\right]}>0.$ Thus BVP (1.1) is HU stable. □

Theorem 4.2. *If the hypotheses (H_1), (H_2) and the inequality (4.3) are satisfied, then a discrete FBVP (1.1) is HUR stable.*

Proof. From inequality (4.2), for $\kappa\in[\sigma-3,\sigma+B]_{\mathbb{N}_{\sigma-3}}$, it follows that

$$\Bigg|v(\kappa)-\frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)}\phi(\ell+\sigma-1,v(\ell+\sigma-1))$$
$$-\frac{\beta A(\kappa)}{\Gamma(\sigma-1)}\sum_{\ell=0}^{B+1}(\sigma+B-g(\ell))^{(\sigma-2)}\phi(\ell+\sigma-1,v(\ell+\sigma-1))$$
$$-\frac{\beta A(\kappa)}{\Gamma(\sigma-2)}\sum_{\ell=0}^{B+2}(\sigma+B-g(\ell))^{(\sigma-3)}\phi(\ell+\sigma-1,v(\ell+\sigma-1))\Bigg|$$
$$\leq\frac{\delta}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)}\Psi(\ell+\sigma-1)$$
$$\leq\lambda\delta\Psi(\kappa+\sigma-1). \tag{4.5}$$

Combining (3.2) and (4.5), for $\kappa \in [\sigma-3, \sigma+B]_{\mathbb{N}_{\sigma-3}}$, we obtain

$$|v(\kappa)-\omega(\kappa)| \leq \lambda\delta\Psi(\kappa+\sigma-1) + \frac{\eta\|v-\omega\|}{\Gamma(\sigma)}\left[\frac{\Gamma(\kappa+1)}{\sigma\Gamma(\kappa+1-\sigma)}\right]$$
$$+\frac{\eta\beta\|v-\omega\||A(\kappa)|}{\Gamma(\sigma-1)}\left[\frac{\Gamma(\sigma+B+1)}{(\sigma-1)\Gamma(B+2)}\right]+\frac{\eta\beta\|v-\omega\||A(\kappa)|}{\Gamma(\sigma-2)}\left[\frac{\Gamma(\sigma+B+1)}{(\sigma-2)\Gamma(B+3)}\right].$$

Then

$$\|v(\kappa)-\omega(\kappa)\| \leq \lambda\delta\Psi(\kappa+\sigma-1)$$
$$+\eta\left[\frac{\Gamma(\sigma+B+1)}{\Gamma(\sigma+1)(B+1)}+\frac{Q\Gamma(\sigma+B+1)}{\Gamma(\sigma)\Gamma(B+2)}+\frac{Q\Gamma(\sigma+B+1)}{\Gamma(\sigma-1)\Gamma(B+3)}\right]\|v-\omega\|.$$

From the above inequality, we have

$$\|v-\omega\| \leq \frac{\lambda\delta\Psi(\kappa+\sigma-1)\Gamma(\sigma-1)\Gamma(B+1)}{\left[\Gamma(\sigma-1)\Gamma(B+1)-\eta\Gamma(\sigma+B+1)\left(\frac{1}{\sigma(\sigma-1)}+\frac{Q}{(B+1)}\left[\frac{1}{(\sigma-1)}+\frac{1}{(B+2)}\right]\right)\right]};$$

where,

$$\frac{\lambda\Gamma(\sigma-1)\Gamma(B+1)}{\left[\Gamma(\sigma-1)\Gamma(B+1)-\eta\Gamma(\sigma+B+1)\left(\frac{1}{\sigma(\sigma-1)}+\frac{Q}{(B+1)}\left[\frac{1}{(\sigma-1)}+\frac{1}{(B+2)}\right]\right)\right]} > 0.$$

Thus BVP (1.1) is HUR stable. □

5. Applications

From the applications of our results, we consider the following examples.

Example 1. *Suppose that $B=0$ with fractional orders $\sigma=2.38$, $\sigma=2.39$ and $\sigma=2.4$. Let $\hbar(\kappa)=\kappa^{(3)}$. Then discrete FBVP (3.1) becomes*

$$\begin{cases} {}_0^{RL}\Delta_\kappa^\sigma\omega(\kappa)=(\kappa+\sigma-1)^{(3)},\ \kappa\in[0,2]_{\mathbb{N}_0}, \\ \omega(\sigma-3)=0,\ \Delta\omega(\sigma-3)=-\Delta^2\omega(\sigma-3),\ \Delta\omega(\sigma)=-\Delta^2\omega(\sigma). \end{cases} \tag{5.1}$$

Using Definition 2.1 and Lemma 2.4. we obtain

$$\frac{1}{\Gamma(\sigma)}\sum_{\ell=0}^{\kappa-\sigma}(\kappa-g(\ell))^{(\sigma-1)}\hbar(\ell+\sigma-1)^{(3)}=\Delta^{-\sigma}(\kappa+\sigma-1)^{(3)}=\frac{\Gamma(4)}{\Gamma(4+\sigma)}\cdot\frac{\Gamma(\kappa+\sigma)}{\Gamma(\kappa-3)}.$$

Similarly, we find

$$\frac{1}{\Gamma(\sigma-1)}\sum_{\ell=0}^{B+1}(\sigma+B-g(\ell))^{(\sigma-2)}\hbar(\ell+\sigma-1)^{(3)}=\frac{\Gamma(4)}{\Gamma(3+\sigma)}\cdot\frac{\Gamma(2\sigma)}{\Gamma(\sigma-2)} \text{ and}$$

$$\frac{1}{\Gamma(\sigma-2)}\sum_{\ell=0}^{B+2}(\sigma+B-g(\ell))^{(\sigma-3)}\hbar(\ell+\sigma-1)^{(3)}=\frac{\Gamma(4)}{\Gamma(2+\sigma)}\cdot\frac{\Gamma(2\sigma)}{\Gamma(\sigma-1)}.$$

The analytical solution of (5.1) is

$$\omega(\kappa)=\frac{\Gamma(4)}{\Gamma(4+\sigma)}\left[\frac{\Gamma(\kappa+\sigma)}{\Gamma(\kappa-3)}\right] +\beta A(\kappa)\left[\frac{\Gamma(4)}{\Gamma(3+\sigma)}\cdot\frac{\Gamma(2\sigma)}{\Gamma(\sigma-2)}+\frac{\Gamma(4)}{\Gamma(2+\sigma)}\cdot\frac{\Gamma(2\sigma)}{\Gamma(\sigma-1)}\right],$$

where β and $A(\kappa)$ are defined as in (3.3). The analytical solutions to different fractional orders σ are shown in Figure 1 & Table 1.

κ	$\omega(\kappa)_{\sigma=2.38}$	$\omega(\kappa)_{\sigma=2.39}$	$\omega(\kappa)_{\sigma=2.4}$	κ	$\omega(\kappa)_{\sigma=2.38}$	$\omega(\kappa)_{\sigma=2.39}$	$\omega(\kappa)_{\sigma=2.4}$
0	20.4270	19.7844	19.1674	1.2	24.3601	23.8697	23.4060
0.4	23.7967	23.2264	22.6815	1.6	22.8293	22.3502	21.8984
0.8	24.8040	24.2838	23.7896	2	20.3962	19.9112	19.4541

TABLE 1. The analytical solutions to Example 1.

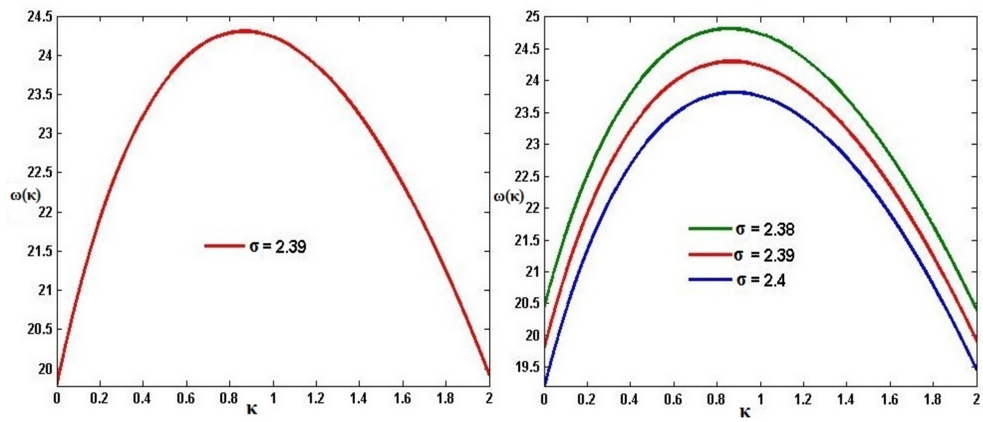

FIGURE 1. The analytical solutions to Example 1.

Example 2. *Suppose that* $\sigma = \frac{5}{2}$ *and* $B = 2$ *with* $\phi(\kappa, \omega) = \dfrac{\cos \omega(\kappa)}{(110+\kappa^2)}$. *Then discrete FBVP (1.1) becomes*

$$
\begin{cases} {}_0^{RL}\Delta_\kappa^{\frac{5}{2}}\omega(\kappa) = \dfrac{\cos \omega\left(\kappa+\frac{3}{2}\right)}{\left[110+\left(\kappa+\frac{3}{2}\right)^2\right]}, \ \kappa \in [0,4], \\ \omega\left(-\frac{1}{2}\right)=0, \ \Delta\omega\left(-\frac{1}{2}\right) = -\Delta^2\omega\left(-\frac{1}{2}\right), \ \Delta\omega\left(\frac{9}{2}\right) = -\Delta^2\omega\left(\frac{9}{2}\right). \end{cases} \tag{5.2}
$$

Since

$$
\frac{\Gamma(\sigma-1)\Gamma(B+1)}{\Gamma(\sigma+B+1)\left[\frac{1}{\sigma(\sigma-1)}+\frac{Q}{(B+1)}\left(\frac{1}{\sigma-1}+\frac{1}{B+2}\right)\right]} \approx 0.0186.
$$

If $\eta = 0.0091 < 0.0186$ *and the inequality*

$$
\left| {}_0^{RL}\Delta_\kappa^{\frac{5}{2}} v(\kappa) - \phi\left(\kappa+\frac{3}{2}, v\left(\kappa+\frac{3}{2}\right)\right)\right| \leq \delta, \quad \kappa \in [0,4]_{\mathbb{N}_0},
$$

hold, then the BVP (5.2) is the HU stable according to Theorem 4.1.

6. Conclusion

The solutions of the discrete FBVP are established with appropriate conditions for different kinds of Ulam stability. For justification, we have presented suitable examples with different solutions to fractional order σ varies in $(2,3]$, which support the efficiency of theoretical results.

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