Advances in Mathematics: Scientific Journal **9** (2020), no.1, 253–265
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.1.21

NEW NANO OPERATOR FOR A NEW NANO TOPOLOGY

HAKEEM A. OTHMAN[1]

ABSTRACT. In this paper, we will use a nano-g-open (nano-g-closed) subset of a nano topological space $(U, \tau_R(X))$ to introduce two new nano operators called nano-H-Closure ($N-H-Cl$ for short) and nano-H-Interior ($N-H-Int$ for short). Using nano-H-Interior we shall define a new nano topological space, namely the nano-H-topological space ($(U, H-\tau_R(X))$ for short).

Moreover, we shall discuss and examiner the fundamental properties of these new nano operators and new nano topology, with emphasis on the transfer of nano regularity conditions on $(U, \tau_R(X))$ to nano separations axiom conditions on $(U, H-\tau_R(X))$. Some applications will be investigated.

1. INTRODUCTION

The notion of $generalized-closed$ sets was introduced in 1970 by Levine in [6], Thivagar in [7,8] introduced nano topological spaces and nano-continuous mapping, and Bhuvaneswari in [1] defined the notion of nano-generalized closed sets in nano topological spaces. Othman in [2–5] has defined new operator in General Topology and Fuzzy Topology and Thivagar introduced nano topological spaces based on the concept of lower approximation, upper approximation, and boundary regions. Some paper such as [7, 9] show that the concept of nano topology can be used as a tool to study certain real-life problems. A nano topology has at least three and at most five nano-open sets

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 54B05, 54C05, 54A10.
Key words and phrases. nano-g-closed, nano topological space, nano-H-Closure, nano-H-Interior, nano-H-topological space, nano separations axioms.

including U, ϕ. So if we want to add some more nano-open sets, we must introduce two new nano operators called nano-H-Closure ($N-H-Cl$ for short) and nano-H-Interior ($N-H-Int$ for short) in order to define a new nano topology finer than the nano topological space. This is called a nano-H-topological space ($(U, H-\tau_R(X))$ for short). Various results related to these new nano operators and nano topology are discussed in this paper.

2. Preliminaries

Throughout this paper U is finite universes and non-empty set; $X \subseteq U$ and $U \backslash R$ denote the families of equivalence classes by equivalence relations R on U and $(U, \tau_R(X))$ or simply by $\tau_R(X)$ we mean a nano topological space. If $\lambda \subseteq U$ then, $N - Int\ \lambda$, $N - cl\ \lambda$ and λ^c denote respectively, the nano-interior of λ, the nano-closure of λ and complement of λ.

Definition 2.1. *[7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.*

(i) $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$ *is the lower approximation of X with respect to R , where $R(x)$ denotes the equivalence class determined by x.*
(ii) $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$ *is the upper approximation of X with respect to R.*
(iii) $B_R(X) = U_R(X) - L_R(X)$ *is the boundary region of X with respect to R.*

Definition 2.2. *[7] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$, then $\tau_R(X)$ satisfies the following conditions:*

(i) ϕ *and* $U \in \tau_R(X)$.
(ii) *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*
(iii) *The union of the elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

That is, $\tau_R(X)$ is a topology on U called the nano-topology on U with respect to X. We call $(U, \tau_R(X))$ as the nano-topological space. The elements of $\tau_R(X)$ are called as nano-open sets and the complement of it are called nano-closed sets.

Definition 2.3. *Let $(U, \tau_R(X))$ be a nano-topological space and the set $\mu \in U$ is called:*

(a) $Nano - generalized - closed$ *[1](n-g-closed set for short) if $cl(\mu) \subseteq \lambda$ whenever $\mu \subseteq \lambda$ and λ is nano-open set.*
(b) $Nano - \alpha - open$ $\big($*resp.* $Nano - \alpha - closed$ $\big)$ *[10] if $\mu \subseteq N - Int(N - Cl(N - Int(\mu)))$* $\big($*resp.* $N - Cl(N - Int(N - Cl(\mu))) \subseteq \mu$ $\big)$.
(c) $Nano - pre - open$ *(resp.* $Nano - pre - closed$ *[10] if $\mu \subseteq N - Int(N - Cl(\mu))$ $\big)$ if* $\big($*resp.* $N - Cl(N - Int(\mu)) \subseteq \mu$ $\big)$.
(d) $Nano - semi - open$ $\big($*resp.* $Nano - semi - closed$ $\big)$ *[10] if $\mu \subseteq N - Cl(N - Int(\mu))$* $\big($*resp.* $N - Cl(N - Int(\mu)) \subseteq \lambda$ $\big)$.
(e) $Nano - \beta - open$ $\big($*resp.* $Nano - \beta - closed$ $\big)$ *[11] if $\mu \subseteq N - Cl(N - Int(N - Cl(\mu)))$* $\big($*resp.* $N - Int(N - Cl(N - Int(\mu))) \subseteq \mu$ $\big)$.

Definition 2.4. *A nano mapping $f : (U, \tau_R(X))) \to (V, \tau_R(Y)))$ is said to be:*

(a) $nano - continuous$ *[8] if the inverse image of each* $nano - open$ *set μ_1 in V is* $nano - open$ *set in U.*
(b) $nano - pre - continuous$ *[10] if the inverse image of each* $nano - open$ *set μ_1 in V is* $nano - pre - open$ *set in U.*
(c) $nano - semi - continuous$ *[10] if the inverse image of each* $nano - open$ *set μ_1 in V is* $nano - semi - open$ *set in U.*
(d) $nano - \alpha - continuous$ *[10] if the inverse image of each* $nano - open$ *set μ_1 in V is* $nano - \alpha - open$ *set in U.*

Definition 2.5. *A Nano topological space $(U, \tau_R(X))$ is called:*

- $Nano - T_0$ *if for all $x \neq y$, there exists an nano-open set $\mu \subseteq U$ such that either $x \in \mu$ and $y \notin \mu$, or $y \in \mu$ and $x \notin \mu$.*
- $Nano - T_1$ *if for all $x \neq y$, there exist an nano-open set $\mu_1,\ \mu_2 \subseteq U$ such that $x \in \mu_1$, $y \notin \mu_1$, and $y \in \mu_2$, $x \notin \mu_2$.*
- $Nano - T_2$ *if for all $x \neq y$, there exist a disjoint nano-open sets $\mu_1,\ \mu_2 \subseteq U$ such that $x \in \mu_1$ and $y \in \mu_2$.*
- *Nano-regular space if for all λ is nano-closed set in U and $x \in U$ such that $x \notin \lambda$, there exist a disjoint nano-open sets $\mu_1,\ \mu_2 \subseteq U$ such that $x \in \mu_1$ and $\lambda \subseteq \mu_2$.*
- $Nano - T_3$ *if $(U, \tau_R(X))$ is* $nano - T_1$ *and nano-regular space.*

- *Nano-Normal space if for all* λ_1 *and* λ_2 *are disjoint nano-closed sets in* U*, there exist a disjoint nano-open sets* $\mu_1,\ \mu_2 \subseteq U$ *such that* $\lambda_1 \subseteq \mu_1$ *and* $\lambda_2 \subseteq \mu_2$*.*
- $Nano - T_4$ *if* $(U, \tau_R(X))$ *is* $nano - T_1$ *and nano-normal space.*

3. New Nano Operator and Nano-H- Topological Space

Definition 3.1. *Let* $(U, \tau_R(X))$ *is a nano topological space,* λ *and* υ *any set in* U*. Then,*

- $Nano - H - Cl\lambda = \bigcap\{\upsilon : \lambda \subseteq\ \upsilon,\ \upsilon\ is\ a\ nano\ g - closed\ set\ of\ U\}$ *is called nano H- closure .*
- $Nano - H - Int\lambda = \bigcup\{\upsilon : \upsilon \subseteq\ \lambda,\ \upsilon\ is\ a\ nano\ g - open\ set\ of\ U\}$ *is called nano H- interior.*

Now we will introduce new nano topology generated by this new operator $nano - H - Int$.

Definition 3.2. *Let* $(U, \tau_R(X))$ *be a nano topological space. The new nano topology generated by* $(nano - H - Int)$*.*

That is, $nano - H - \tau_R(X) = \{\lambda \subseteq U : nano - H - Int(\lambda) = \lambda\}$.

The member of $nano - H - \tau_R(X)$ *is called nano-H-open set, the complement of it is called nano-H-closed set and the family of all nano-H-open (resp. nano-H-closed) sets can be denoted by* $(nano{-}H{-}O(U, X))$ *(resp.* $(nano{-}H{-}C(U, X))$*).*

Definition 3.3. *In a* $(U, \tau_R(X))$*, a subset* λ *is a nano-H-open set if* $\lambda \subseteq nano - Int(\mu)$ *where* $\lambda \subseteq \mu$ *and* $\lambda \in \tau_R^c(X)$*.*

With this new nano topology $(U, H - \tau_R(X))$ we can do something else that we can not do it with nano topology.

Remark 3.1.

(i) *The members of* $nano - H - \tau_R(X)$ *are closed under the intersection properties and the complement (* $H - \tau_R^c(X)$*) are closed under the union properties.*

(ii) *The union of member of* $nano{-}H{-}\tau_R(X)$ *is not be member of* $H{-}\tau_R(X)$ *in general.*

(iii) *The intersection of member of* $H - \tau_R^c(X)$ *is not be member of* $H - \tau_R^c(X)$ *in general.*

(iv) $\tau_R(X) \subseteq H - \tau_R(X)$ *and* $\tau_R^c(X) \subseteq H - \tau_R^c(X)$.

The following example we can clarify the Remark 3.1.

Example 1. *Let* $U = \{1, 2, 3\}$ *and* $X = \{3\}$ *with* $U/R = \{\{3\}\}$*, then*

- $\tau_R(X) = \{\phi, \{3\}, U\}$*, then* $H - \tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, U\}$ *we can see that* $\tau_R(X) \subseteq H - \tau_R(X)$*. If* $\lambda_1 = \{1\}$ *and* $\lambda_2 = \{2\}$ *are nano-H-open sets but* $\lambda_1 \cup \lambda_2 = \{1, 2\}$ *is not nano-H-open set.*
- $\tau_R^c(X) = \{\phi, \{1, 2\}, U\}$*, then* $H - \tau_R^c(X) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, U\}$ *we can see that* $\tau_R^c(X) \subseteq H - \tau_R^c(X)$*. If* $\mu_1 = \{1, 3\}$ *and* $\mu_2 = \{3\}$ *are nano-H-closed sets but* $\mu_1 \cap \mu_2 = \{3\}$ *is not nano-H-closed set.*

We shall add some properties of these new operators.

Proposition 3.1. *Let* λ*,* ν *and* μ *subset from* U *and* $(U, H - \tau_R(X))$ *nano generalize topological space, then*

(i) $nano-H-Int(\phi) = \phi, nano-H-Int(U) = U$ *and* $nano-H-Int(\lambda) \subseteq \lambda$.

(ii) $nano-H-Cl(\phi) = \phi, nano-H-Cl(U) = U$ *and* $\lambda \subseteq nano-H-Cl(\lambda)$.

(iii) *If* λ *is nano-H-open set, then* $nano - H - Int(nano - H - Int(\lambda)) = \lambda$.

(iv) *If* λ *is nano-H-closed set, then* $nano - H - Cl(nano - H - Cl(\lambda)) = \lambda$.

(v) $nano - H - Int(\lambda \cap \mu) = nano - H - Int(\lambda) \cap nano - H - Int(\mu)$.

(vi) $nano - H - Cl(\lambda \cup \mu) = nano - H - Cl(\lambda) \cup nano - H - Cl(\mu)$.

Proof. We will prove only for $nano - H - Int$.

(i) $nano-H-Int(\phi) = \phi, nano-H-Int(U) = U$ and $nano-H-Int(\lambda) \subseteq \lambda$, these directe from (definition 3.1)

(ii) If $\lambda \subseteq \mu,\ \mu \in nano - H - O(U, X)$, then $nano - H - Int(\lambda) \subseteq \mu$ and $nano - H - Int(nano - H - Int(\lambda)) \subseteq \mu$ by (definition 3.1). Then, $nano - H - Int(nano - H - Int(\lambda)) \subseteq \cup\{\mu : \mu \subseteq \lambda : \mu \in nano - H - O(U, X)\} = nano - H - Int(\lambda)$.

(iii) Here we have two cases:

Case (1) If $\lambda \cap \mu \subseteq \nu,\ \nu \in nano - H - O(U, X)$, then $\lambda \subseteq \nu$.
Thus, $nano - H - Int(\lambda) \cap nano - H - Int(\mu) \subseteq \{\nu : \lambda \cap \mu \subseteq \nu \in nano - H - O(U, X)\} = nano - H - Int(\lambda \cap \mu)$.

Case (2) If there is an $x \in nano - H - Int(\lambda \cap \mu)$ with $x \notin nano - H - Int(\lambda) \cap nano - H - Int(\mu)$, then there are nano-H-open set ν_1 and ν_2 with $\lambda \subseteq \nu_1, \mu \subseteq \nu_2$ and $x \notin \lambda \cap \mu$. But $\lambda \cap \mu \subseteq \nu_1 \cap \nu_2 \in nano-H-O(U, X)$, but we know that the intersection of two nano-H-open sets is nano-H-open set contradicting !. Then, $x \in \lambda \cap \mu$.

From case (1) and case (2) we conclude that $nano - H - Int(\lambda \cap \mu) = nano - H - Int(\lambda) \cap nano - H - Int(\mu)$.

□

Remark 3.2. *For any nano subset of U, nano-topological space $(U, \tau_R(X))$ and nano-H-topological space $(U, H - \tau_R(X))$*

(i) *The nano-Closure is nano-closed in $(U, \tau_R(X))$ but nano-H-Closure is not nano-H-closed set in general in $(U, H - \tau_R(X))$.*

(ii) *The nano-Interior is nano-open set in $(U, \tau_R(X))$ but nano-H-Interior is not nano-H-open set in general in $(U, H - \tau_R(X))$.*

By the following example, we can see that

Example 2. *Let $U = \{1, 2, 3\}$ and $X = \{1, 3\}$ with $U/R = \{\{2\}\}$, then $\tau_R(X) = \{\phi, \{2\}, U\}$ and $H - \tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, U\}$ so*

- *we can see that if $\lambda = \{1, 3\} \Rightarrow nano - Int(\lambda) = \phi \in \tau_R(X)$ but $nano - H - Int(\lambda) = \lambda \notin H - \tau_R(X)$.*
- *also we can see that if $\mu = \{1, 2\} \Rightarrow nano - Cl(\mu) = X \in \tau_R(X)$ but $nano - H - Cl(\mu) = \mu \notin H - \tau_R(X)$.*

4. Some Applications

Theorem 4.1. *For any $(U, \tau_R(X))$ and $x \in U$, either $\{x\}$ is nano-H-open set or $\{x\}^c$ is nano-open set.*

Proof. If $\{x\}^c$ is not nano-open set, then the nano closed superset in $\{x\}$ is ϕ. or $\{x\}$ itself so we have two cases:

Case (1): If the nano-closed superset in $\{x\}$ is $\{x\}$, then $\{x\}$ is nano-closed set and $\{x\}^c$ is open contradicting !

Case (2): If the nano-closed superset in $\{x\}$ is ϕ., hence the nano-Interior of $\{x\}$ is containing each nano-closed subset of $\{x\}$, then $\{x\}$ is nano-H-open set.

□

Proposition 4.1. *For any $(U, \tau_R(X))$ and $x \in U$, either $\{x\}$ is nano-closed set or $\{x\}^c$ is nano-H-closed set.*

Definition 4.1. *A nano topological space $(U, \tau_R(X))$ is said to be a nano $\tau_{\frac{1}{2}} - space$ $(nano - \tau_{\frac{1}{2}}\ for\ short)$ if every nano-H-open set is a nano-open set.*

We know that every $(nano - \tau_1 - space)$ is $(nano - \tau_{\frac{1}{2}} - space)$ and every $(nano - \tau_{\frac{1}{2}} - space)$ is $(nano - \tau_0 - space)$,then easy to prove the following theorem.

Theorem 4.2. *Let $(U, \tau_R(X))$ a nano topological space, then the following statements are equivalence.*

(i) $(U, \tau_R(X))$ *is* $(nano - \tau_{\frac{1}{2}} - space)$.
(ii) *Every singleton set in $\tau_R(X)$ is either $nano - open$ or $nano - closed$ set.*
(iii) *Every subset of U is the intersection of all nano-open sets and all nano closed sets containing it.*

Remark 4.1. *Let $(U, \tau_R(X))$ a nano topological space and λ subset from U, then we know that*

(i) *nano-open set is nano-H-open set, then if $\lambda \subseteq U$, then $nano - Int(\lambda) \subseteq nano - H - Int(\lambda) \subseteq \lambda$.*
(ii) *nano-closed set is nano-H-closed set, then if $\lambda \subseteq U$, then $\lambda \subseteq nano - H - Cl(\lambda) \subseteq nano - Cl(\lambda)$.*

We can illustrate that by following example.

Example 3. *Let $U = \{1, 2, 3\}$ and $X = \{1, 3\}$ with $U/R = \{\{1\}, \{2, 3\}\}$, then*

- *$\tau_R(X) = \{\phi, \{1\}, \{2, 3\}, U\}$, then $H - \tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, U\}$ we can see that if $\lambda = \{2\} \Rightarrow nano - Int(\lambda) = \phi$ but $nano - H - Int(\lambda) = \lambda$. So, it is clear that $nano - Int(\lambda) \subseteq nano - H - Int(\lambda) \subseteq \lambda$.*
- *$\tau_R^c(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, U\}$ we can see that if $\lambda = \{1, 3\} \Rightarrow nano - Cl(\lambda) = U$ but $nano - H - Int(\lambda) = \lambda$. So, it is clear that $\lambda \subseteq nano - H - Cl(\lambda) \subseteq nano - Cl(\lambda)$.*

Theorem 4.3. *$(U, \tau_R(X))$ is $nano - \tau_{\frac{1}{2}}$ if and only if $\tau_R(X) \subseteq H - \tau_R(X)$ with equality.*

Proof. If $(U, \tau_R(X))$ is $nano - \tau_{\frac{1}{2}}$ by (Definition 4.1), for any set $\lambda \subseteq U \Rightarrow nano - Int(\lambda) = nano - H - Int(\lambda)$. Hence, $\tau_R(X) = H - \tau_R(X)$.

Conversely, If λ is nano-open set and $nano - Int(\lambda) \subseteq nano - H - Int(\lambda) \subseteq \lambda$ that implies λ is nano-H-closed set. Thus, $\tau_R(X) \subseteq H - \tau_R(X)$. Further, if we assume that $\tau_R(X) = H - \tau_R(X)$ and let $\mu \subseteq U$ be nano-H-open set in $(U, \tau_R(X))$. Then, $\mu = nano - H - Int(\mu)$ and so μ is nano-open set in $\tau_R(X) = H - \tau_R(X)$, thus $(U, \tau_R(X))$ is $nano - \tau_{\frac{1}{2}} - space$. □

Theorem 4.4. *For any* $(U, \tau_R(X))$ *and* $x \neq y$*, then* $nano - H - Cl(x) \neq nano - H - Cl(y)$.

Proof. If $\{x\}$is nano-closed set, $y \notin \{x\} = nano - Cl\{x\} = nano - H - Cl\{x\}$. Otherwise, $y \in \{x\}^c$, a nano-H-closed set by (Theorem 4.1). Then, $y \in nano - H - Cl\{y\} \subseteq \{x\}^c$, and so $x \notin nano - H - Cl\{y\}$. □

Proposition 4.2. *For any* $(U, H - \tau_R(X))$ *is* $nano - T_0 - space$.

Now we can get interesting theorems.

Theorem 4.5. *If any space is* $(U, \tau_R(X))$, $(U, H - \tau_R(X))$*, then it is* $nano - T_{\frac{1}{2}} - space$.

Proof. If $\{x\}$ is nano-closed and nano-H-closed set as well. Otherwise, $\{x\}^c$ is nano-H-closed and $nano - H - Cl(\{x\}^c) = \{x\}^c$, that implies $\{x\}$ is nano-H-open set. By (Theorem 4.2) $(U, H - \tau_R(X))$ is $nano - T_{\frac{1}{2}} - space$. □

Theorem 4.6. *If* $(U, \tau_R(X))$ *is nano regular space, then* $(U, H - \tau_R(X))$ *is* $nano - \tau_3 - space$ *(nano regular space +* $nano - \tau_1 - space$*.)*

Proof. By (Theorem 4.3) $(U, H - \tau_R(X))$ is $nano - \tau_{\frac{1}{2}}$ so it is enough to prove that $(U, H - \tau_R(X))$ is nano regular space.

Let $x \notin \lambda \in H - \tau_R^c(X)$. We have to study two possibility.

(i) Suppose $\{x\} \in \tau_R^c(X)$ since $x \in \lambda = nano - H - Cl(\lambda)$, $x \notin \mu$ for some $\lambda \subseteq \mu$ with μ nano-H-closed in $(U, \tau_R(X))$. But then $\mu \subseteq \{x\}^c \in nano - \tau_R(X)$ and also $nano - cl(\mu) \subseteq \{x\}$. Thus, $x \notin nano - Cl(\mu)$ in the nano-regular space $(U, \tau_R(X))$, so there are disjoint nano-open set U_1 and U_2 in $\tau_R(X)$ such that $x \in U$ and $nano - Cl(\mu) \subseteq U_2$. Hence $x \in U_1 \in H - \tau_R(X)$ and $\lambda \subseteq \mu \subseteq nano - Cl(\mu) \subseteq U_2 \in H - \tau_R(X)$, with $U_1 \cap U_2 = \phi$.

(ii) Suppose now that $\{x\}$ is not nano-closed by Theorem 4.1 $\{x\}^c$ is nano-H-closed and $\{x\} \in H-\tau_R(X)$. We are sure that $\lambda \subseteq (nano-H-Cl(x))^c$ and thus choose $y \in \lambda$ arbitrary. Now we will discuss two cases:
- If $\{x\}$ is nano-closed, $x \notin \{y\}$ but we know that $(U, \tau_R(X))$ is nano-regular space, then there exists U_1 and U_2 such that $x \in U_1 \in \tau_R(X)$, $\{y\} \subseteq U_2 \in \tau_R(X) \subseteq H - \tau_R(X)$ and $U_1 \cap U_2 = \phi$. So, $y \in (nano - H - Cl(x))^c$.
- If $\{x\}$ is not nano-closed by Theorem 4.1 $y \in \{y\} \in H - \tau_R(X)$ with $\{x\} \cap \{y\} = \phi$. In fact that $\lambda \in (nano - H - Cl(x))^c$. But $x \in \{x\} \in \tau_R(X)$ and $\lambda \in (nano - H - Cl(x))^c \in H - \tau_R(X)$ with $\{x\} \cap (nano - H - Cl(x))^c = \phi$.

From (i) and (ii) $(U, H - \tau_R(X))$ is nano regular space.

□

Definition 4.2. *A nano topological space* $(U, \tau_R(X))$ *is said to be a* $nano - R_0 - space$ *if singletons are nano-H-closed set.*

Theorem 4.7. *If* $(U, \tau_R(X))$ *is a* $nano - R_0 - space$*, then* $(U, H - \tau_R(X))$ *is* $nano - \tau_1$

Proof. For all $x \in U$, then $\{x\}$ is nano-H-closed set and so $nano - H - cl(\{x\}) = \{x\}$. If $\{x\}$is nano-closed set, $y \notin \{x\} = nano - Cl\{x\} = nano - H - Cl\{x\}$ or $y \in \{x\}^c$, a nano-H-closed set by (Proposition 4.1). Then, $y \in nano - H - Cl\{y\} \subseteq \{x\}^c$, and so $x \notin nano - H - Cl\{x\}$. □

Remark 4.2. *The strong theorem in a* $nano - \tau_1 - space$ *is failed in* $H - \tau_R(X)$. *That is* $(U, \tau_R(X))$ *is a* $nano - \tau_1 - space$ *if and only if every singleton is nano-closed set.*

We illustrate that by Example 4.

$H - \tau_R^c(X) = \{\phi, \{2, 3\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, U\}$, we can see that $\{1\} \notin H - \tau_R^c(X)$ but $H - \tau_R(X)$ is $nano - \tau_1 - space$.

Now we rephrase that theorem.

Theorem 4.8. $(U, H-\tau_R(X))$ *is a* $nano-\tau_1-space$ *if and only if every singleton is nano-H-open set for all* $x \in U$.

Proof. Let $\{x\}$ and $\{y\}$ are nano-H-closed set and If $(U, H - \tau_R(X))$ is a $nano - \tau_1 - space$, then for $x \neq y$ there exists two nano-H-open set λ_1 and λ_2 such that $x \in \lambda_1$, $y \notin \lambda_1$, $y \in \lambda_2$ and $x \notin \lambda_2$ which implies that $x \in \lambda_1 \subseteq \{y\}^c$ and so $y \in \lambda_1 \subseteq \{x\}^c$, thus $\{y\}^c$ and $\{x\}^c$ are not nano-H-open set which is contradicting. Then, $\{x\}$ and $\{y\}$ are nano-H-open set.

Conversely, if every singleton is nano-H-open set for all $x \in U$, then it is clear that $(U, H - \tau_R(X))$ is a $nano - \tau_1 - space$. □

We can also introduce this interesting theorem.

Theorem 4.9. *Let $(U, H - \tau_R(X))$ be a nano H-topological space. then the following statements are equivalence.*

(i) *$(U, H - \tau_R(X))$ is a $nano - \tau_1 - space$.*
(ii) *For all $x \in U$, every singleton is nano-H-open set*
(iii) *For all $x \in U$, $\{x\}' = \phi$. ($\{x\}'$ is the limit point of $\{x\}$.)*

Remark 4.3. *The most results in this paper about the transfer of properties from $(U, \tau_R(X))$ to $(U, H - \tau_R(X))$ can be summarize by the following figure.*

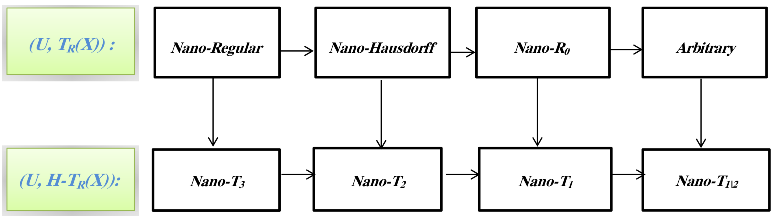

FIGURE 1

Remark 4.4. *The converse of the Theorem 4.6 and 4.7 is not correct in general. We can see that in the following example*

Example 4. *Let $U = \{1, 2, 3\}$ and $X = \{1\}$ with $U/R = \{\{1\}\}$, then*

- *$\tau_R(X) = \{\phi, \{1\}, U\}$, then $H-\tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, U\}$ we can see that*
- *$H - \tau_R(X)$ is nano regular space and $nano - \tau_1 - space$ but $\tau_R(X)$ is not nano regular space.*
- *$H - \tau_R(X)$ is $nano - \tau_1 - space$ but is not $nano - R_0 - space$ since $\{1\}$ is not g-closed set.*

By the new nano operator (nano-H-Interior) we introduce all these new sets in nano-H-topological space $(U, H - \tau_R(X))$ and further work needs to be done on these sets.

Definition 4.3. *Let $(U, \tau_R(X))$ be a nano-topological space and the set $\mu \in U$ is called*

(a) $Nano-H-\alpha-open$ (*resp.* $Nano-H-\alpha-closed$) *if* $\mu \subseteq N-H-Int(N-Cl(N-H-Int(\mu)))$ (*resp.* $N-Cl(N-H-Int(N-Cl(\mu))) \subseteq \mu$).

(b) $Nano-H-pre-open$ *(resp.* $Nano-pre-closed$ *if* $\mu \subseteq N-H-Int(N-Cl(\mu))$) (*resp.* $N-Cl(N-H-Int(\mu)) \subseteq \mu$).

(c) $Nano-H-semi-open$ (*resp.* $Nano-H-semi-closed$) *if* $\mu \subseteq N-Cl(N-H-Int(\mu))$ (*resp.* $N-Cl(N-H-Int(\mu)) \subseteq \mu$).

(d) $Nano-H-\beta-open$ (*resp.* $Nano-H-\beta-closed$) *if* $\mu \subseteq N-Cl(N-H-Int(N-Cl(\mu)))$ (*resp.*$N-H-Int(N-Cl(N-H-Int(\mu))) \subseteq \mu$).

Now we will discuss the relations and converse relations between the sets in nano-H-topological space and nano topological space.

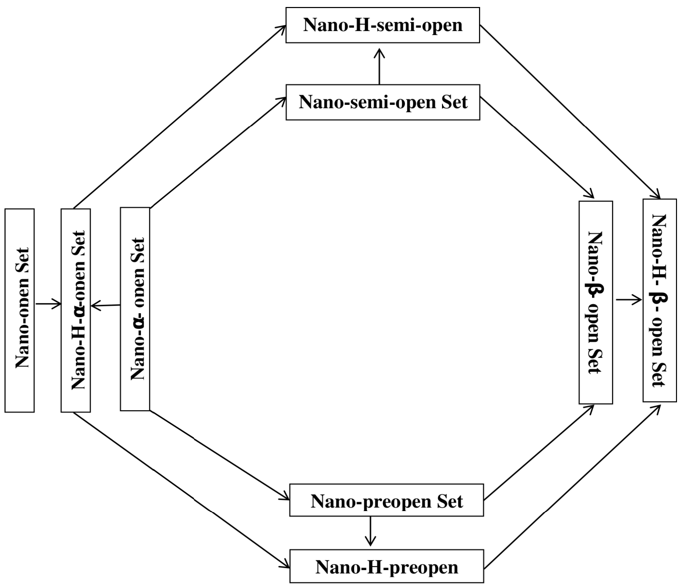

FIGURE 2

Example 5. *Let $U = \{1, 2, 3\}$ and $X = \{1, 3\}$ with $U/R = \{\{2\}\}$, then $\tau_R(X) = \{\phi, \{1\}, U\}$ and $H - \tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, U\}$ so*

(a) *we can see that if $\lambda = \{2, 3\}$, then*
- *λ is $Nano-H-semi-open$ but not $Nano-semi-open$.*
- *λ is $Nano-H-\alpha-open$ but not $Nano-\alpha-open$.*

(b) *also we can see that if* $\mu = \{2\}$*, then*
- μ *is* $Nano - H - pre - open$ **but not** $Nano - pre - open$.
- μ *is* $Nano - H - \beta - open$ **but not** $Nano - \beta - open$.

(c) *if* $\omega = \{1, 3\}$*, then we can see that*
- ω *is* $Nano - H - pre - open$ **but not** $Nano - H - semi - open$.
- ω *is* $Nano - H - \beta - open$ **but not** $Nano - H - \alpha - open$.

Definition 4.4. *A nano mapping* $f : (U, \tau_R(X))) \to (V, \tau_R(Y)))$ *is said to be:*

(a) $nano - H - continuous$ *[8] if the inverse image of each* $nano - H - open$ *set* μ_1 *in* V *is* $nano - H - open$ *set in* U.

(b) $nano - H - pre - continuous$ *[10] if the inverse image of each* $nano - H - open$ *set* μ_1 *in* V *is* $nano - H - pre - open$ *set in* U.

(c) $nano - H - semi - continuous$ *[10] if the inverse image of each* $nano - H - open$ *set* μ_1 *in* V *is* $nano - H - open$ *set in* U.

(d) $nano - H - \alpha - continuous$ *[10] if the inverse image of each* $nano - H - open$ *set* μ_1 *in* V *is* $nano - H - \alpha - open$ *set in* U.

Remark 4.5. *Every nano-continuous mapping is nano-H-continuous mapping but the converse is not true in general we can see that by the following example.*

Example 6. *Let* f *be a mapping between same nano topological (resp. nano-H-topological) spaces* $(U, \tau_R(X))\big(resp.(U, H - \tau_R(X))\big)$ *such that* $U = \{1, 2, 3\}$ *and* $X = \{1\}$ *with* $U/R = \{\{1\}\}$*. then,* $\tau_R(X) = \{\phi, \{1\}, U\}$ *and* $H - \tau_R(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, U\}$ *we can see that*
- *If* $f(1) = 1,\ f(2) = 2\ and\ f(3) = 1$ *and* $\mu = \{1\}$*, then* $f^{-1}(\{1\}) = \{1, 3\}$*, then* f *is nano-H-continuous but not nano-continuous mapping.*

As I said this paper will be the basis for huge numbers of researches in this direction.

REFERENCES

[1] K. BHUVANESHWARI, G. MYTHILI: *Nano generalized closed sets*, International Journal of Scientific and Research Pulications, **4**(5) (2014), 1–3.

[2] H. A. OTHMAN, M.H. PAGE: *On an infra-α-open sets*, Global Journal of Mathematical Analysis, **4**(3) (2016), 12–16.

[3] H. A. OTHMAN: *On Fuzzy supra-preopen sets*, Ann. Fuzzy Math. Inform., **12**(3)(2016), 361–371.

[4] H. A. OTHMAN: *On Fuzzy infra-semopen sets*, Journal of Mathematics, Punjab University, **48**(2) (2016), 1–10.

[5] H. A. OTHMAN: *On Weak and Strong Forms of β-open Sets*, Bulletin of International Mathematical Virtual Institute, **7**(1) (2017), 1–9.

[6] N. LEVINE: *Generalized closed sets in topology*, Rendiconti del Circolo Matematico di Palermo, **19** (1970), 89–96.

[7] M. L. THIVAGAR, R CARMEL: *On Nano Forms of Weakly Open Sets*, International Journal of Mathematics and Statistics Invention, **1**(1) (2013), 31–37.

[8] M. L. THIVAGAR, R. CARMEL: *On Nano Continuity*, Mathematical Theory and Modeling, **3**(7) (2013), 32–37.

[9] M. L. THIVAGAR, R. CARMEL: *Nutrition modeling through nano topology*, Int. Journal of Engineering Research and Applications, **4**(10) (2014), 327–334.

[10] M. L. THIVAGAR, R. CARMEL: *On nano forms of weakly open sets*, Int.J.Math. Stat. Inv., **1**(1) (2013), 31–37.

[11] A. REVATHY, G. ILANGO: *On Nano $\beta-$open sets*, Int. Jr. of Engineering, Contemporary Mathematics and Sciences, **1**(2) (2015), 1–6.

DEPARTMENT OF MATHEMATICS, AL-QUNFUDHAH UNIVERSITY COLLEGE,
UMM AL-QURA UNIVERSITY, KSA.
DEPARTMENT OF MATHEMATICS, RADA'A COLLEGE OF EDUCATION AND SCIENCE,
UNIVERSITY ALBAYDHA, ALBAYDHA, YEMEN.
E-mail address: hakim_albdoie@yahoo.com