ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.1, 289–294
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.1.23

A NOTE ON THE GENERALIZED NILPOTENT ELEMENTS OF A MODULE

SAUGATA PURKAYASTHA[1] AND HELEN K. SAIKIA

ABSTRACT. In this paper, we introduce the notion of the generalized nilpotent element of a module. In [4], the notion of nilpotent element of a module is introduced in the following sense: a non-zero element m of an R-module M is said to be nilpotent if there exists some $a \in R$ such that $a^k m = 0$ but $am \neq 0$ for some $k \in \mathbb{N}$. In our present work we aim to generalize this notion. We have extended this notion to the strongly nilpotent element of a module.

1. INTRODUCTION

In this paper, we assume that R is a ring with unity and M is a unital left R-module. We further let $End(M)$ denote the endomorphism ring of M.

A ring R is said to be completely semiprime if and only if $\forall a \in R$, we get $a = 0$ whenever $a^2 = 0$. As this condition implies that R does not have any non-trivial nilpotent element, such a ring is also termed as reduced ring. Thus, a ring R is not reduced if there is some $a \in R$ satisfying $a^2 = 0$ but $a \neq 0$. This condition has been the key notion to define the nilpotent element of a module as observed in [4].

In [5], it was remarked that the notion of nilpotent element of a module (as in [5]) does not generalize the notion of the nilpotent element of a ring

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 16D10.
Key words and phrases. nilpotent element, strongly nilpotent element, endomorphism ring, anihilator ideal.

in the sense that $m \in {}_RR$ may be nilpotent without $m \in R$ being nilpotent. The same has been justified with the help of an example. In our work, we call an element m of the left R-module M to be generalized nilpotent if and only if $\exists\ f \in End(M)$ such that $a^k f(m) = 0$ but $af(m) \neq 0$ for some $a \in R$ and $k \in \mathbb{N}$. Thus, an element $m \in M$ is generalized nilpotent if the corresponding image under an endomorphism is nilpotent in the sense of [4]. In the realm of this definition, we propose to tackle the above mentioned problem raised in [5] depending upon the existence of such an endomorphism. For example, we consider $R = \mathbb{Z}_8$ and $M = {}_RR$. Then $1 \in M$ is nilpotent in the sense of [4] but $1 \in R$ is not nilpotent. If we consider $f : M \to M$ defined by $f(m) = 2m$, then $f \in End(M)$ and $f(1) = 2$ which is nilpotent in both M and R.

Any undefined terminology or notion may be found in [1] and [3].

2. Generalized Nilpotent Element

We begin by some examples of generalized nilpotent elements of a module defined in the last section.

Example 1. *We exhibit the fact that in the $\mathbb{Z}$-module $\mathbb{Z}_{n^k}$, $n, k \in \mathbb{N}$ the generalized nilpotent elements are precisely the nilpotent ones. For let, $m \in \mathbb{Z}_{n^k}$ be not nilpotent. Thus, we must have $am = 0$ whenever $a^k m = 0$ for $a \in \mathbb{Z}$ and $k \in \mathbb{N}$. But $am = 0$ must imply $af(m) = 0$ whenever $a^k m = 0$ i.e. whenever $a^k f(m) = 0$. Thus, m cannot be generalized nilpotent.*

Example 2. *Following the same argument, one may show that the $\mathbb{Z}_{p^k}$ module $\mathbb{Z}_p$ does not contain any generalized nilpotent element which is not nilpotent.*

Proposition 2.1. *Let M be a torsion R-module. Then the generalized nilpotent elements of M are precisely the nilpotent ones.*

Proof. Let m be a non-zero element of M. As M is torsion, without the loss of generality we may assume that $a^k m = 0$ for some $a \in R$ and $k \in \mathbb{N}$. If m be not nilpotent, then $am = 0$ whenever $a^k m = 0$. Thus we must have $af(m) = 0$ whenever $a^k f(m) = 0$ for $f \in End(M)$. Consequently, m cannot be generalized nilpotent. $\square$

Proposition 2.2. *If M be a torsion free over a reduced ring R. Then M does not have any non-trivial generalized nilpotent element.*

Proof. Let m be a non-zero element of M. If $a^k f(m) = 0$, for $f \in End(M)$ then as M is torsion free, $a^k = 0$. Since R is reduced, we have $a = 0$. Consequently, $af(m) = 0$ and m is not generalized nilpotent. □

Example 3. *The modules* ${}_{\mathbb{Z}}\mathbb{Z}$*,* ${}_{\mathbb{R}}\mathbb{R}$*,* ${}_{\mathbb{Z}}\mathbb{R}$ *do not have any non-trivial generalized nilpotent element.*

In case of general modules over arbitrary rings, we characterize the generalized nilpotency of an element m in terms of the annihilator ideal $(0 : f(m))$ of the image of m under $f \in End(M)$.

Proposition 2.3. *An element* m *of an* R*-module* M *is generalized nilpotent if and only if* $(0 : f(m))$ *is not completely semiprime for* $f \in End(M)$*.*

Proof. If $(0 : f(m))$ be not completely semiprime, then for some $a \in R$ we have $a^2 \in (0 : f(m))$ but $a \notin (0 : f(m))$. This gives $a^2 f(m) = 0$ but $af(m) \neq 0$. Thus, m is generalized nilpotent. The converse is obtained just by taking $k = 2$ in the definition and retracing the above steps. □

From the definition, it is evident that every nilpotent element is generalized nilpotent as for the identity endomorphism $I \in End(M)$, $a^k m = 0$ but $am \neq 0$ gives $a^k I(m) = a^k m = 0$ but $aI(m) = am \neq 0$. We note that nilpotent elements thus form a class of generalized nilpotent elements. We have also seen that for torsion and torsion free modules (over reduced rings), the notion of nilpotency and generalized nilpotency do not differ. We now give an example of the fact that there are classes of modules for which these two notions differ.

Example 4. *we consider the* $\mathbb{Z}$*-module* $\mathbb{Z} \oplus \mathbb{Z}_{p^k}$ *for* $p, k > 1$*. Let us define* $f : \mathbb{Z} \oplus \mathbb{Z}_{p^k} \to \mathbb{Z} \oplus \mathbb{Z}_{p^k}$ *by* $f(a, b) = (0, b)$*. Clearly,* $f \in End(\mathbb{Z} \oplus \mathbb{Z}_{p^k})$*. Moreover,* $f(1, 1) = (0, 1)$ *and* $p^k(0, 1) = (0, 0)$ *but* $p(0, 1) \neq (0, 0)$*. Consequently,* $(1, 1)$ *is generalized nilpotent. On the other hand, as* $a^k(1, 1) \neq 0$ *for all* $a \in \mathbb{Z}$ *and* $k \in \mathbb{N}$*, and hence* $(1, 1)$ *is not nilpotent.*

We now establish a criteria under which a generalized nilpotent element of a module becomes nilpotent.

Proposition 2.4. *Let* $m \in M$ *be a generalized nilpotent element. Then* $m \in M$ *is nilpotent if the short exact sequence* $0 \to M \xrightarrow{f} M$ *is split.*

Proof. Since $m \in M$ is generalized nilpotent, we have $a^k f(m) = 0$ but $af(m) \neq 0$ for some $f \in End(M)$ and for some $k \in \mathbb{N}$. Also using the fact that $0 \to M \xrightarrow{f} M$ is split, we get $g : M \to M$ s.t. $gf = i$, where i is the identity morphism. Thus, we get $a^k f(m) = f(a^k m) = 0$ which in turn gives $g(f(a^k m)) = gf(a^k m) = a^k m = 0$ and $af(m) = f(am) \neq 0$ which gives $g(f(am)) = gf(am) = am \neq 0$. □

3. Strongly generalized nilpotent element

The notion of strongly nilpotent element is intoduced in [4]. An element of an R-module M is said to be strongly nilpotent if for every sequence of the type $a_0, a_1, a_2, ...$ with $a_1 = a$ and $a_{n+1} \in a_n R a_n \ \forall \ n \in \mathbb{N}$, $\exists k \in \mathbb{N}$ such that $a_k m = 0$ but $am \neq 0$. In the same spirit, we define strongly generalized nilpotent element of a module.

Definition 3.1. *An element m of an R-module M to be strongly generalized nilpotent provided for every sequence of the type $a_0, a_1, a_2, ...$ with $a_0 = a$ and $a_{n+1} \in a_n R a_n \forall n \in \mathbb{N}$, $\exists$ some $k > 0$ such that $a_k f(m) = 0$ but $af(m) \neq 0$ for some $f \in End(M)$.*

Proposition 3.1. *Every strongly generalized nilpotent element of an R-module M with $1 \in R$ is generalized nilpotent but not conversely.*

Proof. If $m = 0$, then there is nothing to prove. Let m be a non-zero strongly generalized nilpotent element element of M. Then for every sequence of the type $a_0, a_1, a_2, ...$ with $a_0 = a$ and $a_{n+1} \in a_n R a_n \forall n \in \mathbb{N}$, $\exists$ some $k > 0$ such that $a_k f(m) = 0$ but $af(m) \neq 0$. Thus, in particular, we may choose the sequence $a_0, a_1, a_2, ...$ where $a_0 = a$, $a_1 = a_0.1.a_0 = a.1.a = a^2$, $a_2 = a_1.1.a_1 = a^2.1.a^2 = a^4$,..., $a_n = 2^{2^n}$. As m is strongly generalized nilpotent, $\exists k = 2^{k_0} \in \mathbb{N}$ for $k_0 \in \mathbb{N}$ such that $a^k f(m) = 0$ but $af(m) \neq 0$ and as such m is generalized nilpotent.

Conversely, the element $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}_{p^k}$ (Example 4) is generalized nilpotent but not strongly generalized nilpotent as for the sequence $a_0, a_1, a_2, ...$ where $a_0 = p^k$, $a_2 = a_0.1.a_0 = p^k.1.p^k = p^{2k}$, $a_2 = a_2.1.a_2 = p^{2k}.1.p^{2k} = p^{4k}$,... we see that $a_n f(m) = (0, 0)$. But as $af(m) = p^k f(m) = (0, 0)$, we conclude that m is not strongly generalized nilpotent. □

Corollary 3.1. *The set of all strongly generalized nilpotent elements of an R-module M is properly contained in the set of all generalized nilpotent elements of M.*

Proposition 3.2. *Every strongly nilpotent element of an R-module M is trivially strongly generalized nilpotent. The converse holds provided $f \in End(M)$ is a monomorphism.*

Proof. Let $m \in M$ be strongly nilpotent. Then for every sequence of the type $a_0, a_1, a_2, ...$ with $a_0 = a$ and $a_{n+1} \in a_n R a_n \forall n \in \mathbb{N}$, $\exists$ some $k > 0$ such that $a_k m = 0$ but $am \neq 0$. Then for the identity morphism $I \in End(M)$, $a_k I(m) = a_k m = 0$ but $aI(m) = am \neq 0$. For the converse, we assume that m is a strongly generalized nilpotent element, i.e. for every sequence of the type $a_0, a_1, a_2, ...$ with $a_0 = a$ and $a_{n+1} \in a_n R a_n \forall n \in \mathbb{N}$, $\exists$ some $k > 0$ such that $a_k f(m) = 0$ but $af(m) \neq 0$. Then if $f \in End(M)$ be a monomorphism $a_k f(m) = f(a_k m) = 0$ gives $a_k m = 0$ and $af(m) = f(am) \neq 0$ gives $am \neq 0$. □

4. Generalized nilpotency and envelope

In [2], the notion of the envelope of a subset B of an R-module M for a commutative ring R is defined. For any subset B of M, the envelope of $E(B)$ of B is defined as:

$$E(B) = \{x \in M | x = ra, r^n a \in B, \text{for some } r \in R, a \in M, n \in \mathbb{N}\}$$

We consider the envelope for $B = (0)$. We claim that for $B = (0)$, $E(0) = \{x \in M | r^n x = 0, \text{for some } r \in R, n \in \mathbb{N}\}$. For if $r^n a = 0$ then $r^{n+1} a = r^n (ra) = 0$ i.e. $r^n x = 0$. Consequently, $E(0) = \{x \in M | r^n x = 0, \text{for some } r \in R, n \in \mathbb{N}\}$. Hence the claim.

Proposition 4.1. *Let $I = (0 : f(m))$ be not completely semiprime for $f \in End(M)$ and $m \in M$. If f is one-one, then the set of all generalized nilpotent elements in M is properly contained in $E(0)$.*

Proof. Since I is not completely semiprime, by proposition 2.3 it follows that m is generalized nilpotent. Also, f being one-one, $r^n f(m) = 0$ gives $r^n m = 0$. We thus have $m \in E(0)$. □

The containment in the above proposition is strict in general as can be seen from example 4: for the element $(0, p^{k-1}) \in \mathbb{Z} \oplus \mathbb{Z}_{p^k}$, we have $p(0, p^{k-1}) = (0, 0)$ and thus $(0, p^{k-1}) \in E(0)$. But even though $p^n f(0, p^{k-1}) = p^n(0, p^{k-1})$ for any $n \in \mathbb{N}$, but as $p(0, p^{k-1}) = (0, 0)$, $(0, p^{k-1})$ cannot be generalized nilpotent.

Corollary 4.1. *The set of all strongly generalized nilpotent elements in M is properly contained in $E(0)$.*

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Department of Mathematics
Assm Don Bosco University
Guwahati-781017, Assam, India
E-mail address: sau.pur@rediffmail.com

Department of Mathematics
Gauhati University
Guwahati-781014, Assam, India
E-mail address: hsaikia@yahoo.com