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ENERGY OF SPHERICAL FUZZY GRAPHS

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ABSTRACT. In this paper, the notion of energy extended to spherical fuzzy graph. The adjacency matrix of a spherical fuzzy graph is defined and we compute the energy of a spherical fuzzy graph as the sum of absolute values of eigenvalues of the adjacency matrix of the spherical fuzzy graph. Also, the lower and upper bounds for the energy of spherical fuzzy graphs are obtained.

1. INTRODUCTION

In 1978, Ivan Gutman [8] introduced the energy of a graph as the sum of the absolute values of the eigen values of the adjacency matrix of a graph. The lower and upper bound for the energy of a graph are discussed in [7,10]. Mahmood et al. [11] introduced the concept of spherical fuzzy set which gives an additional strength to the concept of picture fuzzy set by enlarging the space for the grades of all the four parameters. Kifayat et al. [9] studied the geometrical comparison of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, picture fuzzy sets with spherical fuzzy sets. Cen Zuo, et al. [6] introduced the some new concepts of picture fuzzy graph. Yahya Mohamed and Mohamed Ali [12–17] studied some operation on intuitionistic fuzzy graph (IFG) and interval-valued Pythagorean fuzzy graph. Some recent works on energy of various fuzzy graphs can be found in [3–5]. Akram et.al [2] introduced the notion of spherical fuzzy graphs and Abhishek Guleria [1] also introduced generalized

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version spherical fuzzy graphs using T-spherical fuzzy sets. In this paper, the concept of energy of spherical fuzzy graph is studied and we compute the energy of a spherical fuzzy graph using adjacency matrix. Some lower and upper bounds on energy of spherical fuzzy graphs are also obtained.

2. Preliminaries

In this section, some basic definitions are presented which are very useful to understand the concepts of the paper.

Definition 2.1. *[11] A spherical fuzzy set S in U (universe of discourse) is given by $S = \{< \alpha, \mu_S(\alpha), \eta_S(\alpha), \nu_S(\alpha) >: \alpha \in U\}$ where $\mu_S : U \to [0,1], \eta_s : U \to [0,1]$ and $\nu_S : U \to [0,1]$ denote degree of membership, degree of neutral membership and degree of non-membership respectively, and for each $\alpha \in U$ satisfying the condition $0 \leq \mu_S^2(\alpha) + \eta_S^2(\alpha) + \nu_S^2(\alpha) \leq 1, \forall \alpha \in U$.*
The degree of refusal for any spherical fuzzy set S and $\alpha \in U$ is given by $r_S(\alpha) = \sqrt{1 - (\mu_S^2(\alpha) + \eta_S^2(\alpha) + \nu_S^2(\alpha))}$.

Definition 2.2. *[2] Let X be a universal set. A spherical fuzzy graph (SFG) on X is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is a spherical fuzzy set on X with $0 \leq \sigma_1^2(x) + \sigma_2^2(x) + \sigma_3^2(x) \leq 1, \forall x \in X$ and $\mathcal{L}$ is a spherical fuzzy relation in $X \times X$ such that $\mu_1(x,y) \leq \min\{\sigma_1(x), \sigma_1(y)\}, \mu_2(x,y) \leq \min\{\sigma_2(x), \sigma_2(y)\}$, $\mu_3(x,y) \leq \max\{\sigma_3(x), \sigma_3(y)\}$, and satisfying the condition $0 \leq \mu_1^2(x,y) + \mu_2^2(x,y) + \mu_3^2(x,y) \leq 1$ for all $(x,y) \in X \times X$.*

Example 1. *Consider a graph $G^* = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_1u_4, u_1u_3\} \subseteq V \times V$. Let $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ be a spherical fuzzy graph where $\mathcal{N}$ is a spherical fuzzy subset of V and $\mathcal{L}$ is a spherical fuzzy subset of E defined by*

$\mathcal{N}$	u_1	u_2	u_3	u_4
σ_1	0.5	0.4	0.7	0.8
σ_1	0.4	0.5	0.1	0.1
σ_1	0.7	0.3	0.2	0.1

$\mathcal{L}$	u_1u_2	u_2u_3	u_3u_4	u_4u_1	u_1u_3
μ_1	0.3	0.4	0.6	0.4	0.5
μ_2	0.4	0.1	0.1	0.1	0.1
μ_3	0.7	0.3	0.2	0.6	0.7

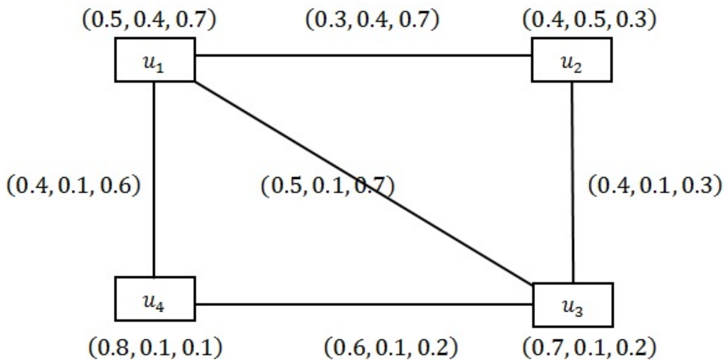

FIGURE 1. Spherical Fuzzy Graph

3. SPECTRUM AND ENERGY OF A SPHERICAL FUZZY GRAPH

Definition 3.1. *The adjacency matrix* $\mathbb{A}(G) = (\mathbb{A}(\mu_1(x_ix_j)), \mathbb{A}(\mu_2(x_ix_j)), \mathbb{A}(\mu_3(x_i, x_j)))$ *of a spherical fuzzy graph* $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ *is defined as a square matrix* $\mathbb{A}(G) = [\alpha_{ij}]$*, where* $\alpha_{ij} = (\mu_1(x_ix_j), \mu_2(x_ix_j), \mu_3(x_i, x_j))$*, where* $\mu_1(x_ix_j), \mu_2(x_ix_j)$ *and* $\mu_3(x_i, x_j)$ *represent the strength of relationship, strength of refusal-relationship and strength of non-relationship between* x_i *and* x_j *respectively.*

Definition 3.2. *The spectrum of adjacency matrix of a spherical fuzzy graph* $\mathbb{A}(G)$ *is defined as* $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$*, where* $\lambda^{(1)}, \lambda^{(2)}$ *and* $\lambda^{(3)}$ *are the sets of eigenvalues of* $\mathbb{A}(\mu_1(x_ix_j)), \mathbb{A}(\mu_2(x_ix_j))$ *and* $\mathbb{A}(\mu_3(x_i, x_j))$ *respectively.*

Definition 3.3. *The energy of a spherical fuzzy graph* $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ *is defined as*

$$\begin{aligned} \mathbb{E}(G) &= (\mathbb{E}(\mu_1(x_ix_j)), \mathbb{E}(\mu_2(x_ix_j)), \mathbb{E}(\mu_3(x_ix_j))) \\ &= (\sum_{\substack{i=1 \\ \alpha_i \in \lambda^{(1)}}}^{n} |\alpha_i|, \sum_{\substack{i=1 \\ \beta_i \in \lambda^{(2)}}}^{n} |\beta_i|, \sum_{\substack{i=1 \\ \gamma_i \in \lambda^{(3)}}}^{n} |\gamma_i|) \end{aligned}$$

Example 2. *The adjacency matrix of a SFG given in Figure 1 is*

$$A(G) = \begin{pmatrix} (0,0,0) & (0.3,0.4,0.7) & (0.5,0.1,0.7) & (0.4,0.1,0.6) \\ (0.3,0.4,0.7) & (0,0,0) & (0.4,0.1,0.3) & (0,0,0) \\ (0.5,0.1,0.7) & (0.4,0.1,0.3) & (0,0,0) & (0.6,0.1,0.2) \\ (0.4,0.1,0.6) & (0,0,0) & (0.6,0.1,0.2) & (0,0,0) \end{pmatrix}.$$

The spectrum and energy of a SFG, given in Figure 1, are as follows:
Spectrum of $\mu_1(x_ix_j) = \{1.15398, -0.721111, -0.433974, 0.00110764\}$
Spectrum of $\mu_2(x_ix_j) = \{0.466165, -0.412921, -0.1, 0.0467558\}$
Spectrum of $\mu_3(x_ix_j) = \{1.34622, -1.00415, -0.34549, 0.00342586\}$
Spectrum of $G = \{((1.15398, 0.466165, 1.34622), (-0.721111, -0.412921, -1.00415),$
$(-0.433974, -0.1, -0.34549), (0.00110764, 0.0467558, 0.00342586))\}$
Energy of $\mathcal{G} = \{2.31017264, 1.0258418, 2.69928586\}$.

4. Lower and Upper Bounds of a Spherical Fuzzy Graph

Theorem 4.1. *Let* $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ *be a spherical fuzzy undirected, loop-less and connected graph with order n and size m and* $\mathbb{A}(\mathcal{G})$ *be its adjacency matrix. If* $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ *and* $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n$ *are the eigen values of* $\mathbb{A}(\mu_1(x_ix_j)), \mathbb{A}(\mu_2(x_ix_j))$ *and* $\mathbb{A}(\mu_3(x_ix_j))$*, respectively, then*

(i) $\sum_{i=1}^{n} \alpha_i = 0$ *for all* $\alpha_i \in \lambda^{(1)}, \sum_{i=1}^{n} \beta_i = 0$ *for all* $\beta_i \in \lambda^{(2)}$ *and* $\sum_{i=1}^{n} \gamma_i = 0$ *for all* $\gamma_i \in \lambda^{(3)}$.

(ii) $\sum_{i=1}^{n} \alpha_i^2 = 2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2$ *for all* $\alpha_i \in \lambda^{(1)}$, $\sum_{i=1}^{n} \beta_i^2 = 2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2$ *for all* $\beta_i \in \lambda^{(2)}$, *and* $\sum_{i=1}^{n} \gamma_i^2 = 2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2$ *for all* $\gamma_i \in \lambda^{(3)}$.

Proof. (i). Let $\mathbb{A}(\mathcal{G})$ be an adjacency matrix with zero diagonal, since there are no loops in the spherical fuzzy graph $\mathcal{G}$. Then the trace of $\mathbb{A}(\mathcal{G})$ is zero and $\mathbb{A}(\mathcal{G})$ is a symmetric matrix. $\sum_{i=1}^{n} \alpha_i = 0$ for all $\alpha_i \in \lambda^{(1)}, \sum_{i=1}^{n} \beta_i = 0$ for all $\beta_i \in \lambda^{(2)}$ and $\sum_{i=1}^{n} \gamma_i = 0$ for all $\gamma_i \in \lambda^{(3)}$. Hence, the sum of eigen values is equal to zero.

(ii). By the trace properties of a matrix, we have
$tr((A(\mu_1(x_ix_j)))^2) = \sum^{n}_{\substack{i=1 \\ \alpha_i \in \lambda^{(1)}}} \alpha_i^2$, where

$$\begin{aligned} tr((A(\mu_1(x_ix_j)))^2) =& (0 + (\mu_1(x_1x_2))^2 + ... + (\mu_1(x_1x_n))^2) \\ &+ (0 + (\mu_1(x_2x_1))^2 + ... + (\mu_1(x_2x_n))^2) + ... \\ &+ (0 + (\mu_1(x_nx_1))^2 + (\mu_1(x_nx_2))^2 + ... + 0) \\ =& 2\sum_{1\leq i<j\leq n} (\mu_1(x_ix_j))^2 . \end{aligned}$$

Hence,

$$\sum_{i=1}^{n} \alpha_i^2 = 2 \sum_{1 \leq i < j \leq n} (\mu_1(x_i x_j))^2 .$$

Similarly, we can show that:

$$\sum_{i=1}^{n} \beta_i^2 = 2 \sum_{1 \leq i < j \leq n} (\mu_2(x_i x_j))^2$$

for all $\beta_i \in \lambda^{(2)}$, and

$$\sum_{i=1}^{n} \gamma_i^2 = 2 \sum_{1 \leq i < j \leq n} (\mu_3(x_i x_j))^2$$

for all $\gamma_i \in \lambda^{(3)}$. □

Theorem 4.2. *Let $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ be a spherical fuzzy graph with n vertices and $\mathbb{A}(\mathcal{G}) = (\mathbb{A}(\mu_1(x_i x_j)), \mathbb{A}(\mu_2(x_i x_j)), \mathbb{A}(\mu_3(x_i, x_j)))$ be the adjacency matrix of $\mathcal{G}$. Then*

$$\sqrt{2 \sum_{1 \leq i < j \leq n} (\mu_1(x_i x_j))^2 + n(n-1)|det(\mathbb{A}(\mu_1(x_i x_j)))|^{\frac{2}{n}}} \leq \mathbb{E}(\mu_1(x_i x_j)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} (\mu_1(x_i x_j))^2},$$

$$\sqrt{2 \sum_{1 \leq i < j \leq n} (\mu_2(x_i x_j))^2 + n(n-1)|det(\mathbb{A}(\mu_2(x_i x_j)))|^{\frac{2}{n}}} \leq \mathbb{E}(\mu_2(x_i x_j)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} (\mu_2(x_i x_j))^2},$$

$$\sqrt{2 \sum_{1 \leq i < j \leq n} (\mu_3(x_i x_j))^2 + n(n-1)|det(\mathbb{A}(\mu_3(x_i x_j)))|^{\frac{2}{n}}} \leq \mathbb{E}(\mu_3(x_i x_j)) \leq \sqrt{2n \sum_{1 \leq i < j \leq n} (\mu_3(x_i x_j))^2}.$$

Proof. **Upper Bound**: Applying Cauchy-Schwarz inequality to the n numbers $(1, 1, ..., 1)$ and $(|\alpha_1|, |\alpha_2|, ..., |\alpha_n|)$ with n entries, we have,

$$\sum_{i=1}^{n} |\alpha_i| \leq \sqrt{n} \sqrt{\sum_{i=1}^{n} |\alpha_i|^2}, \tag{4.1}$$

$$(\sum_{i=1}^{n}\alpha_i)^2=\sum_{i=1}^{n}|\alpha_i|^2+2\sum_{1\le i<j\le n}\alpha_i\alpha_j\,. \tag{4.2}$$

On comparing the coefficients of $\alpha^{(n-2)}$ in the characteristic polynomial,

$$\prod_{i=1}^{n}(\alpha-\alpha_i)=|\mathbb{A}(\mathcal{G})-\alpha I|$$

we get

$$\sum_{1\le i<j\le n}\alpha_i\alpha_j=-\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2\,. \tag{4.3}$$

Substituting (4.3) in (4.2), we obtain,

$$\sum_{i=1}^{n}|\alpha_i|^2=2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2\,. \tag{4.4}$$

Substituting (4.4) in (4.1), we obtain:

$$\begin{aligned}\sum_{i=1}^{n}|\alpha_i|\le\sqrt{n}\sqrt{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2} &= \sqrt{2n\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}\\ \mathbb{E}(\mu_1(x_ix_j)) &\le \sqrt{2n\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}\,.\end{aligned}$$

Similarly, we can show that:

$$\begin{aligned}\mathbb{E}(\mu_2(x_ix_j)) &\le \sqrt{2n\sum_{1\le i<j\le n}(\mu_2(x_ix_j))^2}\,,\\ \mathbb{E}(\mu_3(x_ix_j)) &\le \sqrt{2n\sum_{1\le i<j\le n}(\mu_3(x_ix_j))^2}\,.\end{aligned}$$

Lower Bound:

$$\begin{aligned}(\mathbb{E}(\mu_1(x_ix_j)))^2 &= (\sum_{i=1}^{n}|\alpha_i|)^2=\sum_{i=1}^{n}|\alpha_i|^2+2\sum_{1\le i<j\le n}|\alpha_i\alpha_j|\\ &= 2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2+\frac{2n(n-1)}{2}AM|\alpha_i\alpha_j|\end{aligned}$$

Since $AM\{|\alpha_i\alpha_j|\}\ge GM\{|\alpha_i\alpha_j|\},1\le i<j\le n$, we have

$$E(\mu_1(x_ix_j))\ge\sqrt{2n\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2+n(n-1)GM\{|\alpha_i\alpha_j|\}}\,.$$

Also,

$$\begin{aligned} GM\{|\alpha_i\alpha_j|\} &= (\prod_{1\leq i<j\leq n} |\alpha_i\alpha_j|)^{\frac{2}{n(n-1)}} \\ &= (\prod_{i=1}^{n} |\alpha_i|^{n-1})^{\frac{2}{n(n-1)}} \\ &= (\prod_{i=1}^{n} |\alpha_i|^{n-1})^{\frac{2}{n}} \\ &= |det(A(\mu_1(x_ix_j)))|^{\frac{2}{n}} . \end{aligned}$$

Therefore,

$$\mathbb{E}(\mu_1(x_ix_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2 + n(n-1)|det((\mathbb{A}(\mu_1(x_ix_j)))|^{\frac{2}{n}}} ,$$

and,

$$\begin{aligned} \sqrt{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2 + n(n-1)|det(\mathbb{A}(\mu_1(x_ix_j)))|^{\frac{2}{n}}} &\leq \mathbb{E}(\mu_1(x_ix_j)) \\ &\leq \sqrt{2n\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2} . \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} \sqrt{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2 + n(n-1)|det(\mathbb{A}(\mu_2(x_ix_j)))|^{\frac{2}{n}}} &\leq \mathbb{E}(\mu_2(x_ix_j)) \\ &\leq \sqrt{2n\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2} , \end{aligned}$$

and

$$\begin{aligned} \sqrt{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2 + n(n-1)|det(\mathbb{A}(\mu_3(x_ix_j)))|^{\frac{2}{n}}} &\leq \mathbb{E}(\mu_3(x_ix_j)) \\ &\leq \sqrt{2n\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2} . \end{aligned}$$

□

Theorem 4.3. *Let $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ be a spherical fuzzy graph with n vertices and $\mathbb{A}(\mathcal{G}) = (\mathbb{A}(\mu_1(x_ix_j)), \mathbb{A}(\mu_2(x_ix_j)), \mathbb{A}(\mu_3(x_ix_j)))$ be the adjacency matrix of $\mathcal{G}$. If*

$n \leq 2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2, n \leq 2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2$ *and* $n \leq 2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2$. *Then*

$$\begin{aligned}\mathbb{E}(\mu_1(x_ix_j)) &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2}{n} \\ &+ \sqrt{(n-1)\{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2-(\frac{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2}{n})^2\}},\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\mu_2(x_ix_j)) &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2}{n} \\ &+ \sqrt{(n-1)\{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2-(\frac{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2}{n})^2\}},\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\mu_3(x_ix_j)) &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2}{n} \\ &+ \sqrt{(n-1)\{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2-(\frac{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2}{n})^2\}}.\end{aligned}$$

Proof. If $A(\mathcal{G}) = [a_{ij}]_{n\times n}$ is a symmetric matrix with zero trace, then

$$\alpha_{max} \geq \frac{2\sum_{1\leq i<j\leq n} a_{ij}}{n},$$

where, α_{max} is the minimum eigenvalue of A.
If $A(\mathcal{G})$ is the adjacency matrix of a spherical fuzzy graph $\mathcal{G}$, then
$\alpha_1 \geq \dfrac{2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2}{n}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$.
Moreover,

$$\begin{aligned}\sum_{\substack{i=1\\ \alpha_i\in\lambda^{(1)}}}^{n} \alpha_i^2 &= 2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2 \\ \sum_{\substack{i=1\\ \alpha_i\in\lambda^{(1)}}}^{n} \alpha_i^2 &= 2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2-\alpha_1^2. \qquad (4.5)\end{aligned}$$

Using the Cauchy-Schwarz inequality to the vectors $(1, 1, ..., 1)$ and $(|\alpha_1|, |\alpha_2|, ..., |\alpha_n|)$ with $n-1$ entries, we have:

$$\mathbb{E}(\mu_1(x_ix_j)) - \alpha_1 = \sum_{i=2}^{n} |\alpha_i| \le \sqrt{(n-1)\sum_{i=2}^{n} |\alpha_i|^2}. \tag{4.6}$$

Substituting (4.5) in (4.6), we have:

$$\mathbb{E}(\mu_1(x_ix_j)) - \alpha_1 \le \sqrt{(n-1)(2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2 - \alpha_1^2)}\,,$$

$$\mathbb{E}(\mu_1(x_ix_j)) \le \alpha_1 + \sqrt{(n-1)(2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2 - \alpha_1^2)}\,. \tag{4.7}$$

Now, consider the function,

$$\zeta(x) = x + \sqrt{(n-1)(2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2 - x^2)}$$

decrease on the interval $\left(\sqrt{\frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n}}, \sqrt{2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2)}\right)$,

and also

$$n \le 2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2, 1 \le \frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n}\,.$$

Therefore,

$$\begin{aligned}\sqrt{\frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n}} &\le \frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n}\\ &\le \frac{2\sum_{1\le i<j\le n}\mu_1(x_ix_j)}{n}\\ &\le \lambda_1 \le \sqrt{2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2}\,.\end{aligned}$$

From equation (4.7), we have:

$$\begin{aligned}\mathbb{E}(\mu_1(x_ix_j)) &\le \frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n}\\ &+ \sqrt{(n-1)\{2\sum_{1\le i<j\le n} (\mu_1(x_ix_j))^2 - (\frac{2\sum_{1\le i<j\le n}(\mu_1(x_ix_j))^2}{n})^2\}}\,.\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\mathbb{E}(\mu_2(x_ix_j)) &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2}{n}\\ &+ \sqrt{(n-1)\{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2-(\frac{2\sum_{1\leq i<j\leq n}(\mu_2(x_ix_j))^2}{n})^2\}},\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\mu_3(x_ix_j)) &\leq \frac{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2}{n}\\ &+ \sqrt{(n-1)\{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2-(\frac{2\sum_{1\leq i<j\leq n}(\mu_3(x_ix_j))^2}{n})^2\}}.\end{aligned}$$

□

Theorem 4.4. *Let $\mathcal{G}=(\mathcal{N},\mathcal{L})$ be a spherical fuzzy graph with n vertices. Then* $\mathbb{E}(\mu_1(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n}),\mathbb{E}(\mu_2(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n})$ *and* $\mathbb{E}(\mu_3(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n})$. *That is,* $\mathbb{E}(\mathcal{G})\leq\frac{n}{2}(1+\sqrt{n})$.

Proof. Assume that $\mathcal{G}=(\mathcal{N},\mathcal{L})$ is a spherical fuzzy graph with n vertices.
If $n\leq 2\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2=2\theta$, then by routine calculus, it is easy to show that $f(\theta)=\frac{2\theta}{n}+\sqrt{((1-n)(2\theta-(2\theta/n)^2))}$ is maximized when $\theta=\frac{n^2+\sqrt[n]{n}}{4}$.
Substituting this value of y in the place of $\theta=\sum_{1\leq i<j\leq n}(\mu_1(x_ix_j))^2$.
By theorem (4.3), we must have $\mathbb{E}(\mu_1(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n})$.
Similarly, it is easy to show that $\mathbb{E}(\mu_2(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n})$ and $\mathbb{E}(\mu_3(x_ix_j))\leq\frac{n}{2}(1+\sqrt{n})$. Hence $\mathbb{E}(\mathcal{G})\leq\frac{n}{2}(1+\sqrt{n})$. □

5. Conclusion

Spherical fuzzy sets are an extension of picture fuzzy sets, as they gives enlargement of the space of membership degrees of truthness (a), abstinence (b), and falseness (c) in the unit interval, with the condition $0\leq a^2+b^2+c^2\leq 1$. In this paper, the concept of spectrum and energy of the spherical fuzzy graphs are studied. Also, the lower and upper bounds of the energy of spherical fuzzy graphs are obtained in terms of number of vertices and sum of squares of membership, refusal membership and non-membership values of edges. In the future, we extend this study to (i) interval-valued spherical fuzzy graphs; (ii) hesitant spherical fuzzy hyper graphs; (iii) single-valued neurotrophic spherical fuzzy graphs.

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