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LARGE DEVIATIONS FOR RANDOM EVOLUTION EQUATION IN BESOV-ORLICZ SPACE

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ABSTRACT. In this paper, we develop a large deviations principle for random evolution equations to the Besov-Orlicz space $\mathcal{B}^{v,0}_{M_2,w}$ corresponding to the Young function $M_2(x) = \exp(x^2) - 1$.

1. INTRODUCTION

In recent years, many results on the Brownian motion and diffusion processes in path spaces with stronger topologies than the usual uniform one have been obtained. Freidlin-Wentzel [8] large deviations principle have been developed in Hölder spaces for the Brownian motion in Baldi, P., Ben Arous, G. & Kerkyacharian, G. [1] and for general diffusion processes in Ben Arous, G. & Ledoux, M. [3]. Later on, an extension to Besov spaces was considered in Eddahbi, M., Nzi, M. & Oukinine, Y. [7], Lakhel, E. H. [10] and Roynette [13]. The main purpose of this work is to establish large deviations in Besov-Orlicz spaces for random evolution equation which generalizes the result of Mohamed, M. [11]. His method was using an extension of the principle of contractions.

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We consider $X^\varepsilon = \{X_t^\varepsilon, 0 \leq t \leq 1\}$ the solution of the equation of:

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon, Y_s)ds + \varepsilon^{\frac{1}{2}} \int_0^t \sigma(X_s^\varepsilon, Z_s)dW_s, \tag{1.1}$$

where $x \in \mathbb{R}^d$, W is a standard Brownian motion taking values in $\mathbb{R}^k$, Y is a progressively measurable random process which satisfies some integrability conditions and Z is a random process such that topological support of Z is a compact subset of $\mathcal{B}_{M_2,w}^{v,0}$. Furthermore, W is independent of (Y,Z) and σ, b satisfy some regularity assumptions which we will describe later.

The paper is organized as follows. It is useful in section 2 to present some notions and results on the topology of Besov-Orlicz space. In section 3 we give some definitions and general results, while in section 4 we find the main result of this work. In Section 5, we introduce the regularity of the solution of the equation (1.1). Section 6 is devoted to the proof of our main result. In section 7, we give special cases.

2. Preliminary and Notations

Let $f : [0,1] \to \mathbb{R}^d$ a continuous function.

Let A_{M_2} the Orlicz space on I corresponding to the Young function $M_2(x) = \exp(x^2) - 1$ endowed with the norm

$$\| f \|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau}\left[1 + \int_0^1 M_2(\tau |f(t)|)dt\right] \right\}.$$

For more details on Orlicz spaces we refer for instance to Cieleski, Z., Kerkyacharian, G. & Roynette, B. [5]. The modulus of smoothness for the Orlicz norm is defined by

$$w_{M_2}(f,t) = \sup_{0 \leq h \leq t \leq 1} \| \Delta_h f \|_{M_2} .$$

Let $\mathcal{B}_{M_2,w}^v$ be denote the Besov-Orlicz space of continuous function $f : [0,1] \to \mathbb{R}^u$, $v = d, k, l$ or l such that $\| f \|_{M_2,w} < \infty$. Let us put

$$\| f \|_{M_2,w} = \| f \|_{M_2} + \sup_{0 < t \leq 1} \frac{w_{M_2}(f,t)}{w(t)}$$

where

$$w(t) = \sqrt{t\left(1 + \log \frac{1}{t}\right)}.$$

$\mathcal{B}^{v}_{M_2,w}$ endowed with norm $\| \ . \ \|_{M_2,w}$ is a non separable Banach space. We will use the equivalent of Cieleski, Z., Kerkyacharian, G. & Roynette, B. [5]. Let $\chi_1, \chi_{j,k}, j = 0, 1..., k = 1...2^j, supp\chi_{j,k} = [(k-1)/2^j, k/2^j]$ be the set of Haar functions over the interval $[0,1]$, and let $\varphi_0(t) = 1, \varphi_1(t) = t, \varphi_{j,k}(t) = \int_0^t \chi_{j,k}(s)ds$ be the set of Schauder functions. For all continuous functions $f : [0,1] \to \mathbb{R}^d$, its development in series of Schauder is given by

$$f(t) = f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{n=2^j+1}^{2^{j+1}} \sum_{j,k} f_{j,k}\varphi_{j,k}(t)$$

where $f_0 = f(0), f_1 = f(1) - f(0)$ and

$$f_{j,k} \ = \ 22^{\frac{j}{2}} \left[f\left(\frac{2k-1}{2^{j+1}}\right) - \frac{1}{2}\left(f\left(\frac{2k}{2^{j+1}}\right) + f\left(\frac{2k-2}{2^{j+1}}\right)\right)\right].$$

We will consider a separable subspace $\mathcal{B}^{v,0}_{M_2,w}$ of $\mathcal{B}^{v}_{M_2,w}$ corresponding to the sequences $f_{j,k}$ such that

$$\mathcal{B}^{v,0}_{M_2,w} = \{ f \in \mathcal{B}^{v}_{M_2,w}, \ \| f \|_{M_2,\omega} < \infty, \ \lim_{j\vee p\to\infty} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}} \| f_{j,\cdot} \|_p = 0 \}$$

where

$$\| f \|_p = \left(\sum_{k=1}^{2^j} | f_{j,k} |^p \right)^{\frac{1}{p}}.$$

Theorem 2.1.

1) *Let* $p_0 \geq 0$, $f \in \mathcal{B}^{v}_{M_2,w}$ *if and only if*

$$\max\left\{ |f_0|, |f_1|, \sup_{j\geq 0} \sup_{p\geq p_0} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}} \| f_{j,\cdot} \|_p \right\} < \infty. \tag{2.1}$$

2) $f \in \mathcal{B}^{v,0}_{M_2,w}$ *if and only if*

$$\lim_{j\vee p\to\infty} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}} \| f_{j,k} \|_p = 0. \tag{2.2}$$

We consider the norms

$$\| f \|_{**} = \sup_{0\le s<t\le 1} \frac{| f_t - f_s |}{\omega(t-s)}$$

$$\| f \|_{*} = \max(| f(1) |, \sup_{j\ge 0} \sup_{0\le k\le 2^j} \frac{| f_{j,k} |}{\sqrt{1+j}}).$$

Let us remark that

$$\| f \| \le D_1 \| f \|_{M_2,\omega} \le D_2 \| f \|_{**} \le D_3 \| f \|_{*},$$

for some positive constants D_1, D_2 and D_3, where the last inequality holds only for functions f being null on zero and $\| \, . \, \|$ denotes the uniform norm. The two norms $\| \, . \, \|_*$ and $\| \, . \, \|_{**}$ play an important role in the proof of our results.

3. Definitions and general results

In this section we recall some definitions and results.

Definition 3.1. *A fonction $I : E \longrightarrow [0; +\infty]$ is said to be a rate function if it is lower semicontinuous (lsc) i.e. $\forall f_n \to f$, then $I(f) \le \lim_{n\to+\infty} \inf I(f_n)$. Furthermore, if for any $a < \infty$, $\Gamma_a = \{x \in E, I(x) \le a\}$ is compact, then I is a good rate function.*

Definition 3.2. *For some function I, the probabilities $\{P^\varepsilon\}_{\varepsilon>0}$ satisfy a large-deviation principle with the good rate function I if we have:*

$i)$ For every open subset O of E

$$\liminf_{\varepsilon\longrightarrow 0} \varepsilon \log P_\varepsilon(O) \ge -I(O)$$

$ii)$ For every closed subse F of E

$$\limsup_{\varepsilon\longrightarrow 0} \varepsilon \log P_\varepsilon(F) \le -I(F).$$

Mellouk, M. & Millet, A. [12] have proved the following Schilder theorem

Theorem 3.1. *Let P^ε be the law of ϵW on $\mathcal{B}^{v,0}_{M_2,w}$ with the norm $\| \, . \, \|_{M_2,w}$, then P^ε satisfy the LDP le PGD with the good rate function λ define by*

$$\lambda(g) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{g}(s)|^2 ds, & g \in \mathcal{H}, \\ +\infty, & otherwise. \end{cases} \tag{3.1}$$

Theorem 3.2. *A set* $A \subset \mathcal{B}^{v,0}_{M_2,\omega}$ *has compact closure in* $\mathcal{B}^{v,0}_{M_2,\omega}$ *if and only if the following two conditions hold*

$i)$ $\sup_{f\in A} \parallel f \parallel_{M_2,w} < \infty$;

$ii)$ $\lim_{\delta\to 0} \sup_{f\in A} \omega_{M_2}(f,\delta) = 0$.

Theorem 3.3. *Let* P^ε *be the probability measure family on a Polish space* E *satisfying the LDP with a good rate function* λ *and let* $F : E \to E'$ *be a continuous function. Denote by* $Q^\varepsilon = P^\varepsilon \circ F^{-1}$ *the image measure family of* P^ε *by* F*, then* $\{Q^\varepsilon\}$ *satisfy the LDP with a good rate function* $\tilde{\lambda}$ *define by:*

$$\tilde{\lambda}(y) = \inf_{x:F(x)=y} \lambda(x)$$

with $\inf \emptyset = +\infty$.

Let (E_X, d_X), $(E_Y, d_Y$, (E_Z, d_Z), (E', d') denote Polish spaces and $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that $\{X^\varepsilon, \varepsilon > 0\}$ is a family of random variables with value in E_X. Y is a random variable with value in E_Y. Z is a random variable with values in E_Z. Given a rate function I on E_X and $a > 0$, set

$$\Gamma_a = \{x \in E_X; I(x) \leq a\} \quad \text{and} \quad \Gamma_\infty = \cup_a \Gamma_a.$$

Theorem 3.4. *Let* I *be a good rate function on* E_X*,* F_N*,* $F : \Gamma_\infty \times \Gamma_Y \times E_Z \to E'$*;* $X^\varepsilon_N, X^\varepsilon : \Omega \to E'$ *be applications such that the following hold:*

(a) $\quad i)$ *For all* $a > 0$ *and* $N \geq 1$*,* $F_N/_{\Gamma_a \times SuppY \times SuppZ}$ *is continuous.*

$\quad ii)$ $F_N/_{\Gamma_a \times SuppY \times SuppZ}$ *converges to* $F/_{\Gamma_a \times SuppY \times SuppZ}$ *uniformly as* $N \to +\infty$.

(b) *For each* $a > 0$ *and* $N \geq 1$*,* $F_N\big(\{I \leq a\} \times SuppY \times Z\big)$ *and* $F\big(\{I \leq a\} \times SuppY \times Z\big)$ *are relatively compact in* (E', d').

(c) *For all* $N \geq 1$*,* $\{X^\varepsilon_N; \varepsilon > 0\}$ *satisfies an LDP (as* $\varepsilon \to 0$*) on* E' *with good rate function*

$$I^*_N(\zeta) = \lim_{\rho\to 0} \inf_{\xi \in B'(\zeta,\rho)} I_N(\xi)$$

where $B'(\zeta, \rho)$ *denotes the ball of radius* ρ *centered at* ζ *in* (E', d') *and*

$$I_N(\xi) = \inf\{I(x); \exists (y, z) \in SuppY \times SuppZ \ tel\ que\ F_N(x, y, z) = \xi\} \tag{3.2}$$

(d) $\{X^\varepsilon_N\}$ *are exponentially good approximations of* X^ε*; that is for every* $\delta > 0$

$$\lim_{N\to+\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P\{d'(X^\varepsilon_N, X^\varepsilon) > \delta\} = -\infty.$$

Then $\{X^\varepsilon; \varepsilon > 0\}$ *satisfies an LDP on* E' *with a good rate function*

$$\tilde{I}^*(\zeta) = \lim_{\rho\to 0}\inf_{\xi\in B'(\zeta,\rho)}\tilde{I}(\xi),$$

where $\tilde{I}(\xi) = \inf\{I(x); \exists(y,z) \in SuppY \times SuppZ\ tel\ que\ F(x,y,z) = \xi\}$.

4. Main result

Consider X^ε the solution of:

$$X^\varepsilon_t = x + \int_0^t b(X^\varepsilon_s, Y_s)ds + \varepsilon^{\frac{1}{2}}\int_0^t \sigma(X^\varepsilon_s, Z_s)dW_s, \tag{4.1}$$

Let $\Omega = \mathcal{C}([0,1], \mathbb{R}^k)$ be the space of trajectories of a standard $\mathbb{R}^k$-valued Brownian motion W, P the Wiener mesure and $\mathcal{F}$ the completion of the Borel σ-field of Ω with respect to P. Let $Y = \{Y_t, t \in [0,1]\}$ be a $\mathbb{R}^m$-valued process which is $\{\mathcal{F}_t\}$ progressively measurable. In order to make explicit the LDP rate function for the law of (4.1) in the Besov-Orlicz topology, we suppose that Y is a random variable with values in $L^1([0,1], \mathbb{R}^m)$. Let $Z = \{Z_t, t \in [0,1]\}$ be an $\mathcal{F}_t-$progrssively measurable process taking values in $\mathbb{R}^l$. We assume that $suppZ$ is a compact subset in $\mathcal{B}^{l,0}_{M_2,w}$, and that (Y, Z) and W are independent.
From now on, we suppose that the coefficients σ and b satisfy the following hypotheses (L):

i) $\sigma : \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^d \otimes \mathbb{R}^k$ and $b : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$.

ii) The function $b(x,y)$ is jointly measurable in (x,y) and there exists a constant $C > 0$ such that

$$\mid b(x,y) \mid \leq C(1+ \mid x \mid),$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^m$;

$$\mid b(x_1,y_1) - b(x_2,y_2) \mid \leq C(\mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid),$$

for all $(x_1,x_2) \in \mathbb{R}^d$ and for all $(y_1,y_2) \in \mathbb{R}^m$.

iii) The function $\sigma(x,z)$ is jointly measurable in (x,z) and there exists a constant $C>0$ such that

$$\mid \sigma(x,z) \mid \leq C,$$

for all $(x,z) \in \mathbb{R}^d \times \mathbb{R}^m$;

$$\mid \sigma(x_1,z_1) - \sigma(x_2,z_2) \mid \leq C(\mid x_1 - x_2 \mid + \mid z_1 - z_2 \mid),$$

for all $(x_1,x_2) \in \mathbb{R}^d$ and for all $(z_1,z_2) \in \mathbb{R}^m$.

The existence of a unique solution of (4.1) which is $(\mathcal{F}_t)_{t\in[0,1]}$-adapted and has continuous sample paths is ensured by our assumptions on σ and b.

Theorem 4.1. *Assume (L) is satisfied. Let X^ε be solution of* (4.1). *Then*

$$\mathbb{P}(X^\varepsilon \in \mathcal{B}^{v,0}_{M_2,w}) = 1.$$

Let $\mathcal{H}$ be the Cameron-Martin space associated with the Brownian motion.

$$\mathcal{H} = \left\{ h \in L^2([0,1]) : \text{ there exists } \dot{h} \in L^2([0,1]),\ h_t = \int_0^t \dot{h}_s ds,\ t \in [0,1] \right\}.$$

For $h \in \mathcal{H}([0,1],\mathbb{R}^k)$, $r \in L^1([0,1],\mathbb{R}^m)$ and $u \in suppZ$, we define the Skeleton $S_z(h,r,u) = g$ by

$$g_t = z + \int_0^t b(g_s,r_s)ds + \int_0^t \sigma(g_s,u_s)\dot{h}ds.$$

Define $\bar{\lambda} : \mathcal{B}^{d,0}_{M_2,w} \to [0,+\infty]$ by

$$\bar{\lambda}(\bar{h}) = \inf\{\lambda(h); h \in \mathcal{H}([0,1],\mathbb{R}^d) :$$
$$\text{exists } (r,u) \in suppY \times suppZ \text{ such that } \bar{h} = S(h,r,u)\}.$$

Since $\bar{\lambda}$ is not necessarily lsc (see for example Bezuidenhout [4]).

We introduce its lsc regularization $\bar{\lambda}^*$ defined by

$$\bar{\lambda}^*(\bar{h}) = \lim_{a\to 0} \lim_{\rho \in B_X(\bar{h},a)} \bar{\lambda}(\rho) \tag{4.2}$$

where $B_X(\bar{h},a)$ is the ball of radius a centered at $\bar{h}$ with respect the norm $\|\cdot\|_{M_2,w}$.

Theorem 4.2. *Assume (L). Then*

$i)$ $\bar{\lambda}^*$ *defined by* (4.2) *is a good rate function (with respect the topology of* $\mathcal{B}^{d,0}_{M_2,\omega}$*).*

$ii)$ *The family* $P^\varepsilon = P \circ (X^\varepsilon)^{-1}$ *of the laws of* X^ε *defined by* (4.1) *satisfies an LDP with a good rate function* $\bar{\lambda}^*$ *defined by* (4.2).

5. Regularity of the solution

The main purpose of this section is to prove the theorem 4.1. Suppose that the coefficients σ and b verifies (L) and we show $X_. \in \mathcal{B}^{v,0}_{M_2,w}$ a.s.. It is clear that the process $\{\int_0^t b(X_s, Y_s)ds, t \in I\}$ belongs a.s. to $\mathcal{B}^{v,0}_{M_2,w}$ because $E \mid X \mid < \infty$. Then, it remains to show that the process $\{\int_0^t \sigma(X_s, Z_s)dW_s, t \in I\}$ satisfies (2.1) and (2.2).

Let us put

$$G_t = \int_0^t \sigma(X_s, Z_s)dW_s\,.$$

We shall show that for some p_0, we will check that for any $\alpha < \frac{1}{2}$

$$\sup_{j\geq 0}\sup_{p\geq p_0} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^\alpha}\left(|G_.|^p\right)^{1/p} < \infty \; p.s. \tag{5.1}$$

and

$$\lim_{j\vee p\to\infty} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}}\left(|G_.|^p\right)^{1/p} = 0. \tag{5.2}$$

To check relation (5.1), let $\lambda > 0$, using Chebychev inequality, we get

$$P\left(\frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^\alpha}\left(|G_.|^p\right)^{1/p} > \lambda\right) \leq \frac{\lambda^{-p}2^{-j}}{\sqrt{\frac{p}{2}}(1+j)^{\alpha p}}\left(E|G_.|^p\right)$$

For $p \geq 2$, applying the inequality of Barlow-Yor [2] analyzing the constant C_p appearing in the Burkholder-Davis-Gundy inequality, we have

$$E|G_.|^p \leq CM^p p^{\frac{p}{2}}.$$

Hence

$$P\left(\frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^\alpha}\left(|G_.|^p\right)^{1/p} > \lambda\right) \leq \frac{\lambda^{-p}2^{-j}}{\sqrt{\frac{p}{2}}(1+j)^{\alpha p}}\left(E|G_.|^p\right) \leq \left(\frac{C}{\lambda}\right)^p \frac{1}{(1+j)^{p\alpha}}.$$

Choosing $p_0 \geq \frac{1}{\alpha}$ and λ large enough, the series

$$\sum_{j\geq 0}\sum_{p\geq p_0}\left(\frac{C}{\lambda}\right)^p \frac{1}{(1+j)^{p\alpha}}$$

converges.

To prove (5.2), the exponential inequalities yields that there exist positive constants $K > 0$ such that for all $\lambda > 0$ large enough,

$$P\left(\frac{1}{\sqrt{1+j}}|G_.| > \lambda\right) \leq K \exp\frac{-\lambda^2(1+j)}{K^2}.$$

Therefore, the Borel-Cantelli lemma leads to

$$\sup_{j\geq 1}\frac{1}{\sqrt{1+j}}|G_.| < \infty \;\; p.s.$$

or

$$2^{-\frac{j}{p}}\left(|G_.|^p\right)^{1/p} \leq |G_.|.$$

Thus,

$$\sup_{j\geq 1}\frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}}\left(|G_.|^p\right)^{1/p} \leq \frac{1}{p^{\frac{1}{2}}}\sup_{j\geq 1}|G_.|,$$

and this ends the establishment of (5.2).

6. Proof of the Theorem 4.2

This section is devoted to proving main Theorem 4.2 by means of Theorem (3.4). In order to apply Theorem (3.4), we use the following notation:

$$(E_X, d_X) = \mathcal{B}^{k,0}_{M_2,\omega}$$
$$(E_Y, d_Y) = L^1([0,1],\mathbb{R}^m)$$
$$(E_Z, d_Z) = \mathcal{B}^{l,0}_{M_2,\omega}$$
$$(E', d') = \mathcal{B}^{d,0}_{M_2,\omega}$$

Throughout this section $Y = \{Y_t, 0 \leq t \leq 1\}$ and $Z = \{Z_t, 0 \leq t \leq 1\}$ are the processes defined in Section 4. The rate function I is defined by

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}(s)|^2 ds, & f \in \mathcal{H} \\ +\infty & , \; otherwise. \end{cases}$$

In the following, for $a < \infty$, set $\Gamma_a = \{\lambda \leq a\}$, $\Gamma_\infty = \bigcup_a \Gamma_a$. For $\varepsilon > 0$, $N \geq 1$, set $\underline{t}_N = [Nt]/N$ ($[y]$ is the integer part of y). Let $X^\varepsilon = \{X^\varepsilon(t), 0 \leq t \leq 1\}$ be the solution of (4.1) and $X^\varepsilon_N = \{X^\varepsilon_N(t), 0 \leq t \leq 1\}$ be the solution the sdde

$$dX^\varepsilon_N(t) = b(X^\varepsilon_N(t), Y(t))dt + \sqrt{\varepsilon}\sigma(X^\varepsilon_N(\underline{t}_N), Z\underline{t}_N t))dW_t \tag{6.1}$$

We have to consider two cases

Case 1. b and σ are bounded
For $(r, u) \in L^1([0,1], \mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega}$, define the map

$$F_N(.) : \mathcal{B}^{d,0}_{M_2,\omega} \times L^1([0,1], \mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega} \to \mathcal{B}^{d,0}_{M_2,\omega}$$

by

$$\begin{cases} F_N(\omega, r, u)(t) = & F_N(\omega, r, u)(\frac{k}{N}) + \int_{\frac{k}{N}}^t b(F_N(\omega, r, u)(s), r(s))ds \\ & +\sigma(F_N(\omega, r, u)(\frac{k}{N}), u(\frac{k}{N}))(\omega(t) - \omega(\frac{k}{N})) \text{ for } \frac{k}{N} \leq t \leq \frac{k+1}{N}. \end{cases}$$

Let $g \in \mathcal{H}$ and

$$I(g) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{g}(s)|^2 ds, & g \in \mathcal{H} \\ +\infty, & \text{otherwise.} \end{cases}$$

For $g \in \mathcal{H}$ with $I(g) \leq a$

$$\begin{cases} F_N(\omega, r, u)(t) & = F_N(g, r, u)(0) + \int_0^t b(F_N(g, r, u)(s), r(s))ds \\ & + \int_0^t \sigma(F_N(g, r, u)(\frac{[Ns]}{N}), u(\frac{[Ns]}{N}))\dot{g}ds \text{ for } t \in [0, \infty). \end{cases}$$

Notice that $X^\varepsilon_N(s) = F_N(\sqrt{\varepsilon}W, Y, Z)(s)$ where W is standard brownian motion. Define $I_N(f) = \inf\{\frac{1}{2}I(g); F_N(g) = f\}$ for each $f \in \mathcal{B}^{d,0}_{M_2,\omega}$.

Case 2. for any b and σ

We remove the boundedness assumptionon b and σ.

For $R > 0$, define $m_R := \sup\{|b(x, y), |\sigma(x, y)|, t \in [0, m], |x| \leq R, |y| \leq R\}$ and $b^i_R =:= (-m_R - 1) \vee b_i \wedge (m_R + 1)$, $\sigma^R_{i,j} =:= (-m_R - 1) \vee \sigma_{i,j} \wedge (m_R + 1)$, $1 \leq i, j \leq d$.

Put $b_R := (b^R_1, ..., b^R_d)$ and $\sigma_R := (\sigma^R_{i,j})_{1 \leq i,j \leq d}$. Then $b_R(x, y) = b(x, y)$, $\sigma_R(x, y) = \sigma(x, y)$ for $t \in [0, 1], |x| \leq R, |y| \leq R$. Furthermore b_R, σ_R satisfy the Lipschitz condition (H_1)-(H_2) with the same Lipschitz constant.

For $I(g) \leq \infty$ with $g \in \mathcal{H}$ is absolutely continuous and for $(r,u) \in L^1([0,1],\mathbb{R}^m)\times \mathcal{B}^{l,0}_{M_2,\omega}$, let $F_R(g,r,u)$ be the solution of

$$\begin{cases} F_R(g,r,u)(t) = & F_R(g,r,u)(0) + \int_{\frac{k}{N}}^t b(s, F_R(g,r,u)(s), r(s))ds \\ & + \quad \sigma(s, F_R(g,r,u)(s), u(s))\dot{g}ds \text{ for } t \in [0,\infty). \end{cases}$$

Define $I_R(f) = \inf\{\frac{1}{2}I(g); F_N(g) = f\}$ for each $f \in \mathcal{B}^{d,0}_{M_2,\omega}$.

The existence and uniqueness of the solution of (6.1) follows from hypothesis (H_0)-(H_1)-(H_2) on the coefficients. Furthermore, the trajectories of X^ε and X^ε_N belong almost surely to $\mathcal{B}^{u,0}_{M_2,\omega}$. From now one, to prove the Theorem 4.2, we will follow step by step the assumptionons of Theorem 3.4.

6.1. **Continuity of** F_N**.** We prove that $F_N : \mathcal{B}^{d,0}_{M_2,\omega} \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega} \to \mathcal{B}^{d,0}_{M_2,\omega}$ (resp. F_R) is continuous. Fix $N \geq 1$ and let $(h_1,r_1,u_1),(h_2,r_2,u_2) \in \mathcal{B}^{d,0}_{M_2,\omega} \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega}$; then set $F^{(i)}_N(.) = F_N(h_i,r_i,u_i)$, $i = 1,2$ and $\Psi_N(.) = F^{(1)}_N(.) - F^{(2)}_N(.)$.

Lemma 6.1. *For all $a > 0$, there exists a constant $C = C_N > 0$ such that $\| h_1 \|_{**} \vee \| h_2 \|_{**} \leq a$, we have*

(6.2) $$\| \Psi_N(.) \|_{**} \leq C_N\{\| h_1 - h_2 \|_{**} + \| r_1 - r_2 \|_{L^1} + \| u_1 - u_2 \|_\infty\}.$$

Proof. For $t \in [0,1]$, we have:

$$\begin{aligned} \Psi_N(t) = &\sum_{l=1}^N \big\{\sigma\big(F^{(1)}_N(g_1,r_1,u_1)(l/N), u_1(l/N)\big) \\ &- \sigma\big(F^{(2)}_N(g_2,r_2,u_2)(l/N), u_2(l/N)\big)\big\} \times \{g_2(t) - g_2(l/N)\} \\ &+ \sum_{l=1}^N \big\{\sigma\big(F^{(1)}_N(g_1,r_1,u_1)(l/N), u_1(l/N)\big)\big\} \\ &\times \big\{[g_1(t) - g_1(l/N)] - [g_2(t) - g_2(l/N)]\big\} \\ &+ \int_0^t \big\{b\big(F^{(1)}_N(g_1,r_1,u_1)(s), r_1(s)\big) - b\big(F^{(2)}_N(g_2,r_2,u_2)(s), r_2(s)\big)\big\}ds. \end{aligned}$$

For $0 \leq l \leq N-1$, let $I_{N,l} = [l/N,(l+1)/N]$; then, for $t \in I_{N,l}$, we have

$$\| \Psi_N(t) \| \leq \| \Psi(l/N) \| + \sum_{i=1}^3 \| V_i(t) \|,$$

where

$$V_1(t) = \Big\{\sigma\big(F_N(g_1, r_1, u_1)(l/N), u_1(j/N)\big) - \sigma\big(F_N(g_2, r_2, u_2)(l/N), u_2(l/N)\big)\Big\} \times \{g_2(t) - g_2(l/N)\},$$

$$V_2(t) = \Big\{\sigma\big(F_N(g_1, r_1, u_1)(l/N), u_1(l/N)\big)\Big\} \times \Big\{[g_1(t) - g_1(l/N)] - [g_2(t) - g_2(l/N)]\Big\},$$

$$V_3(t) = \int_{l/N}^{t} \Big\{b\big(F_N(g_1, r_1, u_1)(s), r_1(s)\big) - b\big(F_N(g_2, r_2, u_2)(s), r_2(s)\big)\Big\}ds.$$

The Lipschitz condition on σ implies

$$V_1(t) \leq C\Big\{|\big(F_N(g_1, r_1, u_1)(l/N) - F_N(g_2, r_2, u_2)(l/N)| + |u_1(l/N) - u_2(l/N|\Big\} \times | g_2(t) - g_2(l/N) | \leq C\Big\{ \| \Psi_N(l/N) \| + \| u_1 - u_2 \|_\infty \Big\} \times 2 \| g_2 \|_\infty .$$

Since σ is bounded, we have

$$V_2 \leq \| \sigma \|_\infty \sup_{l/N \leq t \leq (l+1)/N} \Big\{|g_1(t) - g_2(t)| + |g_1(l/N) - g_2((l/N))|\Big\} \leq C \| g_1 - g_2 \|_\infty .$$

The Lipschitz condition on b implies

$$V_3(t) \leq C \| r_1 - r_2 \|_{L^1} + 2C \int_{l/N}^{t} \sup_{\frac{l}{N} \leq v \leq s} \{|F_N(g_1, r_1, u_1)_v - F_N(g_2, r_2, u_2)_v|\}.$$

Grownall's lemma implies that, for $\| g_1 \|_\infty \vee \| g_2 \|_\infty \leq C$,

(6.3) $$\sup_{t \in I_{N,l}} | \Psi(t) | \leq C[| \Psi(l/N) | + \| g_1 - g_2 \|_\infty + \| r_1 - r_2 \|_{L^1} + \| u_1 - u_2 \|_\infty];$$

hence, for $t = (l+1)/N$,

(6.4) $$| \Psi(\frac{l+1}{N}) | \leq C[| \Psi(l/N) | + \| g_1 - g_2 \|_\infty + \| r_1 - r_2 \|_{L^1} + \| u_1 - u_2 \|_\infty].$$

This, in turn, implies that, for $\| g_1 \|_\infty \vee \| g_2 \|_\infty \leq C$, there exists a constant $C_N > 0$ such that

$$\sup_{0 \leq l \leq N-1} | \Psi(\frac{l}{N}) | \leq C_N[\| g_1 - g_2 \|_\infty + \| r_1 - r_2 \|_{L^1} + \| u_1 - u_2 \|_\infty].$$

Finally, (6.3) et (6.4) imply

(6.5) $$\| \Psi_N(.) \|_\infty \leq C_N[\| g_1 - g_2 \|_\infty + \| r_1 - r_2 \|_{L^1} + \| u_1 - u_2 \|_\infty]$$

Now, for $(l-1)/N \leq t \leq l/N$, we have

$$\begin{aligned}\| \Psi_N(.) \|_{**} =& \| \sigma(F_N^{(1)}((l-1)/N), u_1((l-1)/N))g_1(t) \\ & - \sigma(F_N^{(2)}((l-1)/N), u_2((l-1)/N))g_2(t) \\ & + \int_0^t [b(F_N^{(1)}(v), r_1(v)) - b(F_N^{(2)}(v), r_2(v))]dv \|_{**} \\ =& \| \sigma(F_N^{(1)}((l-1)/N), u_1((l-1)/N))\{g_1(t) - g_2(t)\} \\ & + \{\sigma(F_N^{(1)}((l-1)/N), u_1((l-1)/N)) \\ & - \sigma(F_N^{(2)}((l-1)/N), u_2((l-1)/N))\}g_2(t) \\ & + \int_0^t \{[b(F_N^{(1)}(v), r_1(v)) - b(F_N^{(1)}(v), r_2(v))] \\ & + [b(F_N^{(1)}(v), r_2(v)) - b(F_N^{(2)}(v), r_2(v))]\}dv \|_{**} \\ \leq & C \| g_1 - g_2 \|_{**} + C\{\| \Psi_N(.) \|_\infty + \| u_1 - u_2 \|_\infty\} \mid g_2(t) \mid \\ & + C \sup_{0\leq x\leq y\leq 1} \int_x^y \left(\frac{\mid \Psi_N(s) \mid + \mid r_1(s) - r_2(s) \mid}{\omega(y-x)} \right) ds.\end{aligned}$$

Using Cauchy-Schwartz inequality

$$\begin{aligned}\| \Psi_N(.) \|_{**} \leq & C \| g_1 - g_2 \|_{**} + C\{\| \Psi_N(.) \|_\infty + \| u_1 - u_2 \|_\infty\} \mid g_2(t) \mid \\ & + C(\| \Psi_N(.) \|_\infty + \| r_1 - r_2 \|_{L^1}) \sup_{0\leq x\leq y\leq 1} \frac{\sqrt{y-x}}{\omega(y-x)}\end{aligned}$$

Using (6.5) and $\| g_2 \|_\infty < C$, then, we have (6.2). □

6.2. **Uniform convergence of** F_N **to** F **on** $\Gamma_a \times suppY \times suppZ$**.** To verify assertion $(a)(ii)$ of Theorem (3.2), we first prove the following

Lemma 6.2. *For any* $a > 0$*,*

(6.6) $$\sup_N \sup_{\|g\|_{\mathcal{H}\leq a}} \sup_{(r,u)\in suppY\times suppZ} (\| F_N(g,r,u)(.) \|_\infty \vee \| F(g,r,u)(.) \|_\infty) < \infty$$

and

(6.7) $$\lim_{N\to+\infty} \sup_{\|g\|_{\mathcal{H}}\leq a} \sup_{(r,u)\in suppY\times suppZ} \| F_N(g)(.) - F(g)(.) \|_{**} = 0.$$

Proof. The proof of (6.6) is a straightforward application of Gronwall's lemma. We will check (6.7). For $g \in \mathcal{H}$ with $\| g \|_{\mathcal{H}} \leq a$, we have

$$|F(g,r,u)(t) - F(g,r,u)(\underline{s}_N)| \leq \int_{\underline{t}_N}^{t} |b(F(g,r,u)(s), r(s))| ds + \int_{\underline{t}_N}^{t} |\sigma(F(g,r,u)(\underline{s}_N), u(\underline{s}_N)||\dot{g}|(s) ds.$$

Hence hypotheses (H_0)-(H_2) on coefficients together with the Cauchy-Schwartz inequality and (6.6) yield the existence of a constant $C > 0$ depending on $\| \sigma \|_\infty$ and a such that

$$\| F(g,r,u)(t) - F(g,r,u)(\underline{t}_N) \| \leq C\bigg(\frac{1}{\sqrt{N}} \| h \|_{\mathcal{H}} + \int_{\underline{t}_N}^{t} (1 + \| F(g,r,u)(s) \|) ds \bigg) \leq \frac{C}{\sqrt{N}}. \tag{6.8}$$

Furthermore, for $t \in [0,1]$, using $(H_0) - (H_2)$ we have

$$\begin{aligned}
&|F_N(g,r,u)(t) - F(g,r,u)(t)| \\
&= \bigg| \int_0^t \{\sigma(F(g,r,u)(s), u(s)) - \sigma(F(g,r,u)(\underline{s}_N), u(\underline{s}_N))\} \dot{g}(s) ds \\
&\quad + \int_0^t \{\sigma(F(g,r,u)(\underline{s}_N), u(\underline{s}_N)) - \sigma(F_N(g,r,u)(\underline{s}_N), u(\underline{s}_N))\} \dot{g}(s) ds \\
&\quad + \int_0^t \{b(F(g,r,u)(s), u(s)) - b(F_N(g,r,u)(s), u(s)\} \dot{g}(s) ds \bigg| \\
&\leq C\Bigg(\bigg\{ \int_0^t |F(g,r,u)(s) - F(g,r,u)(\underline{s}_N)| + |u(s) - u(\underline{s}_N)| \bigg\} |\dot{g}_s| ds \\
&\quad + \int_0^t |F_N(g,r,u)(\underline{s}_N) - F(g,r,u)(\underline{s}_N)||\dot{g}_s| ds \\
&\quad + \int_0^t |F_N(g,r,u)(s) - F(g,r,u)(s)| ds \Bigg)
\end{aligned}$$

By the Cauchy-Schwartz inequality and (6.8), there exists a constant $C_a > 0$ such that for $\| h \|_{\mathcal{H}} \leq a$,

$$\sup_{v\leq t} |F_N(g,r,u)(v) - F(g,r,u)(v)|^2$$
$$\leq C_a \big\{ 1/N + \sup_{s\in[0,1]} |u(s) - u(\underline{s}_N)|^2 \big\}$$
$$+ \int_0^t \sup_{v\leq s} \mid F_N(g,r,u)(v) - F(g,r,u)(v) \mid^2 ds.$$

Hence, Grownall's lemma yields

(6.9)
$$\sup_{v\leq s} \mid F_N(g,r,u)(v) - F(g,r,u)(v) \mid^2 \leq C_a \big(1/N + \sup_{s\in[0,1]} \mid u(s) - u(\underline{s}_N) \mid^2 \big) \exp C_a$$

Since $suppZ$ is a compact subset $\mathcal{B}^{l,0}_{M_2,\omega}$, Ascoli's theorem implies that

(6.10)
$$\lim_{N} \sup_{s\in[0,1]} \mid u(s) - u(\underline{s}_N \mid = 0;$$

then (6.9) and (6.10) imply

(6.11)
$$\lim_{N\to\infty} \sup_{\|g\|_{\mathcal{H}}\leq a} \sup_{(r,u)\in suppY\times suppZ} \| F_N(g)(.) - F(g)(.) \|_{\infty} = 0.$$

On the other hand

$$\| F_N(g,r,u)(.) - F(g,r,u)(.) \|_{**} \leq C_a \left[\frac{1}{N} + \sup_{s\in[0,1]} |u(\underline{s}_N) - u(s))|^2 \right]$$
$$+ C \sup_{0\leq x<y\leq 1} \int_x^y \frac{(1+|\dot{g}|(s))|F_N(g,r,u)(s) - F(g)(s,r,u)|ds}{\omega(y-x)}.$$

The Cauchy-Schwartz inequality implies

$$\| F_N(g,r,u)(.) - F(g,r,u)(.) \|_{**} \leq C_a \left[\frac{1}{\sqrt{N}} + \sup_{s\in[0,1]} |u(\underline{s}_N) - u(s)|^2 \right]$$
$$+ 2C(1+ \| g \|_{\mathcal{H}}) \| F_N(g,r,u)(.) - F(g,r,u)(.) \|_{\infty} \sup_{0\leq x<y\leq 1} \frac{\sqrt{y-x}}{\omega(y-x)},$$

(6.9) and (6.11) imply (6.7). □

6.3. **Relative compactness.** The aim of this subsection is to prove condition (b) of Theorem (3.4), which follows from the following:

Lemma 6.3. *Let λ be the good rate function defined by* (3.1)*, and K be a relatively compact subset of $\mathcal{B}^{k,0}_{M_2,w}$. Then, for each $N \geq 1$, for $a > 0$, the sets $F_N(K \times suppY \times suppZ)$, $F_N(\{\lambda \leq a\} \times suppY \times suppZ)$ and $F(\{\lambda \leq a\} \times suppY \times suppZ)$ are relatively compact in $\mathcal{B}^{d,0}_{M_2,w}$.*

Proof. Since K is relatively compact, it is bounded in $\mathcal{B}^{u,0}_{M_2,w}$. For $N \geq 1$, for $h \in K$, $r \in suppY$ and $u \in suppZ$. Then by $(H_0) - (H_2)$ it is easy to see that there exists a constant $C_N > 0$ such that

$$\| F_N(h, r, u) \| \quad \leq \quad C_N \| h \|_{\mathcal{H}} + C_N \int_0^t (1 + \sup_{s \leq t} \| F_N(h_s, r_s, u_s) \|) ds.$$

The Grownall's Lemma implies

$$\| F_N(h, r, u) \| \leq C_N \big(\| h \|_{\mathcal{H}} + 1 \big) \exp 1 = K.$$

Then, for $N \geq 1$,

$$\| F_N(h, r, u) \|_{**} \leq C_N \Bigg(\| h \|_{\mathcal{H}} + \sup_{0 \leq x < y \leq 1} \int_x^y \frac{(1 + \| F_N(h_s, r_s, u_s) \|)}{w(| y - x |)} ds \Bigg).$$

The Cauchy-Schwartz inequality and the inequality (6.12) yields:

$$\| F_N(h, r, u) \|_{**} \leq C_N \Bigg(\| h \|_{\mathcal{H}} + (1 + K) \sup_{0 \leq x < y \leq 1} \frac{\sqrt{y - x}}{w(y - x)} \Bigg). \tag{6.12}$$

On the other hand, for $0 < \delta < 1$,

$$\begin{aligned} w_{M_2}(F_N(h, r, u), \delta) &= \sup_{0 \leq h \leq \delta \leq 1} \| \Delta_h F_N \| \\ &\leq C_N \Bigg(w_{M_2}(h, \delta) + \delta (1 + K) \sup_{0 \leq x < y \leq 1} \frac{\sqrt{y - x}}{w(y - x)} \Bigg). \end{aligned} \tag{6.13}$$

Since $h \in K$ is relatively compact in $\mathcal{B}^{k,0}_{M_2,w}$, then (6.12), (6.13) and Ascoli's result that $F_N(K \times suppY \times suppZ)$ is relatively compact $\mathcal{B}^{d,0}_{M_2,w}$. The same arguments prove the relative compactness of $F(\{\lambda \leq a\} \times suppY \times suppZ)$ in $\mathcal{B}^{d,0}_{M_2,w}$, since $\{\lambda \leq a\}$ is a compact subset of $\mathcal{B}^{k,0}_{M_2,w}$. □

6.4. **Large Deviation principle for X_N^ε (as $\varepsilon \to 0$).** For $N \geq 1$, we prove that the family $X_N^\varepsilon \equiv F_N(\sqrt{\varepsilon}W, Y, Z)$ defined by (6.1) satisfies on $\mathcal{B}^{d,0}_{M_2,\omega}$ an LDP, and show that the rate function is of the form (3.2). Since F_N is continuous on $\mathcal{B}^{d,0}_{M_2,\omega} \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega} \to \mathcal{B}^{d,0}_{M_2,\omega}$, we use a version of the contraction principle. Schilder's theorem implies that $\sqrt{\varepsilon}W$ satisfies an LDP on $\mathcal{B}^{k,0}_{M_2,\omega}$ with rate function λ defined by (3.1).

For $N \geq 1$, define

$$\lambda_N(f) = \inf\{\lambda(h), h \in \mathcal{H}([0,1],\mathbb{R}^d) : \\ \exists r \in suppY,\ u \in suppZ \ such\ that\ F_N(h,r,u) = f\},$$

and let $\lambda_N^*(f)$ be its lsc regularisation, i.e., $\lambda_N^*(f) = \lim_{a\to 0} \inf_{g\in B(f,a)} \lambda_N(g)$. An argument similar to that in the proof of Theorem (3.4) shows that λ_N^* is a good rate function, and we check that (X_N^ε) satisfies an LDP with rate function λ_N^*.

We first check the large-deviation lower bound.

Lemma 6.4. *Let O be an open subset of $\mathcal{B}^{d,0}_{M_2,w}$. Then*

$$\liminf_{\varepsilon\to 0} \log \mathbb{P}\{X_N^\varepsilon \in O\} \geq -\inf\{\lambda_N^*(f); f \in O\}$$

.

Proof. Assume $O \neq$, and let $f \in O$ be such that $\lambda_N^*(f) < \infty$; we prove

$$\liminf_{\varepsilon\to 0} \log \mathbb{P}\{X_N^\varepsilon \in O\} \geq -\lambda_N^*(f).$$

By definition, given $\delta, \gamma > 0$, there exist $(h,r,u) \in \mathcal{H}([0,1],\mathbb{R}^d) \times suppY \times suppZ$ such that

$$\| F_N(h,r,u) - f(.) \|_{M_2,\omega} < \gamma \ et\ \lambda(h) \leq \lambda_N^*(f) + \delta.$$

We can choose δ small enough to ensure $B(f,2\delta) \subset O$. The continuity of $F_N(.,r,u)$ on $\mathcal{B}^{k,0}_{M_2,\omega} \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,\omega}$ implies the existence of $\beta > 0$ such that

$$\| Y - r \|_{L^1([0,1],\mathbb{R}^m)} < \beta,\ \| Z - u \|_{M_2,w} < \beta,\ \| \varepsilon^{1/2}W - h \|_{M_2,w} < \beta$$

and

$$\left\{\| \varepsilon^{1/2}W - h \|_{M_2,w} \cap \| Y - r \|_{L^1} \cap \| Z - u \|_{M_2,w} < \beta\right\} \subset \{X_N^\varepsilon \in O\}\,.$$

Since $(r,u) \in suppY \times suppZ$, $P(\| Y - r \|_{L^1} < \beta, \| Z - u \|_\infty < \beta) > 0$ and the independence of W and (Y,Z) yield

$$\begin{aligned}\liminf_{\varepsilon\to 0} \varepsilon \log P\{X_N^\varepsilon \in O\} &\geq \liminf_{\varepsilon\to 0} \varepsilon \log P\{\| \varepsilon^{1/2} W - h \|_{M_2,w} < \beta\} \\ &\geq -\lambda(h) \\ &\geq -\lambda_N^*(f) - \delta.\end{aligned}$$

Letting $\delta \to 0$, we conclude the proof. □

We now prove the large-deviation upper bound.

Lemma 6.5. *Let A be a closed subset of $\mathcal{B}^{d,0}_{M_2,w}$; then*

$$\liminf_{\varepsilon\to 0} \log \mathbb{P}\{X_N^\varepsilon \in A\} \leq -\inf\{\lambda_N^*(f); f \in A\}.$$

Proof. Let

$$H(A) = \{h \in \mathcal{B}^{d,0}_{M_2,w} : \exists r \in suppY, \exists u \in suppZ \ such\ that\ F_N(h,r,u) \in A\}.$$

We have

$$\{X_N^\varepsilon \in A\} = \{F_N(\varepsilon^{1/2} W, Y, Z) \in A\} \subset \{\varepsilon^{1/2} W \in H(A)\}.$$

The Schilder's theorem implies

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(X_N^\varepsilon \in A) \leq -\inf\{\lambda(h); h \in \overline{H(A)}\},$$

where $\overline{H(A)}$ is the closure of $H(A)$ in $\mathcal{B}^{d,0}_{M_2,w}$. Now, let $h \in \overline{H(A)}$; then there exist sequences $h_n \in H(A)$, $r_n \in suppY$ and $u_n \in suppZ$ such that $g_n = F_N(h_n, r_n, u_n) \in A$ and h_n converges to h in $\mathcal{B}^{d,0}_{M_2,w}$. Since h_n is relatively compact in $\mathcal{B}^{k,0}_{M_2,w}$, it follows from Lemma 6.3 that g_n is also relatively compact in $\mathcal{B}^{d,0}_{M_2,w}$; thus (by extracting a subsequence) we may and do assume that g_n converges in $\mathcal{B}^{d,0}_{M_2,w}$, say to g. Note that $g_n \in A$ and A is closed, so that $g \in A$.
Set $\bar{g}_n = F_N(h, r_n, u_n)$; (6.2) implies that $\lim_n \| g_n - \bar{g}_n \|_{M_2,w} = 0$. Finally, for each $h \in \overline{H(A)}$, by definition of λ_N^*:

$$\inf\{\lambda_N^*(f); f \in A\} \leq \lambda_N^*(g) \leq \liminf_{n\to\infty} \lambda_N(\bar{g}_n) \leq \lambda(h).$$

□

Finally, we show that $\{X_N^\varepsilon, \varepsilon > 0\}$ defined by (6.1) are exponentially good approximations of $\{X^\varepsilon, \varepsilon > 0\}$ defined by (4.1).

Let us at first establish the following approximation.

Lemma 6.6. *For any $\delta > 0$, we have*

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\left(\| X^\varepsilon - X_N^\varepsilon \|_{**} > \frac{\delta}{N}\right) = -\infty. \tag{6.14}$$

Proof. Since the drift coeficient b is not neccessarily bounded, to prove (6.14), let us introduce some auxilliary results. For $N \geq 1$, by the Theorem 3.1,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon P\big(\sup_{1\leq k\leq N} |\sqrt{\varepsilon}(W_{k/N} - W_{(k-1)/N})| \geq N\big) \\
&\qquad \leq \limsup_{\varepsilon\to 0} \varepsilon \ln P(\| \sqrt{\varepsilon} W \|_{M_{2,\omega}} \geq N) \\
&\qquad \leq -\inf\left\{\frac{1}{2} \| h \|_{\mathcal{H}}^2; \| h \|_{M_{2,\omega}} \geq N\right\} \\
&\qquad \leq -\frac{1}{2} N^2 .
\end{aligned} \tag{6.15}$$

Indeed, if $h \in \mathcal{H}([0,1], \mathbb{R}^d)$ satisfies $\| h \|_{M_{2,\omega}} \geq N$, the Cauchy-Schwartz inequality implies that $\| h \|_{\mathcal{H}} \geq N$.

Define the set

$$\Gamma_\varepsilon = \{ \sup_{1\leq k\leq N} |\sqrt{\varepsilon}(W_{k/N} - W_{(k-1)/N})| \leq N\} \cap \{\| \sqrt{\varepsilon} W \|_{M_{2,\omega}} \leq N\},$$

by (6.15)

$$\lim_{N\to +\infty} \limsup_{\varepsilon\to 0} \varepsilon \ln P(\Gamma_\varepsilon^c) = -\infty. \tag{6.16}$$

To prove (6.14), set $\Psi_N^\varepsilon(t) = X_N^\varepsilon(t) - X^\varepsilon(t)$ and $\underline{t}_N = \frac{[Nt]}{N}$; then for $t \geq 0$, $\Psi_N^\varepsilon(t)$ satisfies

$$\begin{aligned}
\Psi_N^\varepsilon(t) = &\int_0^t [b(X_N^\varepsilon(s), Y_s) - b(X^\varepsilon(s), Y_s)] ds \\
&+ \sqrt{\varepsilon} \int_0^t [\sigma(X_N^\varepsilon(\underline{s}_N), Z(\underline{s}_N)) - \sigma(X^\varepsilon(s), Z(s))] dW_s.
\end{aligned}$$

For $\rho > 0$, we define

$$\begin{aligned} \tau^{\varepsilon}_{N,\rho}(t) &:= \inf\left\{ t \geq 0; \| \, X^{\varepsilon}_N(t) - X^{\varepsilon}_N(\underline{t}_N) \, \|_{M_2,\omega} \geq \frac{\rho}{N} \right\}, \\ \Psi^{\varepsilon}_{N,\rho}(t) &:= \Psi^{\varepsilon}_N(t \wedge \tau^{\varepsilon}_{N,\rho}), \\ \upsilon^{\varepsilon}_{N,\rho} &:= \inf\left\{ t \geq 0, \| \, \Psi^{\varepsilon}_{N,\rho}(t) \, \|_{M_2,\omega} \geq \frac{\delta}{N} \right\}, \\ \theta^{\varepsilon}_{N,\rho}(t) &= \int_{\Omega} \left\{ \frac{\rho^2}{N^2} + \| \, \Psi^{\varepsilon}_{N,\rho}(t) \, \|^2_{M_2,\omega} \right\}^{\frac{1}{\varepsilon}} dP. \end{aligned}$$

Then clearly

$$P\left(\| \, \Psi^{\varepsilon}_N(.) \, \|_{M_2,\omega} > \frac{\delta}{N} \right) \ \leq \ P(\tau^{\varepsilon}_{N,\rho} < 1) + P(\upsilon^{\varepsilon}_{N,\rho} < 1).$$

We have

$$\begin{aligned} P(\tau^{\varepsilon}_{N,\rho} < 1) &= P(\tau^{\varepsilon}_{N,\rho} < 1, \Gamma_{\varepsilon}) + P(\tau^{\varepsilon}_{N,\rho} < 1, \Gamma^c_{\varepsilon}) \\ &\leq \sum_{k=1}^{N} P\left(\sup_{\frac{k-1}{N} \leq t \leq \frac{k}{N}} \| \, X^{\varepsilon}_N(t) - X^{\varepsilon}_N(\frac{k-1}{N}) \, \|_{M_2,\omega} \geq \frac{\rho}{N}, \Gamma_{\varepsilon} \right) + P(\Gamma^c_{\varepsilon}) \\ &\leq \sum_{k=1}^{N} P\left(\sup_{\frac{k-1}{N} \leq t \leq \frac{k}{N}} \| \int_{\frac{k-1}{N}}^{t} \sqrt{\varepsilon}\sigma(X^{\varepsilon}(s), Z_s) dW_s \, \|_{M_2,\omega} \geq \frac{\rho - k}{N}, \Gamma_{\varepsilon} \right) \\ &\quad + P(\Gamma^c_{\varepsilon}). \end{aligned}$$

where k is a constant > 0.

By the Proposition 6.5 in [10], for all $r > 0$ and $\rho > k$, there exist $\varepsilon > 0$ such that

$$P\left(\sup_{\frac{k-1}{N} \leq t \leq \frac{k}{N}} \| \int_{\frac{k-1}{N}}^{t} \sqrt{\varepsilon}\sigma(X^{\varepsilon}(s), Z_s) dW_s \, \|_{M_2,\omega} \geq \frac{\rho - k}{N}, \Gamma_{\varepsilon} \right) \leq \exp(\frac{-r}{\varepsilon}).$$

So

$$P(\tau^{\varepsilon}_{N,\rho} < 1) \leq N \exp\left(\frac{-r}{\varepsilon} \right) + P(\Gamma^c_{\varepsilon}),$$

and (6.16) implies that

$$\lim_{N \to +\infty} \limsup_{\varepsilon \to 0} \varepsilon \log P(\tau^{\varepsilon}_{N,\rho} \leq 1) = -\infty. \tag{6.17}$$

Since $suppZ$ is a compact subset of $\mathcal{B}^{l,0}_{M_2,\omega}$, for $\rho > 0$ there exist $N_0 \geq 1$ such that, for $N \geq N_0$,

(6.18) $$\sup_{0\leq t\leq 1} \mid Z(t) - Z(\underline{t}_N) \mid \leq \frac{\rho}{N}.$$

For $0 < \varepsilon < \frac{1}{2}$, put $\alpha_N = \frac{\rho}{N}$ and $f_{\varepsilon,\rho}(\Psi^\varepsilon_N(t)) = (\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega})^{1/\varepsilon}$. By the Itô's formula

$$M_t^{N,\rho} := f_{\varepsilon,\rho}(\Psi^\varepsilon_{N,\rho}(t)) - \int_0^{t\wedge\tau^\varepsilon_{N,\rho}} g^\rho_{\varepsilon,N}(s)ds - \alpha_N^{2/\varepsilon}$$

is a martingale, where, if $< ., . >$ is the inner product in $\mathbb{R}^d$,

$$\begin{aligned} g^\rho_{\varepsilon,N}(t) &= \frac{2}{\varepsilon}(\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi(t) \parallel^2_{M_2,\omega})^{\frac{1}{\varepsilon}-1}\langle \Psi^\varepsilon_N(t), b(X^\varepsilon(t), Y(t)) \\ &- b(X^\varepsilon_N(t), Y(t))\rangle + \frac{2}{\varepsilon}(\frac{1}{\varepsilon} - 1)\varepsilon(\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega})^{\frac{1}{\varepsilon}-2} \\ &\times |(\sigma(X^\varepsilon(t), Z(t)) - \sigma(X^\varepsilon_N(\underline{t}_N), Z(\underline{t}_N))|^2 \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2 \\ &+ (\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel_{M_2,\omega})^{1/\varepsilon-1}|(\sigma(t, X^\varepsilon(t), Z(t)) \\ &- \sigma(X^\varepsilon_N(\underline{t}_N), Z(\underline{t}_N))|^2 \end{aligned}$$

For $0 \leq t \leq \tau^\varepsilon_{N,\rho}$, using (6.18), we have, for $N \geq N_0$ and $0 < \varepsilon < \frac{1}{2}$ that there exist $C > 0$ such that

$$\begin{aligned} &\parallel g^\rho_{\varepsilon,N}(.) \parallel_{M_2,\omega} \\ &\leq C\frac{2}{\varepsilon}\left(\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega}\right)^{1/\varepsilon}\left(\frac{\sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel_{M_2,\omega}}{\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega}}\right) \\ &\quad \times \sup_{0\leq t\leq 1} \parallel X^\varepsilon_N(t) - X^\varepsilon(t) \parallel_{M_2,\omega} + C \mid \frac{1}{\varepsilon} - 1 \mid \left\{\alpha_N^2 + \mid Z(\underline{t}_N - Z(t)) \mid^2 \right\} \\ &\quad \times \left(\frac{\sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega}}{\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega}}\right)\left(\alpha_N^2 + \sup_{0\leq t\leq 1} \parallel \Psi^\varepsilon_N(t) \parallel^2_{M_2,\omega}\right)^{1/\varepsilon-1} \end{aligned}$$

$$
\begin{aligned}
&+ C\left\{\alpha_N^2 + \mid Z(\underline{t}_N) - Z(t) \mid^2 \right\}\left(\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2\right)^{1/\varepsilon - 1} \\
&\le \frac{C}{\varepsilon} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t))\left(\frac{\sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right)\alpha_N \\
&+ \mid \frac{1}{\varepsilon} - 1 \mid \left(\frac{\alpha_N^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right) \\
&\quad \left(\frac{\sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right) f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)) \\
&+ C \mid \frac{1}{\varepsilon} - 1 \mid \left(\frac{\mid Z(\underline{t}_N) - Z(t) \mid^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right) \\
&\quad \left(\frac{\sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right) f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)) \\
&+ C\left(\frac{\alpha_N^2 + \mid Z(\underline{t}_N) - Z(t) \mid^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right) f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)) \\
&\le C\left\{\left(\frac{1}{\varepsilon} + 1\right) + \left(1 + \frac{\mid Z(\underline{t}_N) - Z(t) \mid^2}{\alpha_N^2 + \sup_{0\le t\le 1} \parallel \Psi_N^\varepsilon(t) \parallel_{M_2,\omega}^2}\right)\right\} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)) \\
&\le \frac{C}{\varepsilon} f_{\varepsilon,\rho}(\Psi_N^\varepsilon(t)).
\end{aligned}
$$

This, together with Doob's stopping theorem, shows that there exists a constant $K < \infty$, independent of N, ε, ρ and N_0, such that, for $N \ge N_0$,

$$\theta_{N,\rho}^\varepsilon(t) \le \alpha_N^{2/\varepsilon} + \frac{K}{\varepsilon}\int_0^t \theta_{N,\rho}^\varepsilon(s)ds, \quad t \in [0,1]$$

(see for example, Deuschel & Stroock [6], p. 30). Therefore, for $N \ge N_0$

$$\theta_{N,\rho}^\varepsilon(1) \le \exp\left\{\frac{1}{\varepsilon}(K + 2\log\rho - 2\log N)\right\},$$

since, for all $N \geq 1$

$$P(\upsilon^{\varepsilon}_{N,\rho} \leq 1) \leq \left(\frac{\rho^2+\delta^2}{N^2}\right)^{1/\varepsilon} \theta^{\varepsilon}_{N,\rho}(1),$$

we conclude

$$\lim_{\varepsilon\to 0} \sup_N \lim_{\varepsilon\to 0} \sup \varepsilon \log P(\upsilon^{\varepsilon}_{N,\rho} \leq 1) = -\infty. \tag{6.19}$$

This, together with (6.17) and (6.19), implies (6.14). □

Lemma 6.7. *For any $\delta > 0$, we have*

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \sup \varepsilon \log P(\| X^{\varepsilon}(.) - X^{\varepsilon}_R(.) \|_{**} > \delta/R) = -\infty$$

Proof. The proof is similar to that of Lemma 6.6. □

7. Particular case

Case of $\sigma = I_d$. Let $\{X^{\varepsilon}_t, t \in [0,1]\}$ the solution of the differential equation:

$$X^{\varepsilon}_t = x + \int_0^t b(X^{\varepsilon}_s, Y_s)ds + \sqrt{\varepsilon}W_t.$$

In this case, X^{ε} is a continuous functional of Y_t and W_t. Indeed, by posing

$$V^{\varepsilon}_t = X^{\varepsilon}_t - \sqrt{\varepsilon}W_t$$

we have:

$$V^{\varepsilon}_t + \sqrt{\varepsilon}W_t = x + \int_0^t b(V^{\varepsilon}_s + \sqrt{\varepsilon}W_s, Y_s)ds + \sqrt{\varepsilon}W_t$$

so

$$X^{\varepsilon}_t = x + \int_0^t b(V^{\varepsilon}_s + \sqrt{\varepsilon}W_s, Y_s)ds + \sqrt{\varepsilon}W_t$$

then

$$X^{\varepsilon}_t = \Phi(V^{\varepsilon}_t(Y_t), \sqrt{\varepsilon}W_t).$$

Let's determine the continuity of Φ in $\mathcal{B}^{d,0}_{M_2,\omega}$. Let $X_1 = \Phi(V_1(Y_1), \sqrt{\varepsilon}W_1)$ and $X_2 = \Phi(V_2(Y_2), \sqrt{\varepsilon}W_2)$.

$$\begin{aligned}
&\| X_1 - X_2 \|_{**} \\
&= \| \int_0^t b(V_1^\varepsilon(s) + \sqrt{\varepsilon} W_1(s), Y_1(s)) ds - \int_0^t b(V_2^\varepsilon(s) + \sqrt{\varepsilon} W_2(s), Y_2(s)) ds \|_{**} \\
&\leq K \sup_{0 \leq u < v \leq 1} \int_u^v \frac{\| V_1(s) - V_2(s) \|}{\omega(v-u)} ds \\
&\quad + K \sup_{0 \leq u < v \leq 1} \int_u^v \frac{\| Y_1 - Y_2 \|}{\omega(v-u)} ds + K(\sqrt{\varepsilon}+1) \sup_{0 \leq u < v \leq 1} \frac{\| W_1(s) - W_2(s) \|}{\omega(v-u)} ds.
\end{aligned}$$

Thanks to the inequality of Cauchy-Schwartz, we have:

$$\begin{aligned}
\| X_1 - X_2 \|_{**} \leq K \Big[& \| V_1^\varepsilon - V_2^\varepsilon \| \sup_{0 \leq x < y \leq 1} \frac{\sqrt{y-x}}{\omega(y-x)} \\
& + \| Y_1 - Y_2 \| \sup_{0 \leq x < y \leq 1} \frac{\sqrt{y-x}}{\omega(y-x)} \\
& + (\sqrt{\varepsilon}+1) \| W_1 - W_2 \| \sup_{0 \leq x < y \leq 1} \frac{\sqrt{y-x}}{\omega(y-x)} \Big]
\end{aligned}$$

and

$$\begin{aligned}
\| V_1^\varepsilon(.) - V_2^\varepsilon(.) \| &\leq K \sup_{s \leq t} \int_0^t \| V_1^\varepsilon(s) - V_2^\varepsilon(s) \| ds \\
&+ K \int_0^t \sqrt{\varepsilon} \| W_1(s) - W_2(s) \| ds + K \int_0^t \| Y_1(s) - Y_2(s) \| ds.
\end{aligned}$$

Grownall's lemma and Cauchy-Schwartz inequality implies

$$\| V_1^\varepsilon(.) - V_2^\varepsilon(.) \| \leq K \big(\sqrt{\varepsilon} \| W_1 - W_2 \| + \| Y_1 - Y_2 \| \big) \exp K.$$

Thus (7) imply the continuity of Φ in $\mathcal{B}^{d,0}_{M_2,\omega}$. So according to [4], $\{X_t^\varepsilon, t \geq 0\}$ satisfy the LDP with a good rate function λ^*:

$$\lambda^*(\bar{h}) = \lim_{a \to 0} \lim_{\rho \in B_X(\bar{h},a)} \lambda(\rho),$$

where $B_X(\bar{h}, a)$ is the ball of radius a centred at $\bar{h}$ with respect the to norm $\| \, . \, \|_{M_2,\omega}$ and

$$\lambda(\bar{h}) = \inf \big\{ \lambda(h); h \in \mathcal{H}([0,1], \mathbb{R}^d) : \exists r \in suppY \ such\ that\ \bar{h} = \Phi(h,r) \big\}.$$

Case of $\sigma \neq I_d$ and $Z = 0$(in $\mathbb{R}$). Let $\{X_t^\varepsilon, t \in [0,1]\}$ the solution of the differential equation:

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon, Y_s)ds + \sqrt{\varepsilon}\int_0^t \sigma(X_s)dW_s.$$

In this case, X^ε is a functional continuous of Y_t and W_t. Following the idea of [9], there exist a continuous process V_{Y_t}, $\mathcal{F}_t$-adapted, almost certainly locally Lipschitz in t, unique to a near indistinguishability such that;

$$X_t^\varepsilon = \Psi(V_{Y_t}, \sqrt{\varepsilon}W_t) \ \ \forall t \geq 0 \, p.s. \tag{7.1}$$

where the map $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is solution of ordinary differential equations

$$\forall \alpha, \beta \in \mathbb{R}\frac{\partial \Psi}{\partial \beta}(\alpha, \beta) = \sigma(\Psi(\alpha, \beta)); \ \Psi(\alpha, 0) = \alpha,$$

Ψ exists since σ is lipschitz.

When more, σ is class $\mathbf{C}^{1,b}$ (i.e. class $\mathbf{C}^1$ and σ' boundeed), $\{V_{Y_t}, t \geq 0\}$ is, for almost all ω, differentiable in t and solution of ordinary differential equation:

$$\begin{cases} V'_{Y_t}(\omega) = & \exp\Big(- \int_0^{\sqrt{\varepsilon}W_t(\omega)} \sigma'(\Psi(V_{Y_t}(\omega), s)ds\Big)\Big(- \frac{\varepsilon}{2}\sigma'\sigma(\Psi(V_{Y_t}(\omega), \sqrt{\varepsilon}W_t(\omega)) \\ & +b(\Psi(V_{Y_t}(\omega), \sqrt{\varepsilon}W_t(\omega)), Y_t)\Big) \\ D_{Y_0}(\omega) = & X_0(\omega) \end{cases}$$

Show (7.1).
By using the Ito's formula

$$\begin{aligned} X_t^\varepsilon - X_0 = \sqrt{\varepsilon}\int_0^t \frac{\partial \Psi}{\partial \beta}(V_{Y_s}, \sqrt{\varepsilon}W_s)dW_s + \int_0^t \frac{\partial \Psi}{\partial \alpha}(V_{Y_s}, \sqrt{\varepsilon}W_s)V'_{Y_s}ds \\ + \frac{1}{2}\int_0^t \frac{\partial^2 \Psi}{\partial \alpha \partial \beta}(V_{Y_s}, \sqrt{\varepsilon}W_s)d < V_{Y_s}, \sqrt{\varepsilon}W_s > \\ + \frac{1}{2}\int_0^t \frac{\partial^2 \Psi}{\partial \alpha^2}(V_{Y_s}, \sqrt{\varepsilon}W_s)d < V_{Y_s} > \\ + \frac{1}{2}\int_0^t \frac{\partial^2 \Psi}{\partial \beta^2}(V_{Y_s}, \sqrt{\varepsilon}W_s)d < \sqrt{\varepsilon}W_s > . \end{aligned}$$

Or

$$d < V_{Y_s}, \sqrt{\varepsilon}W_s >= 0 \ \ and \ \ d < V_{Y_s} >= 0$$

because V_{Y_s} has not a martingale part. Then

$$X_t^\varepsilon - X_0 = \sqrt{\varepsilon}\int_0^t \frac{\partial\Psi}{\partial\beta}(V_{Y_s},\sqrt{\varepsilon}W_s)dW_s + \int_0^t \frac{\partial\Psi}{\partial\alpha}(V_{Y_s},\sqrt{\varepsilon}W_s)V'_{Y_s}ds$$
$$+\frac{\varepsilon}{2}\int_0^t \frac{\partial^2\Psi}{\partial\beta^2}(V_{Y_s},\sqrt{\varepsilon}W_s)ds.$$

By using the lemma 2 in [9] (page 101), we have

$$\frac{\partial\Psi}{\partial\alpha}(\alpha,\beta) = \exp\left(\int_0^\beta \sigma'(\Psi(\alpha,s))ds\right) \text{ and } \frac{\partial^2\Psi}{\partial\beta^2}(\alpha,\beta) = \sigma'.\sigma(\Psi(\alpha,\beta))$$

so

$$X_t^\varepsilon - X_0 = \sqrt{\varepsilon}\int_0^t \sigma(\Psi(V_{Y_s},\sqrt{\varepsilon}W_s))dW_s + \int_0^t \exp\left(\int_0^{W_t}\sigma'(\Psi(V_{Y_s},u))du\right)$$
$$\times\exp\left(-\int_0^{W_t}\sigma'(\Psi(V_{Y_s},u))du\right)b(\Psi(V_{Y_s},\sqrt{\varepsilon}W_s),Y_s)ds$$
$$+\frac{1}{2}\int_0^t\left\{\varepsilon\sigma'\sigma(\Psi(V_{Y_s},\sqrt{\varepsilon}W_s) - \exp\left(\int_0^{W_t}\sigma'(\Psi(V_{Y_s},u))du\right)\right.$$
$$\left.\times\exp\left(-\int_0^{W_t}\sigma'(\Psi(V_{Y_s},u))du\right)\varepsilon\sigma'\sigma(\Psi(V_{Y_s},\sqrt{\varepsilon}W_s)\right\}ds$$
$$= \sqrt{\varepsilon}\int_0^t\sigma(X_s^\varepsilon)dW_s + \int_0^t b(X_s^\varepsilon,Y_s)ds.$$

Show that Ψ is continuous in $\mathcal{B}^{d,0}_{M_2,\omega}$. For all $(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in\mathbb{R}^2\times\mathbb{R}^2$ we have:

$$\|\ \Psi_1(\alpha_1,\beta) - \Psi_2(\alpha_2,\beta)\ \|$$
$$=\|\ \big(\alpha_1 + \int_0^\beta \sigma(\Psi(\alpha_1,s))ds\big) - \big(\alpha_2 + \int_0^\beta \sigma(\Psi(\alpha_2,s))ds\big)\ \|$$
$$\leq\|\ \alpha_1-\alpha_2\ \| + K\int_0^\beta \|\ \Psi(\alpha_1,s)-\Psi(\alpha_2,s)\ \|\ ds$$

By Grownall's lemma, we have if $\|\ \beta\ \|\leq W$

$$\|\ \Psi_1(\alpha_1,\beta) - \Psi_2(\alpha_2,\beta)\ \|\leq \big(\ \|\ \alpha_1-\alpha_2\ \|\ \big)\exp KW.$$

So

$$\|\ \Psi_1(V^1_{Y_t},\sqrt{\varepsilon}W_t) - \Psi_2(V^2_{Y_t},\sqrt{\varepsilon}W_t)\ \|\leq \big(\ \|\ V^1_{Y_t} - V^2_{Y_t}\ \|\ \big)\exp K\ \|\ W\ \|\ .$$

On the one hand

$$\begin{aligned}\| \Psi_1(V^1_{Y_.}, \sqrt{\varepsilon}W_.) - \Psi_2(V^2_{Y_.}, \sqrt{\varepsilon}W_.) \|_{**} &\leq \| V^1_{Y_.} - V^2_{Y_.} \|_{**} \\ &+ \sup_{0\leq u<v\leq 1} \int_u^v \frac{K \| \Psi_1(V^1_{Y_s}, \sqrt{\varepsilon}W_s) - \Psi_2(V^2_{Y_s}, \sqrt{\varepsilon}W_s) \|}{\omega(v-u)} ds.\end{aligned}$$

By using Cauchy-Schwartz inequality

$$\begin{aligned}\| \Psi_1(V^1_{Y_.}, \sqrt{\varepsilon}W_.) - \Psi_2(V^2_{Y_.}, \sqrt{\varepsilon}W_.) \|_{**} &\leq \| V^1_{Y_.} - V^2_{Y_.} \|_{**} \\ &+ K \| \Psi_1(V^1_{Y_s}, \sqrt{\varepsilon}W_s) - \Psi_2(V^2_{Y_s}, \sqrt{\varepsilon}W_s) \| \sup_{0\leq u<v\leq 1} \frac{\sqrt{v-u}}{\omega(v-u)}.\end{aligned}$$

By using

$$\forall x \in \mathbb{R} \qquad | \sigma'(x) | \leq K$$

we have

$$\begin{aligned}&\| V'_{Y_1}(t) - V'_{Y_2}(t) \| \\ &\leq \exp(K \| W \|)\big\{K\varepsilon\big(K \| \Psi(V_{Y_1}(t), W(t)) - \Psi(V_{Y_2}(t), W(t)) \| + M\big) \\ &\quad + K \| \Psi_1(V_{Y_1}(t), W(t)) - \Psi_2(V_{Y_2}(t), W(t)) \| + \| b(0) \| + K \| Y_1 - Y_2 \| \big\} \\ &\leq \exp(K \| W \|)\big(KM + \| b(0) \| + K \| Y_1 - Y_2 \|\big) \\ &\quad + (\exp(K \| W \|))^2(\varepsilon K^2 + K) \| V_{Y_1}(t) - V_{Y_2}(t) \| \\ &= A + B \| V_{Y_1}(t) - V_{Y_2}(t) \|,\end{aligned}$$

then

$$\| V_{Y_1}(t) - V_{Y_2}(t) \| \leq (\| X_0 \| + At) + B \int_0^t \| V_{Y_1}(t) - V_{Y_2}(t) \| ds \qquad \forall t \in [0,1]$$

and

$$\| V_{Y_1}(t) - V_{Y_2}(t) \| \leq (\| X_0 \| + A) \exp B.$$

So

$$\| V_{Y_1}(.) - V_{Y_2}(.) \|_{**} \quad \leq \quad (\| X_0 \| + At) + B \sup_{0\leq u<v\leq 1} \int_u^v \frac{\| V_{Y_1}(s) - V_{Y_2}(s) \|}{\omega(v-u)} ds.$$

Thanks to the Cauchy-Schwartz inequality, we have

$$\| V_{Y_1}(.) - V_{Y_2}(.) \|_{**} \leq (\| X_0 \| + At) + B \| V_{Y_1} - V_{Y_2} \| \sup_{0\leq u<v\leq, 1} \frac{\sqrt{v-u}}{\omega(v-u)}$$

which implies the continuity of Ψ in $\mathcal{B}^{d,0}_{M_2,\omega}$. So according to [4], $\{X^\varepsilon_t, t \geq 0\}$ satisfy a LDP with a good rate function λ^* defined by

$$\lambda^*(\bar{h}) = \lim_{a\to 0} \lim_{\rho\in B_X(\bar{h},a)} \lambda(\rho),$$

where $B_X(\bar{h}, a)$ is the ball of radius a centred at $\bar{h}$ with respect to the norm $\| \,.\, \|_{M_2,\omega}$ and

$$\lambda(\bar{h}) = \inf \big\{\lambda(h); h \in \mathcal{H}([0,1], \mathbb{R}^d) : \exists r \in suppY \ such\ that\ \bar{h} = \Psi(h, r)\big\}.$$

Case of $\sigma \neq I_d$ and $Z = 0$ (in $\mathbb{R}^d$). Suppose the vector fields $\sigma = (\sigma_1, ..., \sigma_r)$ switch two by two, i.e., $\forall i' \neq i$ $(i, i' \in \{1, ..., r\})$, the Lie bracket $[\sigma_i, \sigma_{i'}] \equiv 0$. (According to the terminology of [9], the matrix $\sigma = (\sigma_1, ..., \sigma_r)$ verify the Frobenius's conditions).

Let

$$X^\varepsilon_t = x + \int_0^t b(X^\varepsilon_s, Y_s)ds + \sqrt{\varepsilon} \int_0^t \sigma(X_s)dW_s.$$

In this case, X^ε is again a continuous functional of Y_t and W_t. As in the previous case, there exist a process $\{V_{Y_t}, t \geq 0\}$, $\mathcal{F}_t$-adapted, almost certainly locally Lipschitz in t and check the equivalent equation next:

$$X^\varepsilon_t = \Theta(D_{Y_t}, \sqrt{\varepsilon}W_t), \tag{7.2}$$

where $\Theta : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is the application determined by the resolution of the differential equation:

$$\forall \alpha, \beta \in \mathbb{R} \frac{\partial \Theta}{\partial \beta_i}(\alpha, \beta) = \sigma_i(\Theta(\alpha, \beta)), \forall i = 1, ..., r;\ \Theta(\alpha, 0) = \alpha\ \ \forall\ (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^r.$$

This solution exists under the Frobenius's conditions, cf [9]; $V^\varepsilon_{Y_t}$ is solution of ordinary differential equation solution

$$\begin{cases} V'_{Y_t}(\omega) = & \exp\Big(- \int_0^{\sqrt{\varepsilon}W_t(\omega)} \sigma'(\Theta(V_{Y_t}(\omega), s)ds\Big) \\ & \cdot \ \ \Big(- \frac{\varepsilon}{2} tr\big(\sigma'\sigma(\Theta(V_{Y_t}(\omega), \sqrt{\varepsilon}W_t(\omega))\big) \\ & + \ \ b(\Theta(V_{Y_t}(\omega), \sqrt{\varepsilon}W_t(\omega)), Y_t)\Big) \\ V_{Y_0}(\omega) = & X_0(\omega) \end{cases}$$

Show (7.2).
By using the Ito's formula,

$$X_t^\varepsilon - X_0 = \sqrt{\varepsilon}\int_0^t \frac{\partial\Theta}{\partial\beta}(V_{Y_s},\sqrt{\varepsilon}W_s)dW_s + \int_0^t \frac{\partial\Theta}{\partial\alpha}(V_{Y_s},\sqrt{\varepsilon}W_s)V'_{Y_s}ds + \frac{\varepsilon}{2}\int_0^t tr\big(\frac{\partial^2\Theta}{\partial\beta^2}(V_{Y_s},\sqrt{\varepsilon}W_s)\big)ds.$$

By using the lemma 18 in [9] (page 119),

$$\frac{\partial\Theta}{\partial\alpha}(\alpha,\beta) = \exp\left(\int_0^\beta \sigma'(\Theta(\alpha,s))ds\right) \text{ and } \frac{\partial^2\Theta}{\partial\beta^2}(\alpha,\beta) = \sigma'.\sigma(\Theta(\alpha,\beta))$$

then,

$$\begin{aligned} X_t^\varepsilon - X_0 &= \sqrt{\varepsilon}\int_0^t \sigma(\Theta(V_{Y_s},\sqrt{\varepsilon}W_s))dW_s + \int_0^t \exp\left(\int_0^{W_t} \sigma'(\Theta(V_{Y_s},u))du\right) \\ &\quad\times \exp\left(-\int_0^{W_t} \sigma'(\Theta(V_{Y_s},u))du\right) b(\Theta(V_{Y_s},\sqrt{\varepsilon}W_s),Y_s)ds \\ &\quad+ \frac{1}{2}\int_0^t \left\{\varepsilon tr\big(\sigma'\sigma(\Theta(V_{Y_s},\sqrt{\varepsilon}W_s)\big) - \exp\left(\int_0^{W_t} \sigma'(\Theta(V_{Y_s},u))du\right)\right. \\ &\quad\left.\times \exp\left(-\int_0^{W_t} \sigma'(\Theta(V_{Y_s},u))du\big)\varepsilon tr\big(\sigma'\sigma(\Theta(V_{Y_s},\sqrt{\varepsilon}W_s)\right)\right\}ds \\ &= \sqrt{\varepsilon}\int_0^t \sigma(X_s^\varepsilon)dW_s + \int_0^t b(X_s^\varepsilon,Y_s)ds. \end{aligned}$$

And then Θ is continuous in $\mathcal{B}^{d,0}_{M_2,\omega}$ (the proof is similar to that of the case b)) then $\{X_t^\varepsilon, t\geq 0\}$ satisfy a LDP with a good rate function λ^* defined by

$$\lambda^*(\bar{h}) = \lim_{a\to 0}\lim_{\rho\in B_X(\bar{h},a)} \lambda(\rho),$$

where $B_X(\bar{h},a)$ is the ball of radius a centred at $\bar{h}$ with respect to the norm $\|\,.\,\|_{M_2,\omega}$ and

$$\lambda(\bar{h}) = \inf\Big\{\lambda(h); h\in\mathcal{H}([0,1],\mathbb{R}^d) : \exists r\in suppY \text{ such that } \bar{h} = \Theta(h,r)\Big\}.$$

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