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SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS BOUNDED BY CONICAL DOMAIN

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ABSTRACT. The main objective of this present investigation is to derive the Fekete Szegö inequality for the function belongs to the defined class in the open unit disk with respect to symmetric and conjugate points bounded by conical region. As a special consequences of our result, we extracted some interesting results through corollaries.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions which are analytic and univalent in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$.

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In 1959, Sakaguchi [3] introduced a subclass S_s^* of function $f \in \mathcal{A}$ with respect to symmetric points in $\mathbb{D}$ and satisfying the condition

$$\Re\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > 0, \quad z \in \mathbb{D}.$$

In 1977, Das and Singh [2] have discussed about the necessary and sufficient condition for $f \in \mathcal{A}$, which are convex with respect to symmetric points is denoted by C_s, such as

$$\Re\left\{\frac{(2zf'(z))'}{(f(z)-f(-z))'}\right\} > 0, \quad z \in \mathbb{D}.$$

El-Ashwah and Thomas [4], introduced two other subclasses S_c^* consisting of functions starlike with respect to conjugate points as

$$\Re\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\} > 0, \quad z \in \mathbb{D}.$$

Also Janteng et al. [5] defined the subclass C_c consisting of the function convex with respect to conjugate points and satisfying the condition

$$\Re\left\{\frac{(2zf'(z))'}{(f(z)+\overline{f(\bar{z})})'}\right\} > 0, \quad z \in \mathbb{D}.$$

Now consider the conic region Ω_k, $k \geq 0$,

$$\Omega_k = \{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\}.$$

This class was introduced and studied by Kanas and Wisniowska [6]. The above domain represents the right half plane for $k = 0$, a hyperbola for $0 < k < 1$, a parabola for $k = 1$ and an ellipse for $k > 1$.

The functions which play the role of extremal functions for these conic regions are given as

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} & \text{for } k = 0, \\ 1 + \dfrac{2}{\pi^2}\log^2\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right) & \text{for } k = 1, \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left(A(k)\operatorname{arctanh}\arctan\sqrt{z}\right) & \text{for } k \in (0,1), \\ 1 + \dfrac{2}{k^2-1}\sin^2\left(\dfrac{\pi}{2K(t)}\mathcal{K}\left(\dfrac{\sqrt{z}}{\sqrt{t}}, t\right)\right) & \text{for} k > 1. \end{cases}$$

where $z \in \mathbb{U}$, $A(k) = \dfrac{2}{\pi} \arccos k$, $\mathcal{K}(w,t)$ is the Legendre elliptic integral of the first kind,

$$\mathcal{K}(w,t) = \int_0^w \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}, \qquad K(t) = \mathcal{K}(1,t),$$

and $t \in (0,1)$ is chosen such that $k = \cosh \dfrac{\pi K'(t)}{2K(t)}$.
By virtue of,

$$p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z) \text{ or } p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z)\,,$$

and the properties of the domains, we have

$$\Re(p(z)) > \Re(p_k(z)) > \frac{k}{k+1}\,.$$

Using the concepts of symmetric and conjugate points, we now define the following:

Definition 1.1. *If $f \in \mathcal{A}$ is in the class $\mathrm{M}_s(\alpha, p_k)$ and $0 \le \alpha < 1$, must satisfy the condition*

$$\frac{2[zf^{'}(z) + \alpha z^2 f^{''}(z)]}{(1-\alpha)\left(f(z) - f(-z)\right) + \alpha z \left(f(z) - f(-z)\right)^{'}} \prec p_k(z), \quad z \in \mathbb{D}\,.$$

Definition 1.2. *If $f \in \mathcal{A}$ is in the class $\mathrm{M}_c(\alpha, p_k)$ and $0 \le \alpha < 1$, must satisfy the condition*

$$\frac{2[zf^{'}(z) + \alpha z^2 f^{''}(z)]}{(1-\alpha)\left(f(z) + \overline{f(\bar{z})}\right) + \alpha z \left(f(z) + \overline{f(\bar{z})}\right)^{'}} \prec p_k(z), \quad z \in \mathbb{D}\,.$$

In this paper, we find the coefficient estimates and Fekete Szegö problem for the above defined subclasses. In order to prove our main results we require the following Lemma:

Lemma 1.1. *[8] Let $k \in [0, \infty)$, be fixed and p_k be the Riemann map of $\mathbb{D}$ onto Ω_k, satisfying $p_k(0) = 1$ and $\Re\{p_k^{'}(0)\} > 0$.*

If $p_k(z) = 1 + P_1(z) + P_2(z) + \ldots, \quad (z \in \mathbb{D})$, *then*

$$P_1(z) = \begin{cases} \dfrac{2A^2}{1-k^2} & \textit{for } 0 < k < 1, \\ \dfrac{8}{\pi^2} & \textit{for } k = 1, \\ \dfrac{\pi^2}{4(k^2-1)K^2(t)(1+t)\sqrt{t}} & \textit{for } k > 1. \end{cases} \tag{1.2}$$

where $A = (2/\pi)\arccos k$ *while* k *and* $K(t)$ *are defined as* $k = \cosh(\pi K'(t)/(4K(t)))$. *Here* $K(t)$ *is Legendre's complete elliptic integral of first kind see [1, 7].*

Lemma 1.2. *[9] If* $p(z)$ *is of the form* $p(z) = 1 + c_1 z + c_2^2 + \cdots$ *be analytic in* $\mathbb{D}$ *with positive real part then the following sharp inequality holds,*

$$|c_2 - \nu c_1^2| \leq 2\max\{1, |2\nu - 1|\}, \quad \text{where} \quad \nu \in \mathbb{C}\,.$$

2. MAIN RESULTS

Theorem 2.1. *Let* $k \in [0, \infty)$, $0 \leq \alpha < 1$ *and if* $f(z)$ *of the form* (1.1) *belongs to the class* $\mathrm{M}_s(\alpha, p_k)$ *then for any complex number* μ,

$$|a_3 - \mu a_2^2| \leq \frac{P_1}{2(1+2\alpha)}\, max\left\{1, \left|\frac{P_2}{P_1} - \mu\frac{P_1(1+2\alpha)}{2(1+\alpha)^2}\right|\right\}. \tag{2.1}$$

Proof. Let $f \in \mathrm{M}_s(\alpha, p_k)$, and we consider the function $q(z)$ as

$$q(z) = \frac{2[zf'(z) + \alpha z^2 f''(z)]}{(1-\alpha)\left(f(z) - f(-z)\right) + \alpha z\left(f(z) - f(-z)\right)'} \qquad (z \in \mathbb{D})$$

then we have,

$$q(z) \prec p_k(z), \quad z \in \mathbb{D} \tag{2.2}$$

where

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots.$$

Using the relation (2.2), we see that the function $p(z)$ given by

$$p(z) = \frac{1 + p_k^{-1}(q(z))}{1 - p_k^{-1}(q(z))} = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic and has positive real part in the open unit disk $\mathbb{D}$. We also have

$$\begin{aligned} q(z) &= p_k\left(\frac{p(z)-1}{p(z)+1}\right) \\ &= 1+\frac{P_1c_1}{2}z+\left(\frac{P_1c_2}{2}-\frac{P_1c_1^2}{4}+\frac{P_2c_1^2}{4}\right)z^2+\dots \end{aligned} \tag{2.3}$$

After the simple calculation, we get

$$q(z) = 1 + 2(1+\alpha)a_2 z + 2(1+2\alpha)a_3 z^2 + \dots \tag{2.4}$$

By equating the like powers of z in (2.3) and (2.4) we get

$$a_2 = \frac{P_1c_1}{4(1+\alpha)},$$

$$a_3 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \frac{1}{2}\left(1-\frac{P_2}{P_1}\right)c_1^2\right].$$

For any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \frac{1}{2}\left(1-\frac{P_2}{P_1}\right)c_1^2\right] - \mu\frac{P_1^2c_1^2}{16(1+\alpha)^2}.$$

Then,

$$a_3 - \mu a_2^2 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \nu c_1^2\right],$$

where

$$\nu = \frac{1}{2}\left(1-\frac{P_2}{P_1}+\mu\frac{P_1(1+2\alpha)}{2(1+\alpha)^2}\right).$$

Where P_1 and P_2 are defined in (1.2).

By taking absolute value on both sides and using Lemma 1.2 we obtain our required result as in (2.1). □

Corollary 2.1. *When $\alpha = 0$ the class $\mathrm{M}_s(\alpha, p_k)$ reduced to $\mathrm{M}_s(p_k)$. Hence*

$$a_2 = \frac{P_1c_1}{4}, \qquad a_3 = \frac{P_1}{4}\left[c_2 - \frac{1}{2}\left(1-\frac{P_2}{P_1}\right)c_1^2\right],$$

and for any complex number μ we have,

$$|a_3 - \mu a_2^2| \le \frac{P_1}{2}\max\left\{1, \left|\frac{P_2}{P_1} - \mu\frac{P_1}{2}\right|\right\}.$$

Corollary 2.2. *When $\alpha = 1$ the class $\mathrm{M}_s(\alpha, p_k)$ reduced to $\mathrm{M}_s(1, p_k)$. Hence*

$$a_2 = \frac{P_1 c_1}{8}, \qquad a_3 = \frac{P_1}{12}\left[c_2 - \frac{1}{2}\left(1 - \frac{P_2}{P_1}\right)c_1^2\right],$$

and for any complex number μ we have,

$$|a_3 - \mu a_2^2| \leq \frac{P_1}{6}\max\left\{1, \left|\frac{P_2}{P_1} - \mu\frac{3P_1}{8}\right|\right\}.$$

Theorem 2.2. *Let $k \in [0, \infty)$, $0 \leq \alpha < 1$ and if f of the form* (1.1) *belongs to the class $\mathrm{M}_c(\alpha, p_k)$ then for any complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{P_1}{2(1+2\alpha)}\max\left\{1, \left|\frac{P_2}{P_1} + P_1 - 2\mu\frac{P_1(1+2\alpha)}{(1+\alpha)^2}\right|\right\}. \tag{2.5}$$

Proof. Let $f \in \mathrm{M}_c(\alpha, p_k)$, and we consider the function $q(z)$ as

$$q(z) = \frac{2[zf'(z) + \alpha z^2 f''(z)]}{(1-\alpha)\left(f(z) + \overline{f(\bar{z})}\right) + \alpha z\left(f(z) + \overline{f(\bar{z})}\right)'} \qquad (z \in \mathbb{D})$$

then we have,

$$q(z) \prec p_k(z), \quad z \in \mathbb{D}.$$

where

$$p_k(z) = 1 + P_1 z + P_2 z^2 + \cdots.$$

After the simple calculation, we get

$$q(z) = 1 + (1+\alpha)a_2 z + [2(1+2\alpha)a_3 - (1+\alpha)a_2^2]z^2 + \ldots \tag{2.6}$$

From (2.3) and (2.6) we get

$$a_2 = \frac{P_1 c_1}{2(1+\alpha)},$$

$$a_3 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1\right)c_1^2\right].$$

For any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1\right)c_1^2\right] - \mu\frac{P_1^2 c_1^2}{4(1+\alpha)^2}.$$

Then,

$$a_3 - \mu a_2^2 = \frac{P_1}{4(1+2\alpha)}\left[c_2 - \eta c_1^2\right],$$

where
$$\eta = \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1 + 2\mu \frac{P_1(1+2\alpha)}{(1+\alpha)^2}\right).$$
Where P_1 and P_2 are defined in (1.2).

By taking absolute value on both sides and using Lemma 1.2 we obtain our required result as in (2.5). □

Corollary 2.3. *When $\alpha = 0$ the class $\mathrm{M}_c(\alpha, p_k)$ reduced to $\mathrm{M}_c(p_k)$. Hence*
$$a_2 = \frac{P_1 c_1}{2}, \qquad a_3 = \frac{P_1}{4}\left[c_2 - \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1\right)c_1^2\right],$$
and for any complex number μ we have,
$$|a_3 - \mu a_2^2| \leq \frac{P_1}{2}\max\left\{1, \left|\frac{P_2}{P_1} - \mu\frac{P_1}{2}\right|\right\}.$$

Corollary 2.4. *When $\alpha = 1$ the class $\mathrm{M}_c(\alpha, p_k)$ reduced to $\mathrm{M}_c(1, p_k)$. Hence*
$$a_2 = \frac{P_1 c_1}{4}, \qquad a_3 = \frac{P_1}{12}\left[c_2 - \frac{1}{2}\left(1 - \frac{P_2}{P_1} - P_1\right)c_1^2\right],$$
and for any complex number μ we have,
$$|a_3 - \mu a_2^2| \leq \frac{P_1}{6}\max\left\{1, \left|\frac{P_2}{P_1} - \mu\frac{3P_1}{2}\right|\right\}.$$

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