ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.1, 471–488
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.1.37

DYNAMICS OF A PREY PREDATOR DISEASE MODEL WITH HOLLING TYPE FUNCTIONAL RESPONSE AND STOCHASTIC PERTURBATION

M. N. SRINIVAS, G. BASAVA KUMAR, AND V. MADHUSUDANAN[1]

ABSTRACT. The present research article constitutes Holling type II and IV diseased prey predator ecosystem and classified into two categories namely susceptible and infected predators.We show that the system has a unique positive solution. The deterministic and stochastic nature of the dynamics of the system is investigated. We check the existence of all possible steady states with local stability. By using Routh-Hurwitz criterion we showed that the positive equilibrium point E_7 is locally asymptotically stable if $x^* > \sqrt{m_1}$ .Moreover condition of the global stability of positive equilibrium point E_7 are also entrenched with help of Lyupunov theorem. Some Numerical simulations are carried out to illustrate our analytical findings.

1. INTRODUCTION AND PRELIMINARIES

In eco-environmental studies prey-predator schemes are always evergreen and wide range influential and significant in modelling. Modelling again is a powerful tool and a single solution for any sort of complex situations. To formulate any complex problem into known terms and analysing in any discipline is meant by modelling according to researchers. Analysis in their disciplines like physics, chemistry, biology, medical and medicines - modelling plays a effective scientific tool in the research industry. For modelling mathematical

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 34C07, 34D23, 92D25.
Key words and phrases. prey-predator, stochastic, equilibrium points, stability, disease model.

and numerical techniques are added concepts to analyse and conclude some remarkable results in any research topic. One of the most popular and the standard model in mathematical bio research industry is Lotka-Volterra. The interactions between prey and predator is always an existing and exiting topics to study deeply and at the same time which is an interesting and challenging evergreen research trends in the bio-mathematical works. Many scientists studied the dynamics of the prey predator schemes in environmental science and contributed some inspirational and extendable topics and problems formulated and developed some quality articles in the collection of population models [3, 4] and [7–12].

In ecological industry prey-predator schemes are always takes up the significant parts and the greatest standard model to analyse prey-predator schemes is Lotka-Volterra. The dealing between prey and predator existing in the same atmosphere is an interesting field in the bio-mathematical works [1, 14]. Many scientists studied the dynamics of the prey predator schemes in environmental science and contributed to the growth of the population models [8]. Moreover, the connection between the illness and the prey-predator structure is a subject of substantial attention, and the synthesis of epidemiology and environmental science is a reasonably innovative division of eco-epidemiology. It is a point that the predator is extra exposed to the diseased prey because the diseased prey may become feebler and less dynamic so that prey may be effortlessly caught by the predator, and the similar theory studied by [6, 8].

Most of the research works in disease models are under the primary assumption that the disease is spreaded over prey populace majorly. But the disease entered and developed in prey populace results many changes in the environmental elements and predator populace significantly. The natural scenario is quite complex and mixed with so many dangerous and easily transferable diseases are there which can change the total structure and the entire ecological system. To catch up those elemental changes and corresponding developments (both growth and decay) in view of diseases and infections and in view of population balance in the structure is a wide range of challenging research area. Many Scientists [2, 3, 5, 9, 11] worked on disease models with a predefined constraint which is that infection enters and infects prey populace. Recently many developments [1, 4, 12, 13, 15] came in disease models with time constraint called delay, and a new phenomenal technique called functional response called Holling type responses and its types and so on. How disease structures are framed as infected prey and differently infected predator, Holling type functional responses also framed as types based on prey and some are differently based on predator.

Now a days environmental factor plays a significant role on daily routines of every living creatures. In biology particularly each and every minute changes and effects plays major role in the entire system. Those environmental fluctuations can be studied further using stochastic perturbation techniques with the help of some mathematical tools or methodologies like Wiener process, Fourier transforms and so on. In any sort of biological phenomenon or problem can be easily formulated by modelling and further detail analysis and remarkable conclusions can be drawn with the help of so many applied mathematical tools.

In the current article the problem is little different and the assumption carried out while defining the basic structure is the predators are infected by disease, which is a new perspective of challenging trending theme. Many authors studied disease structures under time constraint influence (delay) as well as structures with functional responses. This current research study is more interesting and different from the previous studies so far done by others. The current problem framed, which is combination of two types of functional response and on different predefined assumption the basic structure is formulated and analysed. Another value-added content to this study is the stochastic perturbation observations and illustrations.

This paper is organized as follows: In Sect. 2, formulation of the mathematical model is presented. In Sect. 3, positivity and boundedness of the system has been obtained. In Sect. 4, the stability criterion of the system is discussed at all the feasible equilibrium states and obtained the conditions for the existence of Hopf bifurcation at the disease-free and endemic equilibrium states. In Sect. 5, the local and global stabilities are analysed. In Sect. 6, the stochastic perturbation technique is used to study and observe the environmental fluctuations theoretically and finally checked with graphical views. In Sect. 7, the entire numerical simulations are framed in support of our analytical findings with a nice graphical view. Graphical views are drawn by using MATLAB software. Finally, the results have been concluded in the conclusion section.

2. MATHEMATICAL MODEL

We considered and developed a modified disease oriented model with the following assumptions

(i) Let $x(t)$ represents the density of prey population at time t and it is assumed that the disease spreads only among the predator,$y_1(t)$ and $y_2(t)$ denotes susceptible and infected predator population at time t, and the total predator population is $N(t) = y_1(t) + y_2(t)$.

(ii) In the absence of predators, the prey population grows logistically with intrinsic growth rate α_0 with environmental carrying capacity $\alpha_0\beta_0^{-1}$. Let $\gamma_1$and $k\gamma_1$ are the predation rates for susceptible and infected predator. Because of disease infection, the predation ability for the infected predator is a bit less than that for the susceptible predator; therefore,$0 < k < 1$.The disease transmission rate is θ.

(iii) Here α_1, α_2 are the per capita growth rates of each predator population. The parameters m_1, m_2 respectively represent the half saturation constants of prey and predator populations.

In addition, the authors introduce intra specific competition among the predators sound and infected sub populations with parameters γ_2 and γ_3 for which $\gamma_2 > \gamma_3$

$$\frac{dx}{dt} = \alpha_0 x - \beta_0 x^2 - \frac{\gamma_1 x y_1}{m_1 + x^2} - \frac{k\gamma_1 x y_2}{m_1 + x^2}\,, \tag{2.1}$$

$$\frac{dy_1}{dt} = \alpha_1 y_1 - \frac{\gamma_2 y_1 (y_1 + y_2)}{m_2 + x} - \theta y_1 y_2\,, \tag{2.2}$$

$$\frac{dy_2}{dt} = \theta y_1 y_2 + \alpha_2 y_2 - \frac{\gamma_3 y_2 (y_1 + y_2)}{m_2 + x}\,, \tag{2.3}$$

with the initial conditions

$$x(0) \geq 0, y_1(0) \geq 0, y_2(0) \geq 0\,. \tag{2.4}$$

3. POSITIVE INVARIANCE AND BOUNDEDNESS

Feasibility or biologically positivity studies aim to objectively and rationally uncover the strength of the proposed model in the given environment. Biologically positive insures the population never become negative and population always survive. The following theorems ensure that the positivity and boundedness of the system (2.1) - (2.3).

Theorem 3.1. *All solutions of* $x(t), y_1(t), y_2(t)$ *the system* (2.1)-(2.3) *with the initial condition* (2.4) *are positive for all* $t \geq 0$.

Proof. From (2.1)-(2.3) it is observed that

$$\frac{dx}{x} = \left[\alpha_0 - \beta_0 x - \frac{\gamma_1 y_1}{m_1 + x^2} - \frac{k\gamma_1 y_2}{m_1 + x^2}\right] dt = \phi_1(x, y_1, y_2)dt \quad (\text{say; })$$
$$\frac{dy_1}{y_1} = \left[\alpha_1 - \frac{\gamma_1(y_1 + y_2)}{m_2 + x} - \theta y_2\right] dt = \phi_2(x, y_1, y_2)dt \ (\text{say;})$$
$$\frac{dy_2}{y_2} = \left[\theta y_1 + \alpha_2 - \frac{\gamma_3(y_1 + y_2)}{m_2 + x}\right] dt = \phi_3(x, y_1, y_2)dt \ (\text{say.})$$

Integrating in the region $[0, t]$ we get $x(t) = x(0)\exp\left(\int \phi_1(x, y_1, y_2)dt\right) > 0$, $y_1(t) = y_1(0)\exp\left(\int \phi_2(x, y_1, y_2)dt\right) > 0$, $y_2(t) = y_2(0)\exp\left(\int \phi_3(x, y_1, y_2)dt\right) > 0$,for all t.Hence, all solutions starting from interior of the first octant (InR^3_+) remain positive in it for future time. □

Theorem 3.2. *All the non-negative solutions of the model system* (2.1)-(2.3) *that initiate in* $\Re^3_+$ *are uniformly bounded.*

Proof. Let $x, y_1(t), y_2(t)$ be any solution of the system (2.1)-(2.3). Since, from equation (2.1), $\frac{dx}{dt} \leq x(\alpha_0 - \beta_0 x)$, we have $\lim_{t\to\infty} \sup x(t) \leq \frac{\alpha_0}{\beta_0}$. Let $w = x + y_1 + y_2$. Differentiate with respect to t we get

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt} \tag{3.1}$$

Substituting the equation (2.1)-(2.3) in above equation (3.1),we get

$$\frac{dw}{dt} + \varsigma w \leq \frac{(\alpha_0 + \varsigma)^2}{4\beta_0} + (\alpha + \varsigma)y_1 + (\alpha + \varsigma)y_2$$
$$\frac{dw}{dt} + \varsigma w \leq l \quad \text{since} \ \frac{(\alpha_0 + \varsigma)^2}{\beta_0} = l(say))$$

Applying Lemma on differential inequalities, we obtain

$$0 \leq w(x, y_1, y_2) \leq (l/\varsigma)\left(1 - e^{-\varsigma t}\right) + \left(w(x(0), y_1(0), y_2(0))/e^{\varsigma t}\right)$$

and for $t \to \infty$ we have $0 \leq w(x, y_1, y_2) \leq (l/\varsigma)$. Thus all solutions of system (2.1)-(2.3) enter into the region

$\Gamma = \left\{(x, y_1, y_2) \in R^3_+ : 0 \le x \le \frac{\alpha_0}{\beta_0}, 0 \le w \le (l/\varsigma) + \varepsilon, \forall \varepsilon > 0\right\}$.
This completes the proof. □

4. EXISTENCE OF EQUILIBRIUM POINTS WITH FEASIBILITY CONDITION

It can be checked that the system (2.1)-(2.3) has 8 non-negative equilibria and four of them namely $E_0\,(0,0,0)\,, E_1\left(\frac{\alpha_0}{\beta_0}, 0, 0\right), E_2\left(0, \frac{\alpha_1 m_2}{\gamma_2}, 0\right), E_3\left(0, 0, \frac{\alpha_2 m_2}{\gamma_3}\right)$ always exists. We show that the existence of other equilibrium as follows.
Existence of $E_4(\bar{x}, \bar{y}_1, 0)$**:**Here $\bar{x}, \bar{y}_1$ are the positive solutions of the following algebraic equations:

$$\alpha_0 - \beta_0 x - \frac{\gamma_1 y_1}{m_1 + x^2} = 0\,, \tag{4.1}$$

$$\alpha_1 - \frac{\gamma_2 y_1}{m_2 + x} = 0\,. \tag{4.2}$$

Solving (4.1) and (4.2) we get

$$\bar{y}_1 = \alpha_1 (m_2 + \bar{x})/\gamma_2\,.$$

Substituting the value of the value of $\bar{y}_1$ in equation (4.1) we get cubic polynomial is of the form

$$A\bar{x}^3 + B\bar{x} + C\bar{x} + D = 0\,,$$

where $A = \gamma_2\beta_0, B = -\gamma_2\alpha_0, C = (\gamma_2\beta_0 m_1 + \gamma_1\alpha_1), D = (\gamma_1\alpha_1 m_2 - \gamma_2\alpha_0 m_1)\,.$
Clearly $\bar{x}$ is positive if

$$\gamma_1\alpha_1 m_2 < \gamma_2\alpha_0 m_1\,.$$

Existence of $E_5(\tilde{x}, 0, \tilde{y}_2)$**:** Here $\tilde{x}, \tilde{y}_2$ are the positive solutions of the following algebraic equations.

$$\alpha_0 - \beta_0 x - \frac{k\gamma_1 y_2}{m_1 + x^2} = 0\,, \tag{4.3}$$

$$\alpha_2 - \frac{\gamma_3 y_2}{m_2 + x} = 0\,. \tag{4.4}$$

Solving equations (4.3) and (4.4), we get

$$\tilde{y}_2 = \alpha_2 (m_2 + \bar{x})/\gamma_3\,.$$

Substituting the value of the value of $\tilde{y}_2$ in equation (4.4) we get cubic polynomial is of the form

$$A\left(\tilde{x}\right)^3 + B\left(\tilde{x}\right)^2 + C\left(\tilde{x}\right) + D = 0\,,$$

where $A = \gamma_3\beta_0, B = -\gamma_3\alpha_0, C = (\gamma_3\beta_0 m_1 + k\gamma_1\alpha_2), D = (k\gamma_1\alpha_2 m_2 - \gamma_3\alpha_0 m_1)\,.$ Clearly $\tilde{x}$ is positive if

$$k\gamma_1\alpha_2 m_2 < \gamma_3\alpha_0 m_1\,.$$

Existence of $E_6(0, \hat{y}_1, \hat{y}_2)$**:**Here $\hat{y}_1, \hat{y}_2$ are the positive solutions of the following algebraic equations.

$$\alpha_1 - \frac{\gamma_2(y_1 + y_2)}{m_2 + x} - \theta y_2\,, \tag{4.5}$$

$$\theta y_1 + \alpha_2 - \frac{\gamma_3(y_1 + y_2)}{m + x}\,. \tag{4.6}$$

Solving equations (4.5) and (4.6) we get

$$\hat{y}_1 = \frac{\alpha_1\gamma_3 - \alpha_2(\gamma_2 + m_2\theta)}{\theta(m_2\theta + \gamma_2 - \gamma_3)}; \hat{y}_2 = \frac{\alpha_1(\gamma_3 - m_2\theta) - \alpha_2\gamma_2}{\theta(m_2\theta + \gamma_2 - \gamma_3)}\,.$$

Clearly $\hat{y}_1$ is positive if

$$\alpha_1\gamma_3 < \alpha_2(\gamma_2 + m_2\theta), (m_2\theta + \gamma_2 < \gamma_3)\,.$$

Clearly $\hat{y}_2$ is positive if

$$\gamma_3 < m_2\theta\,.$$

Existence of $E_7(\hat{x}, \hat{y}_1, \hat{y}_2)$**:** We find the steady states of the system (2.1)-(2.3) by equating the derivatives on the left hand sides to zero and solving the resulting algebraic equations. This gives three possible steady states, namely:

$$\alpha_0\left(m_1 + x^2\right) - \beta_0 x\left(m_1 + x^2\right) - \gamma_1 y_1 - k\gamma_1 y_2 = 0\,,$$

$$\alpha_1\left(m_2 + x\right) - \gamma_2(y_1 + y_2) - \theta y_2\left(m_2 + x\right) = 0\,, \tag{4.7}$$

$$\theta y_1\left(m_2 + x\right) + \alpha_2\left(m_2 + x\right) - \gamma_3(y_1 + y_2) = 0\,. \tag{4.8}$$

Solving equations (4.7) and (4.8) ,we get

$$f(y_1, y_2) = 0\,, \tag{4.9}$$

$$f(y_1, y_2) = \gamma_3(\alpha_1 - \theta_1 y_2) - \gamma_2(\theta y_1 + \alpha_2)\,,$$

where $y_1 \to 0$ then $y_2 \to y_a$. The equation (4.9) becomes $y_a = (\gamma_3\alpha_1 - \gamma_2\alpha_2) / (\theta_1\gamma_3)$, we note that $y_a > 0$,$\gamma_3\alpha_1 > \gamma_2\alpha_2$. Solving equations (4.7) and (4.8),we get

$$f(x, y_2) = 0\,, \tag{4.10}$$

$$f(x, y_2) = \alpha_0(m_1 + x^2) - \beta_0 x(m_1 + x^2) - \frac{\gamma_1}{\gamma_2}\left(\gamma_2 y_2 + (m_2 + x)(\theta_1 y_2 - \alpha_1)\right) - k\gamma y_2\,,$$

where $x \to 0$ then $y_2 \to y_b$.
Equation (4.10) becomes $y_b = \left(\frac{\gamma_1\alpha_1 m_2 + \gamma_2\alpha_0 m_1}{\gamma_1\theta_1 m_2 + \gamma_1\gamma_2(1+k)}\right)$
We note that $y_b > 0$, $\gamma_1\alpha_1 m_2 + \gamma_2\alpha_0 m_1 > 0$ and $\gamma_1\theta_1 m_2 + \gamma_1\gamma_2(1 + k) > 0$.
From equation (4.10), we have $x \to x_a$ then $y_2 \to 0$.
Then $\beta_0 x^3 - \alpha_0 x^2 + \left(\beta_0 m_1 + \frac{\gamma_1\alpha_1}{\gamma_2}\right) x - \alpha_0 m_1 \frac{\gamma_1\alpha_1}{\gamma_2} = 0$,
which is in the form of

$$A_1 x^3 + A_2 x^2 + A_3 x + A_4 = 0\,, \tag{4.11}$$

where $A_1 = \beta_0 > 0, A_2 = -\alpha_0, A_3 = \left(\beta_0 m_1 + \frac{\gamma_1\alpha_1}{\gamma_2}\right), A_4 = -\alpha_0 m_1 \frac{\gamma_1\alpha_1}{\gamma_2}$ Equation (4.11) has a unique positive solution $x = x^*$ if the following inequalities hold:

$$\beta_0 m_1 + \frac{\gamma_1\alpha_1}{\gamma_2} > 0\,.$$

5. STABILITY ANALYSIS

(i) The variational matrix of equilibrium point at $E_0(0, 0, 0)$ is

$$J(E_0) = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are α_0,α_1 and α_2. In this case all the eigenvalues are positive. Therefore, equilibrium point $E_0(0, 0, 0)$ is unstable.

(ii) The variational matrix of equilibrium point at $E_1\left(\frac{\alpha_0}{\beta_0}, 0, 0\right)$ is

$$J(E_1) = \begin{pmatrix} -\alpha_0 & \frac{\gamma_1\alpha_0\beta_0}{\alpha_0^2 + m_1\beta_0^2} & \frac{-k\gamma_1\alpha_0\beta_0}{\alpha_0^2 + m_1\beta_0^2} \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $-\alpha_0$,α_1 and α_2. In this case two of the eigenvalues are positive and one will be negative.Therefore, equilibrium point $E_1\left(\frac{\alpha_0}{\beta_0}, 0, 0\right)$ is stable in x direction and unstable in $y_1 - y_2$.

(iii) The variational matrix of equilibrium point at $E_2\left(0, \frac{\alpha_1 m_2}{\gamma_2}, 0\right)$ is

$$J(E_2) = \begin{pmatrix} -\frac{\gamma_1\alpha_1 m_2}{\gamma_2 m_1} & 0 & 0 \\ \frac{\alpha_1^2}{\gamma_2} & -\alpha_1 & \frac{-\alpha_1(\gamma_2+\theta m_2)}{\gamma_2} \\ 0 & 0 & \frac{\alpha_2\gamma_2-\alpha_1(\theta m_2-\gamma_3)}{\gamma_2} \end{pmatrix}.$$

The eigenvalues of $J(E_2)$ all are negative, if

$$\alpha_2\gamma_2 < \alpha_1(\theta m_2 - \gamma_3), \theta m_2 > \gamma_3 .$$

In this case the point $E_2\left(0, \frac{\alpha_1 m_2}{\gamma_2}, 0\right)$ is stable in x - y_1 - y_2 direction.

(iv) The variational matrix of equilibrium point at $E_3\left(0, 0, \frac{\alpha_2 m_2}{\gamma_3}\right)$ is

$$J(E_3) = \begin{pmatrix} \frac{\alpha_0 m_1 - k\gamma_1\alpha_2 m_2}{m_1} & 0 & 0 \\ 0 & \frac{\alpha_1\gamma_3-\alpha_2(\gamma_2+\theta m_2)}{\gamma_3} & 0 \\ \frac{\alpha_2^2}{\gamma_3} & \frac{\alpha_2(\theta m_2-\gamma_3)}{\gamma_3} & -\alpha_2 \end{pmatrix}.$$

The eigenvalues of $J(E_3)$ all are negative, if

$$\alpha_0 m_1 < k\gamma_1\alpha_2 m_2, \alpha_1\gamma_3 < \alpha_2(\gamma_2 + \theta m_2) .$$

In this case,the point $E_3\left(0, 0, \frac{\alpha_2 m_2}{\gamma_3}\right)$ is stable in x - y_1 - y_2 direction.

(v) The variational matrix of equilibrium point at $E_4(\bar{x}, \bar{y}_1, 0)$ is

$$J(E_4) = \begin{pmatrix} \alpha_0 - 2\beta_0\bar{x} - \frac{\gamma_1\bar{y}_1(m_1-\bar{x}^2)}{(m_1+\bar{x}^2)^2} & -\frac{\gamma_1\bar{x}}{(m_1+\bar{x}^2)} & -\frac{k\gamma_1\bar{x}}{(m_1+\bar{x}^2)} \\ \frac{\gamma_2\bar{y}_1^2}{(m_2+\bar{x})^2} & -\frac{\gamma_2\bar{y}_1}{m_2+\bar{x}} & -\frac{\gamma_2\bar{y}_1}{m_2+\bar{x}} - \theta\bar{y}_1 \\ 0 & 0 & \alpha_3 + \theta\bar{y}_1 - \frac{\gamma_3\bar{y}_1}{m_2+\bar{x}} \end{pmatrix}.$$

The eigenvalues of $J(E_4)$ all are negative, if $(\alpha_3 + \theta\bar{y}_1)(m_2 + \bar{x}) < \gamma_3\bar{y}_1$

$$\alpha_0(m_1 + \bar{x}^2)^2 < 2\beta_0\bar{x}(m_1 + \bar{x}^2)^2 + \gamma_1\bar{y}_1(m_1 - \bar{x}^2) .$$

In this case the point $E_4(\bar{x}, \bar{y}_1, 0)$ is stable in x - y_1 - y_2 direction.

(vi) The variational matrix of equilibrium point at $E_5(\tilde{x}, 0, \tilde{y}_2)$ is

$$J(E_5) = \begin{pmatrix} \alpha_0 - 2\beta_0\tilde{x} - \frac{k\gamma_1\tilde{y}_2(m_1-\tilde{x}^2)}{(m_1+\tilde{x}^2)} & -\frac{\gamma_1\tilde{x}}{(m_1+\tilde{x}^2)} & -\frac{k\gamma_1\tilde{x}}{(m_1+\tilde{x}^2)} \\ 0 & \alpha_2 - \frac{\gamma_2\tilde{y}_2}{m_2+\tilde{x}} - \theta\tilde{y}_2 & 0 \\ \frac{\gamma_3\tilde{y}_2^2}{(m_2+\tilde{x})^2} & \theta\tilde{y}_2 - \frac{\gamma_3\tilde{y}_2}{m_2+\tilde{x}} & \frac{-\gamma_3\tilde{y}_2}{m_2+\tilde{x}} \end{pmatrix}.$$

The eigenvalues of $J(E_5)$ all are negative, if $(\alpha_2 - \theta\tilde{y}_2)(m_2 + \tilde{x}) < \gamma_2\tilde{y}_2$

$$\alpha_0(m_1 + \tilde{x}^2)^2 < 2\beta_0\tilde{x}(m_1 + \tilde{x}^2)^2 + k\gamma_1\tilde{y}_2(m_1 - \tilde{x}^2) .$$

In this case the point $E_5(\tilde{x}, 0, \tilde{y}_2)$ is stable in x - y_1 - y_2 direction.

(vii) The variational matrix of equilibrium point at $E_6(0,\hat{y}_1,\hat{y}_2)$ is

$$J(E_6)=\begin{pmatrix} \alpha_0-\frac{k\gamma_1\hat{y}_2}{m_1} & 0 & 0\\ \frac{\gamma_2\hat{y}_1(\hat{y}_1+\hat{y}_2)}{m_2^2} & \alpha_2-\frac{\gamma_2(2\hat{y}_1+\hat{y}_2)}{m_2}-\theta\hat{y}_2 & -\frac{\gamma_2\hat{y}_2}{m_2}-\theta\hat{y}_2\\ \frac{\gamma_2\hat{y}_2(\hat{y}_1+\hat{y}_2)}{m_2^2} & \theta\hat{y}_2-\frac{\gamma_3\hat{y}_2}{m_2} & \alpha_3+\theta\hat{y}_2-\frac{\gamma_3(\hat{y}_1+2\hat{y}_2)}{m_2}\end{pmatrix}.$$

The eigenvalues of $J(E_6)$ all are negative, if $\alpha_2-\theta\hat{y}_2(m_2)<\gamma_2 2\hat{y}_1+\hat{y}_2$

$$\alpha_3+\theta\hat{y}_2 m_2<\gamma_3\hat{y}_1+2\hat{y}_2\,.$$

In this case the point $E_6(0,\hat{y}_1,\hat{y}_2)$ is stable in x - y_1 - y_2 direction. Now to investigate the local stability of interior equilibrium point $E_7^*(x^*,y_1^*,y_2^*)$, we first find the variation matrix $M(E^*)$ at equilibrium point

$$M\left(E^*\right)=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= \alpha_0-2\beta_0-\gamma_1(ky_2+y_1)\left(\frac{m_1-x^2}{(m_1+x^2)^2}\right);a_{12}=\left(\frac{-\gamma_1 x}{(m_1+x^2)}\right);a_{13}=\left(\frac{-\gamma_1 kx}{(m_1+x^2)}\right)\\ a_{21} &= \left(\frac{\gamma_2(y_1^2+y_1y_2)}{(m_1+x^2)^2}\right);a_{22}=\alpha_1-\frac{\gamma_2(2y_1+y_2)}{(m_1+x)}-\theta y_2;a_{23}=-\left(\theta+\frac{\gamma_2}{m_2+x}\right);\\ a_{31} &= \left(\frac{\gamma_3(y_2^2+y_1y_2)}{(m_1+x)^2}\right);a_{32}=y_2\left(\theta-\frac{\gamma_3}{m_2+x}\right);a_{33}=\theta y_1+\alpha_2-\frac{\gamma_3(y_1+2y_2)}{(m_2+x)}\,. \end{aligned}$$

The corresponding characteristic equation is

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0\,,$$

where

$$\begin{aligned} A_1 &= -(a_{11}+a_{22}+a_{33});\quad A_2=a_{11}(a_{22}+a_{33})+a_{33}(a_{22}-a_{12}a_{21})-(a_{23}a_{32}+a_{13}a_{31});\\ A_3 &= -a_{11}(a_{22}a_{33}-a_{23}a_{32})+a_{21}(a_{12}a_{33}-a_{13}a_{32})+a_{31}(a_{13}a_{22}-a_{12}a_{23})\,. \end{aligned}$$

Therefore an application of Routh-Hurwitz condition shows that the following conditions are satisfied:

$A_1>0$, if $x^*-m_1>0\Rightarrow x^*>\sqrt{m_1}$ and $\alpha_0+\alpha_1+\alpha_2+\theta y_1^*<0$,

$A_3>0$ if $y_1+\alpha_2<0$,$(m_2+x^*)>\gamma_2$ and $y_2(\gamma_2y_1+1)+y_1(1+\gamma_2)<0$,

$A_1A_2-A_3>0$ if $\alpha_1+\alpha_2<0$ and $x^*>\sqrt{m_1}$.

Hence, positive equilibrium point $E_7(x^*,y_1^*,y_2^*)$ is locally stable under the condition.

Global stability :

Theorem 5.1. *The co-existence equilibrium point E_7 is globally asymptotically stable if $Q_1 > 0, Q_2 > 0,\ Q_3 > 0$ where $Q_1, Q_2,\ Q_3$ are defined latter.*

Proof. Let us define the function

$$L(x, y_1, y_2) = L_1((x, y_1, y_2)) + L_2((x, y_1, y_2)) + L_3((x, y_1, y_2)) \ ,$$

where

$$L_1 = x - x^* - x^* \ln \frac{x}{x^*} \ , L_2 = y_1 - y_1^* - y_1^* \ln \frac{y}{y_1^*} \ , \ L_3 = y_2 - y_2^* - y_2^* \ln \frac{y}{y_2^*} \,.$$

It is to be shown that L is a Lyapunov function and L vanishes at E_7 and it is positive for all $x, y_1, y_2 > 0$. Hence E_7 represents its global minimum. We have

$$\begin{aligned} \frac{dL}{dt} &= \\ &- \left[\begin{array}{c} A(x - x^*)^2 + B(y_1 - y_1^*)^2 + c(y_2 - y_2^*)^2 + 2H(x - x^*)(y - y_1^*) \\ +2F(y - y_1^*)(y_2 - y_2^*) + 2G(y_2 - y_2^*)(x - x^*) \end{array} \right] \\ &= -V^T R V \,. \end{aligned}$$

Here $V = ((x - x^*) \ , (y_1 - y_1^*) \ , (y_2 - y_2^*))^T$ and R is symmetric quadratic form given by

$$R = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix} ,$$

with the entries that are functions only of the variable x and

$$A = b_1 - \frac{c_1(y_1^* + k y_2^*)(x x^* - m_1)}{((x^*)^2 + k_1)(x^2 + m_1)} \ , B = \frac{\gamma_2}{x + m_2} \ , C = \frac{\gamma_3}{x + m_2} \ , F = \frac{\gamma_2 + \gamma_3}{2(x + m_2)} ,$$

$$H = \frac{1}{2}\left[\frac{\gamma_1 x}{x^2 + m_1} - \frac{\gamma_2(y_1^* + y_2^*)}{(x^* + m_2)(x + m_2)}\right] \ , G = \frac{1}{2}\left[\frac{k\gamma_1 x}{x^2 + m_1} - \frac{\gamma_3(y_1^* + y_2^*)}{(x^* + m_2)(x + m_2)}\right].$$

If the matrix R is positive definite, then $\frac{dL}{dt} < 0$ so, all the principal minors of Q namely, $Q_1 \equiv A \ , \ Q_2 \equiv AB - H^2 \ ,$

$$Q_3 \equiv C\left(AB - H^2\right) + G(FH - BG) + F(GH - AF) \ ,$$

to be positive , i.e. $Q_1 > 0, Q_2 > 0,\ Q_3 > 0$ □

6. STOCHASTIC ANALYSIS

Deterministic models are stable with a cyclic behaviour in the common period for the sizes of species population. Moreover these models may be inadequate for capturing the exact variability in nature. In predator-prey model the random fluctuations are also undeniably arising from either environmental variability or internal species. In fact, biological systems are inherently random in nature and noise play a vital role in the structure and function of such system. These random fluctuations result in changing some degree of parameter in the deterministic environment. Now we allow stochastic perturbations of the variables x^*, y_1^*, y_2^* around their values at the positive equilibrium E_7. We consider the white noise stochastic perturbations which are proportional to the distances of x, y_1, y_2 from x^*, y_1^*, y_2^*. So the stochastically perturbed system (2.1)-(2.3) is given by:

$$\text{(6.1)} \qquad \begin{cases} dx = (\alpha_0 x - \beta_0 x^2 - \dfrac{\gamma_1 x y_1}{m_1 + x^2} - \dfrac{k\gamma_1 x y_2}{m_1 + x^2})dt + \sigma_1(x - x^*)d\xi_t^1 \\ dy_1 = (\alpha_1 y_1 - \dfrac{\gamma_2 y_1(y_1 + y_2)}{m_2 + x} - \theta y_1 y_2)dt + \sigma_2(y_1 - y_1^*)d\xi_t^2 \\ dy_2 = (\theta y_1 y_2 + \alpha_2 y_2 - \dfrac{\gamma_3 y_2(y_1 + y_2)}{m_2 + x})dt + \sigma_3(y_2 - y_2^*)d\xi_t^3 \end{cases},$$

where $\sigma_i, i = 1, 2, 3$ are real constants, $\xi_t^i = \xi_t(t), i = 1, 2, 3$ are independent from each other standard Wiener processes [2]. We wonder whether the dynamical behaviour of model (2.1)-(2.3) is robust with respect to such a kind of stochasticity by investigating the asymptotic stability behaviour of the equilibrium E_7 for the system (6.1) and comparing the results with those obtained for (2.1)-(2.3). Consider (6.1) as the Ito stochastic differential system. To analyse the stochastic stability of E_7, we consider the linear system of (6.1) around E_7 as follows:

$$\text{(6.2)} \qquad du(t) = f(u(t))dt + g(u(t))d\xi(t)$$

where $u(t) = Col(u_1(t), u_2(t)); f(u(t)) = J(u(t))$,

$$f(u(t)) = \begin{bmatrix} -\beta_0 x^* + \frac{2\gamma_1(x^*)^2 y_1^*}{(m_1+(x^*)^2)^2} + \frac{2k\gamma_1(x^*)^2 y_2^*}{(m_1+(x^*)^2)^2} & \frac{-\gamma_1 x^*}{(m_1+(x^*)^2)} & \frac{-k\gamma_1 x^*}{(m_1+(x^*)^2)} \\ \frac{\gamma_2(y_1^*+y_2^*)y_1^*}{(m_2+x^*)^2} & \frac{-\gamma_2 y_1^*}{(m_2+x^*)} & \frac{-\gamma_2 y_1^*}{(m_2+x^*)} - \theta y_1^* \\ \frac{-\gamma_3(y_1^*+y_2^*)y_2^*}{(m_2+x^*)^2} & \theta y_2^* - \frac{\gamma_3 y_2^*}{(m_2+x^*)} & \frac{-\gamma_3 y_2^*}{(m_2+x^*)} \end{bmatrix};$$

$$g(u(t)) = \begin{bmatrix} \sigma_1 u_1 & 0 & 0 \\ 0 & \sigma_2 u_2 & 0 \\ 0 & 0 & \sigma_3 u_3 \end{bmatrix}; \; d\xi(t) = col\,(\xi_1(t), \xi_2(t), \xi_3(t));$$

$$u_1 = x - x^*, u_2 = y_1 - y_1^*, u_3 = y_2 - y_2^*.$$

Let $U = \{(t \geq t_0) \times R^n, t_0 \in R^+\}$. Hence $V_2 \in C_2^0(U)$ is a continuous function with respect to t and a twice continuously differentiable function with respect to u. With reference to [2], we have:

$$LV(t,u) = \frac{\partial V(t,u)}{\partial t} + f^T(u)\frac{\partial V(t,u)}{\partial u} + \frac{1}{2}Tr\left(g^T(u)\frac{\partial V^2(t,u)}{\partial u^2}g(u)\right),$$

where T means transposition

$$\frac{\partial V}{\partial u} = col\left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2}, \frac{\partial V}{\partial u_3}\right)^T; \frac{\partial V^2(t,u)}{\partial u^2} = col(\frac{\partial V^2}{\partial u_i \partial u_j}); i,j = 1,2,3.$$

Theorem 6.1. *If there exists a function $V_2 \in C_2^0(U)$ satisfying the following*

(6.3) $$K_1 |u|^p \leq V(t,u) \leq K_2 |u|^p; LV(t,u) \leq -K_3 |u|^p \;\; K_i > 0, p > 0.$$

Then the trivial solution of (6.2) *is exponentially p-stable for t .*

Theorem 6.2. *Suppose that*

$$\sigma_1^2 < 2\left[\beta_0 x^* - \frac{2\gamma_1 (x^*)^2 (y_1^* + k y_2^*)}{(m_1 + (x^*)^2)^2}\right], \sigma_2^2 < 2\left[\frac{\gamma_2 y_1^*}{(m_2 + x^*)}\right], \sigma_3^2 < 2\left[\frac{\gamma_3 y_2^*}{(m_2 + x^*)}\right]$$

then the zero solution of (6.2) *is asymptotically mean square stable.*

Proof. Let us consider the Lyapunov function

$$V(u) = \frac{1}{2}\left(w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2\right), w_i > 0,$$

where w_i are real positive constants to be chosen in the following. It is easy to check that inequalities (6.3) hold true with p = 2. Further more the inequalities in (6.3) are true when p = 2 and we have:

$$\begin{aligned} LV(t,u) &= \\ w_1 &\left((-\beta_0 x^* + \frac{2\gamma_1 (x^*)^2 y_1^*}{(m_1 + (x^*)^2)^2} + \frac{2k\gamma_1 (x^*)^2 y_2^*}{(m_1 + (x^*)^2)^2})u_1 + (\frac{-\gamma_1 x^*}{(m_1 + (x^*)^2)})u_2 + (\frac{-k\gamma_1 x^*}{(m_1 + (x^*)^2)})u_3\right) u_1 \\ &+ w_2 \left(\left(\frac{\gamma_2 (y_1^* + y_2^*) y_1^*}{(m_2 + x^*)^2}\right) u_1 + \left(\frac{-\gamma_2 y_1^*}{(m_2 + x^*)}\right) u_2 + \left(\frac{-\gamma_2 y_1^*}{(m_2 + x^*)} - \theta y_1^*\right) u_3\right) u_2 \end{aligned}$$

$$+ \quad w_3 \left(\left(\frac{-\gamma_3(y_1^* + y_2^*)y_2^*}{(m_2 + x^*)^2} \right) u_1 + \left(\theta y_2^* - \frac{\gamma_3 y_2^*}{(m_2 + x^*)} \right) u_2 + (\frac{-\gamma_3 y_2^*}{(m_2 + x^*)}) u_3 \right) u_3$$

$$+\frac{1}{2} Tr \left(g^T(u) \frac{\partial V^2(t,u)}{\partial u^2} g(u) \right) . \tag{6.4}$$

We can easily observe that

$$\frac{\partial^2 V}{\partial u^2} = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} ,$$

and hence

$$g^T(u) \frac{\partial V^2(t,u)}{\partial u^2} g(u) = \begin{bmatrix} w_1\sigma_1^2 u_1^2 & 0 & 0 \\ 0 & w_2\sigma_2^2 u_2^2 & 0 \\ 0 & 0 & w_3\sigma_3^2 u_3^2 \end{bmatrix} ,$$

with

$$\frac{1}{2} Tr \left(g^T(u) \frac{\partial V^2(t,u)}{\partial u^2} g(u) \right) = \frac{1}{2} \left(w_1\sigma_1^2 u_1^2 + w_2\sigma_2^2 u_2^2 + w_3\sigma_3^2 u_3^2 \right) . \tag{6.5}$$

If we choose $x^* w_1 = y_1^* w_2 = y_2^* w_3$ in equation (6.4) along equation (6.5) we get:

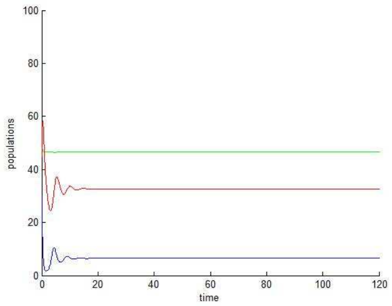
$$\begin{aligned} LV(t,u) &= -w_1 \left[\beta_0 x^* - \frac{2\gamma_1 (x^*)^2 y_1^*}{[m_1 + (x^*)^2]^2} - \frac{2k\gamma_1 (x^*)^2 y_2^*}{[m_1 + (x^*)^2]^2} - \frac{1}{2}\sigma_1^2 \right] u_1^2 \\ & \quad -w_2 \left[\frac{\gamma_2 y_1^*}{(m_2 + x^*)} - \frac{1}{2}\sigma_2^2 \right] u_2^2 - w_3 \left[+ \frac{\gamma_3 y_2^*}{(m_2 + x^*)} - \frac{1}{2}\sigma_3^2 \right] u_3^2 . \end{aligned}$$

This is negative definite function. Hence the proof is completed based on theorem 6.2. □

7. NUMERICAL SIMULATIONS

In this segment, we validate and justify our mathematical findings by computer simulations with help of MATLAB software considering different sets of parameter values as follows:

Example 1. *For the parameters* $\alpha_0 = 4.5; \beta_0 = 0.075; \gamma_1 = 2.8; \gamma_2 = 1.872; \gamma_3 = 1.95;$ $\mathrm{k} = 0.047;$$\theta = 0.0937;$ $\mathrm{m}_1 = 100; m_2 = 160; \alpha_1 = 3.8; \alpha_2 = 1.05;$.



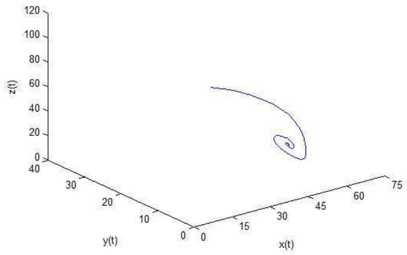
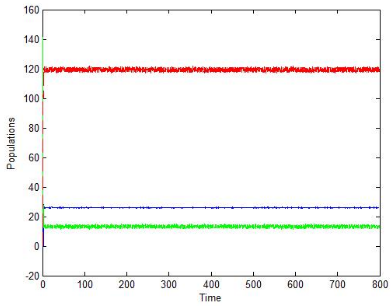


Figure 1A Figure 1B

Figure 1A represents the numerical solution of system (2.1)-(2.3) and Figure 1B represents phase portrait diagram among species

Example 2. *For the parameters* $\alpha_0 = 2; \beta_0 = 0.075; \gamma_1 = 2.8; \gamma_2 = 2; \gamma_3 = 1.95; \mathrm{k} = 0.047; \mathrm{m}_1 = 100; \theta = 0.0937; m_2 = 160; \alpha_1 = 1.5; \alpha_2 = 0.15;\ \sigma_1 = 0.02, \sigma_2 = 0.02, \sigma_3 = 0.02.$



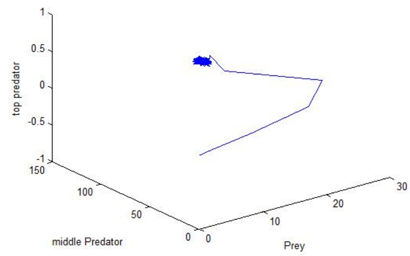


Figure 2A Figure 2B

Figure 2A represents the dynamics exhibits by stochastic system (6.1) and Figure 2B represents the corresponding phase portrait diagram among species

Example 3. *For the parameters* $\alpha_0 = 2; \beta_0 = 0.075; \gamma_1 = 2.8; \gamma_2 = 2; \gamma_3 = 1.95; \mathrm{k} = 0.047; \mathrm{m}_1 = 100; \theta = 0.0937; m_2 = 160; \alpha_1 = 1.5; \alpha_2 = 0.15;\ \sigma_1 = 0.08, \sigma_2 = 0.08, \sigma_3 = 0.08$.

Figure 3A represents the dynamics exhibits by stochastic system and Figure 3B represents the corresponding phase portrait diagram among species.

Example 4. *For the parameters* $\alpha_0 = 2; \beta_0 = 0.075; \gamma_1 = 2.8; \gamma_2 = 2; \gamma_3 = 1.95; \mathrm{k} = 0.047; \mathrm{m}_1 = 100; \theta = 0.0937; m_2 = 160; \alpha_1 = 1.5; \alpha_2 = 0.15;$ *with*

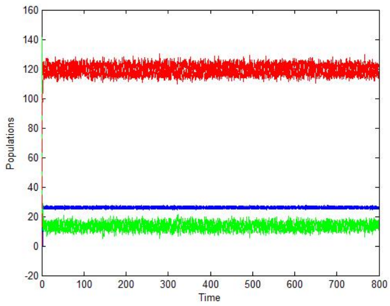


Figure 3A

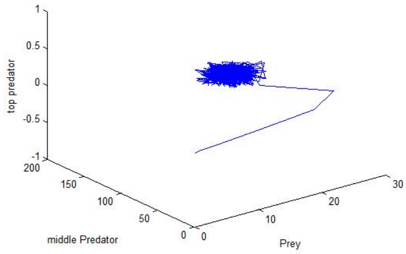


Figure 3B

densities $x(0) = 0.15, y_1(0) = 0.33, y_2(0) = 0.01$ *with High medium strength of noise* $\sigma_1 = 0.2, \sigma_2 = 0.2, \sigma_3 = 0.2$.

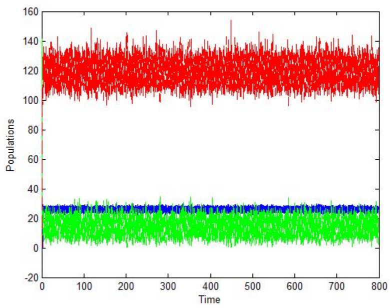


Figure 4A

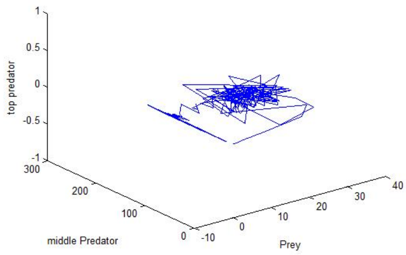


Figure 4B

Figure 4A represents the dynamics exhibits by stochastic system and Figure 4B represents the corresponding phase portrait diagram among species.

8. CONCLUDING REMARKS

In this paper we have studied Leslie-Gower prey-predator model along with Sokel-Howell functional reaction around the interior steady state. we have studied Dynamics of prey and diseased in the predator population. We verified the local stability of all existence possible equilibrium points. Also we ratify the global stability of the system with Lyupunov theorem and the graphical view shows in Figures 1-2.We examined the system by introducing stochastic perturbations. By using Lyapunov function we proved the stochastic differential equation of system is asymptotically mean square stable. In stochastic system, population variations have a great role for the stochastic stability. The noise in the equation results in a big variance of fluctuations around the equilibrium point which suggests that our system oscillates with respect to the noisy environment. From the numerical simulation we conclude that the inclusion of stochastic perturbation creates a noteworthy variation in the intensity of populations. Due to change of responsive parameters chaotic dynamics with different level variances of oscillations are obtained and showed in Figures 3-8.

REFERENCES

[1] A. JOHRI: *Study of a prey-predator model with diseased prey*, Int. J. Contemp. Math. Sci., **7**(9-12) (2012), 489-498.

[2] B. DUBEY: *A prey-predator model with a reserved area*, Nonlinear Anal Model Control, **12**(4) (2010), 479-494.

[3] H. SINGH, J. DHAR, H.S.BHATTI: *Discrete-time bifurcation behavior of a prey-predator system with generalized predator*, Adv Diff Eq., **2015**(1) (2015), 1-15.

[4] H.W.HETHCOTE, W. WANG, L. HAN, Z.MA: *A predator-prey model with infected prey*, Theor. Popul. Biol., **66**(3) (2004), 259-268.

[5] J. DHAR, K.S. JATAV: *Mathematical analysis of a delayed stage structured predator-prey model with impulsive diffusion between two predators territories*, Ecol Complex, **16** (2013), 59-67.

[6] J. DHAR, H. SINGH, H.S. BHATTI: *Discrete-time dynamics of a system with crowding effect and predator partially dependent on prey*, Appl Math Comput., **252** (2015), 324-335.

[7] J. MOORE: *Parasites and the behavior of animals*, Oxford University Press, New York Murray, 2002.

[8] J.P. TRIPATHI, S. ABBAS, M. THAKUR: *Dynamical analysis of a prey-predator model with beddington-deangelis type function response incorporating a prey refuge*, Nonlinear Dyn., **80**(1-2) (2015), 177-196.

[9] J. M. JESCHKE, M. KOPP, R. TOLLRIAN: *Predator functional responses: discriminating between handling and digesting prey*, EcolMonogr, **72**(1) (2002), 95-112.

[10] K. A. HADELER, H. FREEDMAN: *Predator-prey populations with parasitic infection*, J. Math. Biol., **27**(6) (1989), 609-631.

[11] M. SEN, M. BANERJEE, A. MOROZOV: *Bifurcation analysis of a ratio-dependent prey-predator model with the allee effect*, Ecol Complex, **11** (2012), 12-27.

[12] N.H. KHANH: *Stability analysis of an influenza virus model with disease resistance,* J. Egypt. Math. Soc., **24** (2016), 193-199.
Chaos Solitons Fractals, 47, 42-53.

[13] S. KANT, V. KUMAR: *Stability analysis of predator-prey system with migrating prey and disease infection in both species,* Appl. Math. Model., **42** (2017), 509-539.

[14] V. VOLTERRA: *Fluctuations in the abundance of a species considered mathematically*, Nature **118** (1926), 558-560.

[15] X. LIU, C. WANG: *Bifurcation of a predator-prey model with disease in the prey,* Nonlinear Dyn., **62**(4) (2010), 841-850.

DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY
VELLORE, INDIA
E-mail address: mnsrinivaselr@gmail.com

DEPARTMENT OF MATHEMATICS
VELLORE INSTITUTE OF TECHNOLOGY
VELLORE, INDIA
E-mail address: basavakumarg@gmail.com

DEPARTMENT OF MATHEMATICS
S A ENGINEERING COLLEGE
CHENNAI, INDIA
E-mail address: mvmsmaths@gmail.com