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REMEMBERING RAMANUJAN

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ABSTRACT. Mathematicians like Guido Grandi, Ernesto Cesaro and others found novel way of assigning finite sum to divergent series. This created a new scope of understanding leading to analytic continuation of real valued functions. One among such methods was called "Ramanujan Summation" proposed by Indian Mathematician Srinivasa Ramanujan. In this paper, I try to highlight how Ramanujan could have possibly arrived at those values by looking through his notebook jottings and extending further to provide Geometrical meaning behind those values obtained by him. Finally, I provide a novel way to arrive at the general formula obtained by Ramanujan regarding his summation of zeta function.

1. INTRODUCTION

Srinivasa Ramanujan an Indian mathematician was considered to be one of the greatest mathematicians of 20^{th} century for his extra ordinary contributions to mathematics spanning nearly 11 sub-disciplines. In particular, he is considered to be a magician in the areas of Summability Theory, Integrals, Elliptic Functions, Continued Fractions, Theta Functions to name a few. Ramanujan produced 3872 theorems and conjectures in his short lifetime spanning just

[1]*National Awardee for Popularizing Mathematics among masses*

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32 years, 4 months and 5 days. During this short time, he produced extraordinary formulas whose truth baffle even current mathematicians and make them wonder how he would have even thought about them in first place.

With the help of famous English mathematician G.H. Hardy at Trinity College, Cambridge University, England, Ramanujan did research for five years in the period from 1914 to 1919. He produced seven papers in collaboration with Hardy during this period. Each one of these is considered to be a gem in mathematics. Much of what Ramanujan did in Analytic Number Theory became a basic tool for solving many unsolved problems. In this regard, Ramanujan's contribution became very essential for the development of mathematics particularly in Analytic Number Theory during last century. More importantly, Ramanujan's life and contributions made other scientists of India to believe that they can also produce such high quality research at other parts of globe and scale greater heights in mountain of mathematics.

One of the curious contributions that Ramanujan made was called "Ramanujan Summation" which comes under Summability Theory in particular the category of "Cesaro Summation". These summations became prominent for the last three to four centuries and huge amount of research is being carried out even today in this area, see [1–10].

It is well known that Ramanujan Summation is related to the Bernoulli numbers. In particular, using this idea, while seeking patronage of other mathematicians before he left to England, Ramanujan wrote in almost all of his letters that "Under my theory, I found that

$$
\begin{aligned}
&1+2+3+4+\cdots=\frac{-1}{12}\,,\\
&1^2+2^2+3^2+4^2+\cdots=0\,,\\
&1^3+2^3+3^3+4^3+\cdots=\frac{1}{240}\,.\\
&\cdots\cdots
\end{aligned}
$$

For usual mathematicians, these results seemed to be completely unobvious and incorrect, because in all the equations presented above, the left hand side we have positive numbers added up but in right hand side we find answers either negative number or 0 or a number which is less than 1. How come this makes any meaning on earth? This was the first opinion of any mathematician to whom Ramanujan had sent his work. In fact, Professor J.M. Hill of University College of London while seeing these equations mentioned the following to Professor Griffith:

"Mr. Ramanujan has fallen into the pitfalls of the very difficult subject of Divergent series. Otherwise he could not have got the erroneous results you send me

$$\begin{aligned} 1+2+3+4\ \&\ c &= \frac{-1}{12} \\ 1^2+2^2+3^2+4^2\ \&\ c &= 0 \\ 1^3+2^3+3^3+4^3\ \&\ c &= \frac{1}{240}\,, \end{aligned}$$

all the 3 series have infinity for their sums $\cdots$"

In this paper, I will try to explain the Geometric meaning of how Ramanujan churned out these equations and what does it try to convey to the mathematical world. Today, research mathematicians knew the mathematical significance of these ideas, but seldom normal mathematicians could have any idea about why and how these equations must be true. In this aspect, this paper will provide a new insight in to what probably could have made Ramanujan to produce such weird equations.

2. Summing through Ramanujan's way

In the three notebooks that Ramanujan wrote while in India before leaving to England, he mentioned the following in chapter VIII (see [1]), of his first notebook: Let us take the sum $1+2+3+4+5+\ \&\ c$. Let C be its constant. Then:

$$\begin{aligned} C &= \ 1+2+3+4\ \&\ c \\ 4C &= \qquad 4 \qquad +8 \quad +\&c \\ -3C &= 1-2+3-4\ \&\ c = \frac{1}{(1+1)^2} = \frac{1}{4} \end{aligned}$$

Therefore $C = -\dfrac{1}{12}$.

According to this calculation, we get $1+2+3+4+\ \&c = -\dfrac{1}{12}$ or in the notation that we use today we have $1+2+3+4+\cdots = -\dfrac{1}{12}$. We observe that Ramanujan wrote 4, 8 and other numbers in the second line by leaving one gap between each term in the series $4C$ as compared to that of C in the first line. I will first try to explain how Ramanujan obtained this value and generalize that idea to arrive at a final expression that we know as of today.

First let us try to understand the basic aspect of summing the following infinite series

$$
\begin{aligned}
&1+1+1+1+\cdots\\
&1+2+3+4+\cdots\\
&1^2+2^2+3^2+4^2+\cdots\\
&1^3+2^3+3^3+4^3+\cdots\\
&1^4+2^4+3^4+4^4+\cdots\\
&1^5+2^5+3^5+4^5+\cdots\\
&1^6+2^6+3^6+4^6+\cdots\\
&\cdots\ \cdots\ \cdots\\
&1^k+2^k+3^k+4^k+\cdots
\end{aligned}
$$

It is known from basic analysis that all the above series diverges to plus infinity as the sum increase as large as possible as we sum the terms for each of the above series. In this paper I will provide ways to compute the values of above series through novel method and try to connect it with Ramanujan Summation method which led him to wrote those answers in his letters.

3. Definition

The zeta function defined by $\zeta(s) = \dfrac{1}{1^s}+\dfrac{1}{2^s}+\dfrac{1}{3^s}+\dfrac{1}{4^s}+\cdots = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s}$ was studied by great Swiss mathematician Leonhard Euler for all real variables s. With this convention, the infinite series listed in 2, are respectively $\zeta(0), \zeta(-1), \zeta(-2), \zeta(-3), \cdots, \zeta(-k)$. Euler worked extensively and obtained nice expressions for computing $\zeta(2k)$.

Later, German mathematician Bernhard Riemann considered the variable s to be complex number and extended the real valued Euler zeta function to complex valued Riemann zeta function. Thus if s is a complex number then the series $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is called Riemann zeta function. Notice in 2, Ramanujan proved rather in non-rigorous way that $\zeta(-1) = -\frac{1}{12}$. I will try to present methods concerning calculation of $\zeta(0), \zeta(-1), \zeta(-2), \zeta(-3), \cdots$ and try to explain the geometric meaning behind those values.

4. Constructing Functions

Here, I try to generate functions representing certain class of rational expressions. We begin with the expression $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$. We need to determine a rational expression whose Maclaurin's series expansion is the given expression.

Let $S_0(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$, and then we have

$$\begin{aligned} S_0(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots \\ -xS_0(x) &= \quad\ -x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots \end{aligned}$$

Subtracting these two equations, we get $(1 + x)S_0(x) = 1$.

Hence we get

$$S_0(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots = \frac{1}{1+x} \tag{4.1}$$

In general, let $S_k(x) = 1^k - 2^k x + 3^k x^2 - 4^k x^3 + 5^k x^4 - 6^k x^5 + 7^k x^6 - \cdots$.

I will provide methods to generate functions for $S_k(x)$ corresponding to certain integer values of k and use it for further exploration. Equation (4.1) provide such value when $k = 0$.

Now, we will find $S_1(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \cdots$ For this, in equation (4.1), we should multiply $S_0(x)$ by x and differentiating term by term to get

$$xS_0(x) = x - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots = \frac{x}{1+x},$$

$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots = \frac{1}{(1+x)^2}.$$

Hence

$$S_1(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots = \frac{1}{(1+x)^2}\,. \tag{4.2}$$

Now to compute $S_2(x) = 1^2 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - 6^2x^5 + 7^2x^6 - \cdots$ using (4.2), we first multiply $S_1(x)$ by x and differentiate term by term to get:

$$\begin{aligned} xS_1(x) &= x - 2x^2 + 3x^3 - 4x^4 + 5x^5 - 6x^6 + 7x^7 - \cdots = \frac{x}{(1+x)^2}\,, \\ & 1^2 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - 6^2x^5 + 7^2x^6 - \cdots = \frac{1-x}{(1+x)^3}\,. \end{aligned}$$

Hence,

$$S_2(x) = 1^2 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - 6^2x^5 + 7^2x^6 - \cdots = \frac{1-x}{(1+x)^3}\,. \tag{4.3}$$

Extending this process for $S_3(x)$ and $S_4(x)$ we find that

$$\begin{aligned} xS_2(x) &= 1^2x - 2^2x^2 + 3^2x^3 - 4^2x^4 + 5^2x^5 - 6^2x^6 + 7^2x^7 - \cdots = \frac{x-x^2}{(1+x)^3}\,, \\ S_3(x) &= 1^3 - 2^3x + 3^3x^2 - 4^3x^3 + 5^3x^4 - 6^3x^5 + 7^3x^6 - \cdots = \frac{1-4x+x^2}{(1+x)^4}\,, \end{aligned}$$

(4.4)

$$\begin{aligned} xS_3(x) &= 1^3x - 2^3x^2 + 3^3x^3 - 4^3x^4 + 5^3x^5 - 6^3x^6 + 7^3x^7 - \cdots = \frac{x-4x^2+x^3}{(1+x)^4}\,, \\ S_4(x) &= 1^4 - 2^4x + 3^4x^2 - 4^4x^3 + 5^4x^4 - 6^4x^5 + 7^4x^6 - \cdots = \frac{1-11x+11x^2-x^3}{(1+x)^5}\,. \end{aligned}$$

(4.5)

In similar fashion, we can get the following equations:

$$\begin{aligned} S_5(x) &= 1^5 - 2^5x + 3^5x^2 - 4^5x^3 + 5^5x^4 - 6^5x^5 + 7^5x^6 - \cdots \\ &= \frac{1-26x+66x^2-26x^3+x^4}{(1+x)^6} \end{aligned} \tag{4.6}$$

$$\begin{aligned} S_6(x) &= 1^6 - 2^6x + 3^6x^2 - 4^6x^3 + 5^6x^4 - 6^6x^5 + 7^6x^6 - \cdots \\ &= \frac{1-57x+302x^2-302x^3+57x^4-x^5}{(1+x)^7} \end{aligned} \tag{4.7}$$

$$(4.8)\qquad \begin{aligned} S_7(x) &= 1^7 - 2^7x + 3^7x^2 - 4^7x^3 + 5^7x^4 - 6^7x^5 + 7^7x^6 - \cdots \\ &= \frac{1 - 120x + 1191x^2 - 2416x^3 + 1191x^4 - 120x^5 + x^6}{(1+x)^8} \end{aligned}$$

From the equations (4.1) to (4.8) derived above, we observe the following facts:

$S_k(x)$ for each value of $k = 0, 1, 2, 3, 4, 5, \cdots$ is an rational expression such that:

(i) The denominator is $(1+x)^{k+1}$.

(ii) The numerator is a polynomial of degree $k-1$ such that its coefficients are symmetric with opposite signs for even values of k and same signs for odd values of k.

(iii) Let $S_k(x) = \dfrac{P_k(x)}{(1+x)^{k+1}}$. Then $P_k(x)$ is a polynomial of degree $k-1$ such that $P_0(1) = 1$ and $P_k(1) = 0$ for $k = 2, 4, 6, 8, 10, \cdots$

We will now find ways to compute $P_k(1)$ for odd values of k say $k = 1, 3, 5, 7, 9, 11, \cdots$ For doing this, we need the following special type of numbers.

5. Bernoulli Numbers

Definition 5.1. *Bernoulli Numbers are numbers which occur as coefficients of* $\dfrac{x^n}{n!}$ *in the Taylor's series expansion of* $\dfrac{x}{e^x - 1}$ *about* $x = 0$. *We denote the nth Bernoulli Number by* B_n.

Thus by definition we get

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \qquad (5.1)$$

We notice that the constant term of $\dfrac{x}{e^x - 1}$ is 1 and so from (5.1) it follows that $B_0 = 1$.

Now to determine other Bernoulli Numbers we use the coefficients of Pascal's triangle to obtain the following equations:

$$\begin{aligned} 0 &= 1B_0 + 2B_1 \\ 0 &= 1B_0 + 3B_1 + 3B_2 \\ 0 &= 1B_0 + 4B_1 + 6B_2 + 4B_3 \end{aligned}$$

$$0 = 1B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4$$
$$0 = 1B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5$$
$$\cdots\cdots\cdots$$

In general we have

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0\,. \tag{5.2}$$

Since $B_0 = 1$, from $0 = 1B_0 + 2B_1$, we have $B_1 = -\frac{1}{2}$. Similarly substituting the values of $B_0 = 1, B_1 = -\frac{1}{2}$ in $0 = 1B_0 + 3B_1 + 3B_2$, we get $B_2 = \frac{1}{6}$. Using successive values of B_n in (5.2) we get the following few values of Bernoulli Numbers:

$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0,$ $B_8 = -\frac{1}{30}, B_9 = 0, B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}, B_{13} = 0, B_{14} = \frac{7}{6},$ $B_{15} = 0, B_{16} = -\frac{3617}{510}, \cdots$

From the above values we observe that except for $B_1, B_n = 0$ for all odd values of n.

Furthermore, we see that $B_{4n-2} > 0, B_{4n} < 0, n = 1, 2, 3, 4, 5, 6, \cdots$

6. Connection

We now make the connection between $P_k(1)$ for values of $k = 1, 3, 5, 7, 9, 11, \cdots$ and the Bernoulli Numbers B_n.

Since, $S_1(x) = \frac{P_1(x)}{(1+x)^2}$. From (4.2), we see that $P_1(x) = 1$. Hence $P_1(1) = 1$. Also $B_2 = \frac{1}{6}$ Hence:

$$P_1(1) = 1 = 6 \times \frac{1}{6} = 6B_2 = 2^2 \times (2^2 - 1) \times \frac{B_2}{2}\,. \tag{6.1}$$

Similarly, from equations (4.4), (4.6) and (4.8) we get the following equations:

$$P_3(1) = -2 = 60 \times -\frac{1}{30} = 60B_4 = 2^4 \times (2^4 - 1) \times \frac{B_4}{4}\,, \tag{6.2}$$

(6.3) $$P_5(1) = 16 = 672 \times \frac{1}{42} = 672B_6 = 2^6 \times (2^6 - 1) \times \frac{B_6}{6}\,,$$

(6.4) $$P_7(1) = -272 = 8160 \times \frac{-1}{30} = 8160B_8 = 2^8 \times (2^8 - 1) \times \frac{B_8}{8}\,.$$

Thus from equations (6.1) to (6.4), we see that:

(6.5) $$P_{2k-1}(1) = 2^{2k} \times (2^{2k} - 1) \times \frac{B_{2k}}{2k}\,.$$

Equation (6.5) provides the desired connection between Bernoulli Numbers and $P_k(1)$ for all odd values of k. We already knew that

(6.6) $$P_0(1) = 1 \text{ and } P_k(1) = 0 \text{ for } k = 2, 4, 6, 8, 10, \cdots$$

Thus equations (6.5) and (6.6) provide the values of $P_k(1)$ for all whole numbers k.

7. Computing zeta function values

Making use of the equations derived above, we now compute the values $\zeta(0), \zeta(-1), \zeta(-2), \zeta(-3), \cdots$

For doing this, we use the idea of Ramanujan as presented in 2.

$$\zeta(0) = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots$$
$$2\zeta(0) = \quad 2 \quad + 2 \quad + 2 \quad + 2 + \cdots$$

Subtracting these two equations, and using equation (4.1), we get:

$$-\zeta(0) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots = S_0(1) = \frac{1}{2}\,.$$

Hence,

(7.1) $$\zeta(0) = 1 + 1 + 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}\,.$$

Now using the fact that $S_k(1) = \dfrac{P_k(1)}{2^{k+1}}$ we now proceed in similar fashion to determine the more general value of Riemann Zeta function at negative integers as follows:

$$\begin{aligned} \zeta(-k) &= 1^k + 2^k + 3^k + 4^k + 5^k + 6^k + 7^k + 8^k + \cdots \\ 2^{k+1}\zeta(-k) &= 2 \times 2^k \quad + 2 \times 4^k \quad + 2 \times 6^k \quad + 2 \times 8^k + \cdots \end{aligned}$$

Subtracting these two equations, we get:

$$(1-2^{k+1})\zeta(-k) = 1^k - 2^k + 3^k - 4^k + 5^k - 6^k + 7^k - 8^k + \cdots = S_k(1) = \frac{P_k(1)}{2^{k+1}},$$

$$\zeta(-k) = \frac{P_k(1)}{2^{k+1}\times(1-2^{k+1})} = -\frac{P_k(1)}{2^{k+1}\times(2^{k+1}-1)}. \tag{7.2}$$

From equations (6.5) and (6.6) we know the values of $P_k(1)$ for each positive integer k. In particular, from equation (6.6), we know that $P_k(1) = 0$ for all values of $k = 2, 4, 6, 8, 10, \cdots$ Using this information in (7.2) we see that

$$\zeta(-k) = 1^k + 2^k + 3^k + 4^k + 5^k + 6^k + 7^k + 8^k + \cdots = 0, k = 2, 4, 6, 8, 10, 12, \cdots \tag{7.3}$$

If k is odd, say $k = 2m-1$, then we can use equation (6.5) in equation (7.2) to get

$$\zeta(1-2m) = -\frac{P_{2m-1}(1)}{2^{2m}\times(2^{2m}-1)} = -\frac{2^{2m}\times(2^{2m}-1)\times\frac{B_{2m}}{2m}}{2^{2m}\times(2^{2m}-1)} = -\frac{B_{2m}}{2m}. \tag{7.4}$$

Equations (7.3) and (7.4) provide complete values of the zeta function for all negative integer values of k. In particular from (7.3) and (7.4) we can write the following equations:

$$1^{2m} + 2^{2m} + 3^{2m} + 4^{2m} + 5^{2m} + 6^{2m} + \cdots = 0, \tag{7.5}$$

$$1^{2m-1} + 2^{2m-1} + 3^{2m-1} + 4^{2m-1} + 5^{2m-1} + 6^{2m-1} + \cdots = -\frac{B_{2m}}{2m}, \tag{7.6}$$

where $m = 1, 2, 3, 4, 5, 6, 7, 8, \cdots$

Equations (7.5) and (7.6) are called Ramanujan Summation for Divergent Series. These equations are probably the reason why Ramanujan had written so in his letter correspondences to most of the mathematicians before his potential is fully recognized. But Ramanujan never gave any clue about how he arrived at these equations anywhere in his writings.

8. Geometric Meaning of Zeta Values

By definitions of Riemann Zeta function, $S_k(x)$ and the way Ramanujan summation is performed, we have

$$(1-2^{k+1})\zeta(-k) = 1^k - 2^k + 3^k - 4^k + 5^k - 6^k + 7^k - 8^k + \cdots = \lim_{x\to 1} S_k(x)$$

$$\zeta(-k) = \frac{1}{1-2^{k+1}} \lim_{x\to 1} S_k(x)\,. \tag{8.1}$$

We first observe that $S_k(x) = \dfrac{P_k(x)}{(1+x)^{k+1}}$ is continuous at all points except at $x = -1$ and the denominator of $S_k(x)$ is finite as $x \to 1$. Further $\zeta(-k)$ is evaluated using equation (8.1) by knowing the limit as $x \to 1$ in $S_k(x)$. But since $S_k(x)$ is continuous at $x = 1$ we should have $\lim_{x\to 1} S_k(x) = S_k(1)$. Hence equation (8.1) can be written as

$$\zeta(-k) = \frac{S_k(1)}{1-2^{k+1}}\,. \tag{8.2}$$

Now we try to identify values of $\zeta(-k)$ for each k through respective graphs shown below:

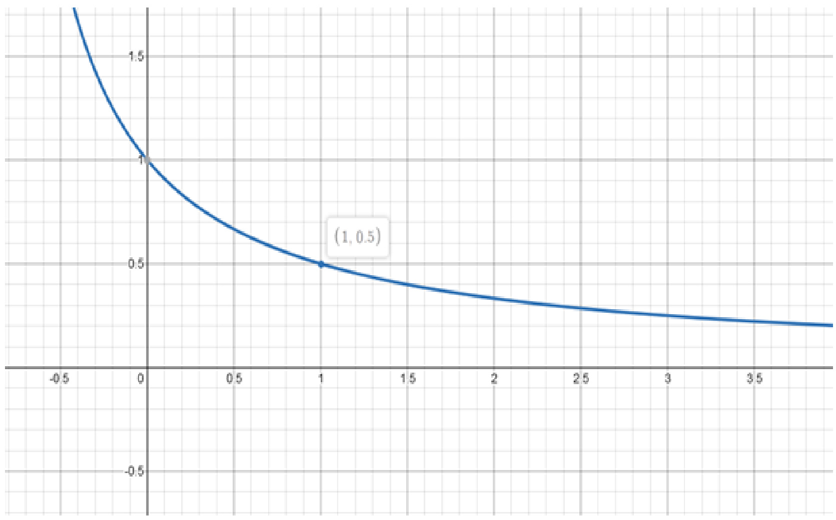

FIGURE 1. Graph of $S_0(x) = \dfrac{1}{1+x}$ near $x = 1$

We notice that $S_0(1) = 0.5 = \dfrac{1}{2}$.

Hence by equation (8.2) we get $\zeta(0) = -S_0(1) = -\dfrac{1}{2}$.

Thus $\zeta(0) = 1+1+1+1+1+1+\cdots = -\dfrac{1}{2}$ which is equation (7.1) derived above.

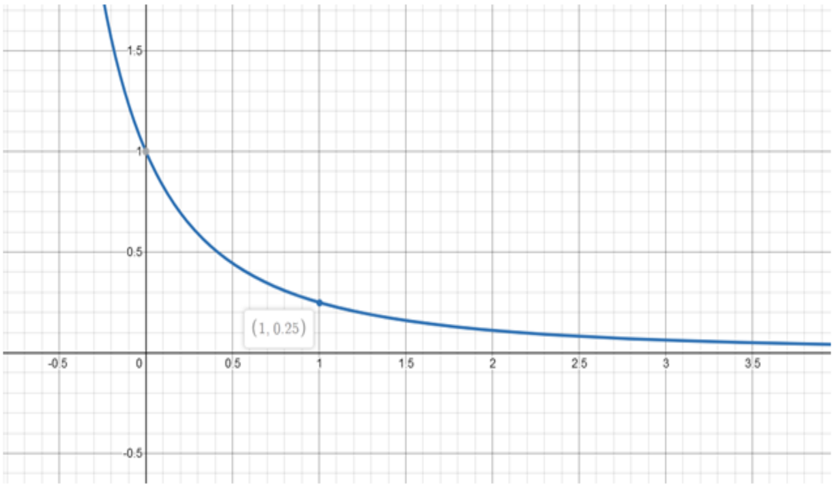

FIGURE 2. Graph of $S_1(x) = \dfrac{1}{(1+x)^2}$ near $x = 1$

We notice from Figure 2 that $S_1(1) = 0.25 = \frac{1}{4}$. Hence by equation (8.2) we get

$$\zeta(-1) = \frac{-S_1(1)}{3} = \frac{-\frac{1}{4}}{3} = -\frac{1}{12}\,.$$

Thus

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12} = -\frac{B_2}{2}\,,$$

which agrees with equation (7.6) when $m = 1$. We notice from Figure 3 that $S_2(1) = 0$. Hence by equation (8.2) we get

$$\zeta(-2) = \frac{-S_2(1)}{7} = 0\,.$$

Thus

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + 4^2 + \cdots = 0\,.$$

which agrees with equation (7.5) when $m = 1$. We notice from Figure 4 that $S_3(1) = -0.125 = -\frac{1}{8}$. Hence by equation (8.2) we get

$$\zeta(-3) = \frac{-S_3(1)}{15} = \frac{-\left(-\frac{1}{8}\right)}{15} = \frac{1}{120}\,.$$

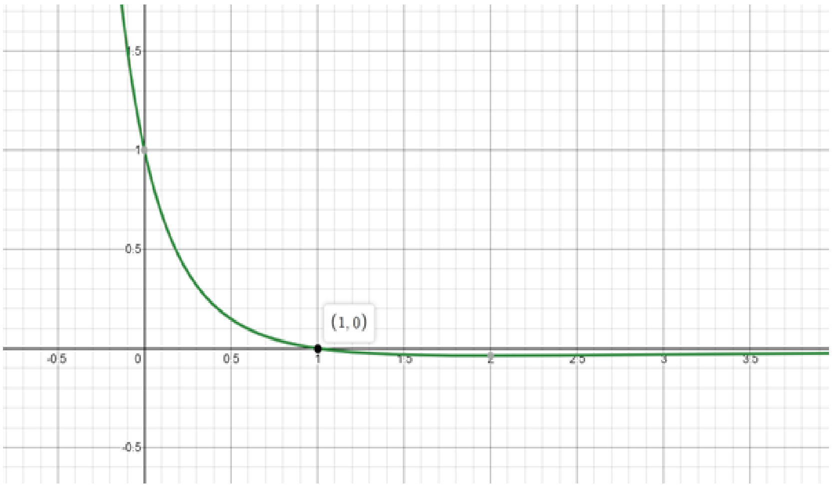

FIGURE 3. Graph of $S_2(x) = \dfrac{1-x}{(1+x)^3}$ near $x = 1$

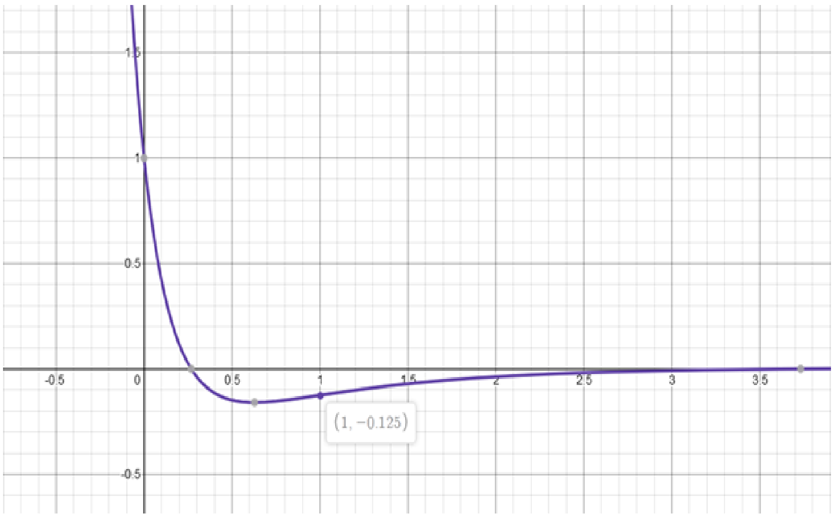

FIGURE 4. Graph of $S_3(x) = \dfrac{1-4x+x^2}{(1+x)^4}$ near $x = 1$

Thus

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120} = -\frac{B_4}{4}\,,$$

which agrees with equation (7.6) when $m = 2$.

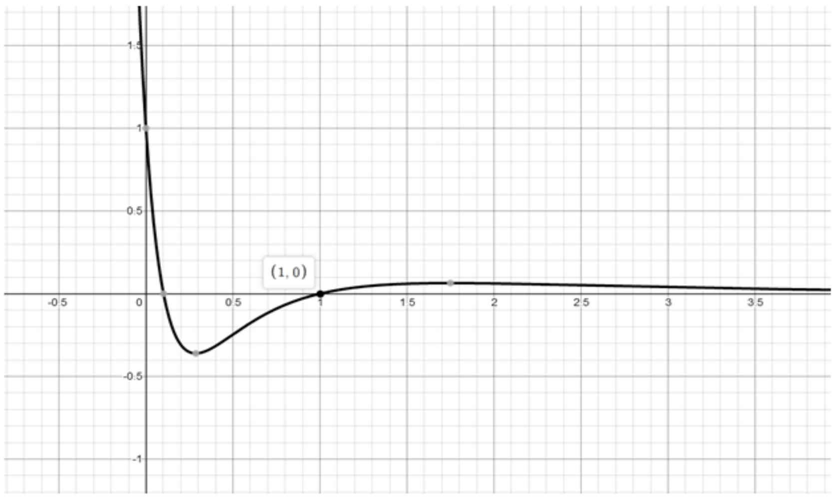

FIGURE 5. Graph of $S_4(x) = \dfrac{1 - 11x + 11x^2 - x^3}{(1+x)^5}$ near $x = 1$

We notice from Figure 5 that $S_4(1) = 0$. Hence by equation (8.2) we get

$$\zeta(-4) = \frac{-S_4(1)}{31} = 0\,.$$

Thus

$$\zeta(-4) = 1^4 + 2^4 + 3^4 + 4^4 + \cdots = 0\,,$$

which agrees with equation (7.5) when $m = 2$.

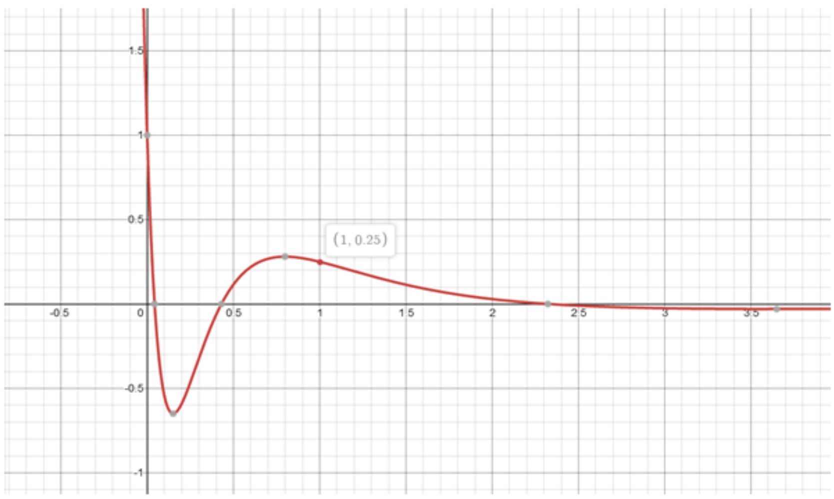

FIGURE 6. Graph of $S_5(x) = \dfrac{1 - 26x + 66x^2 - 26x^3 + x^4}{(1+x)^6}$ near $x = 1$

We notice that $S_5(1) = 0.25 = \dfrac{1}{4}$. Hence by equation (8.2) we get

$$\zeta(-5) = \frac{-S_5(1)}{63} = \frac{-\left(\dfrac{1}{4}\right)}{63} = -\frac{1}{252}\,.$$

Thus

$$\zeta(-5) = 1^5 + 2^5 + 3^5 + 4^5 + \cdots = -\frac{1}{252} = -\frac{B_6}{6}\,,$$

which agrees with equation (7.6) when $m = 3$.

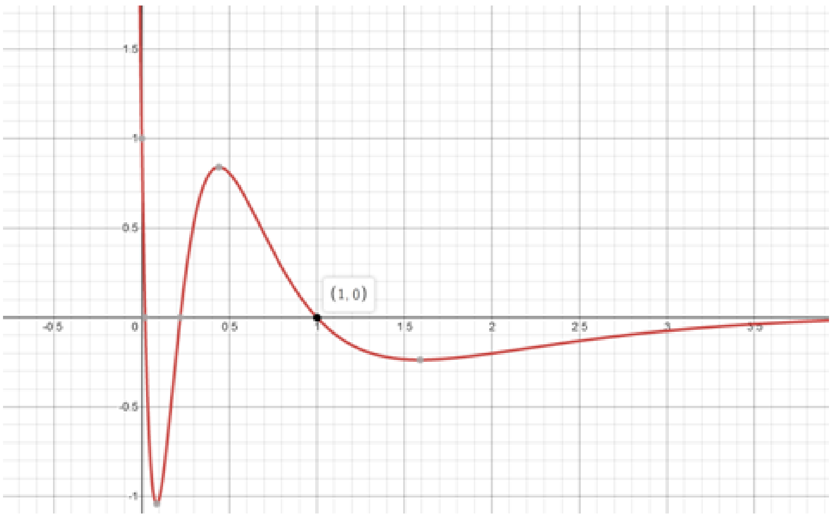

FIGURE 7. Graph of $S_6(x) = \dfrac{1 - 57x + 302x^2 - 302x^3 + 57x^4 - x^5}{(1+x)^7}$ near $x = 1$

We notice that $S_6(1) = 0$. Hence by equation (8.2) we get

$$\zeta(-6) = \frac{-S_6(1)}{127} = 0\,.$$

Thus

$$\zeta(-6) = 1^6 + 2^6 + 3^6 + 4^6 + \cdots = 0\,,$$

which agrees with equation (7.5) when $m = 3$.

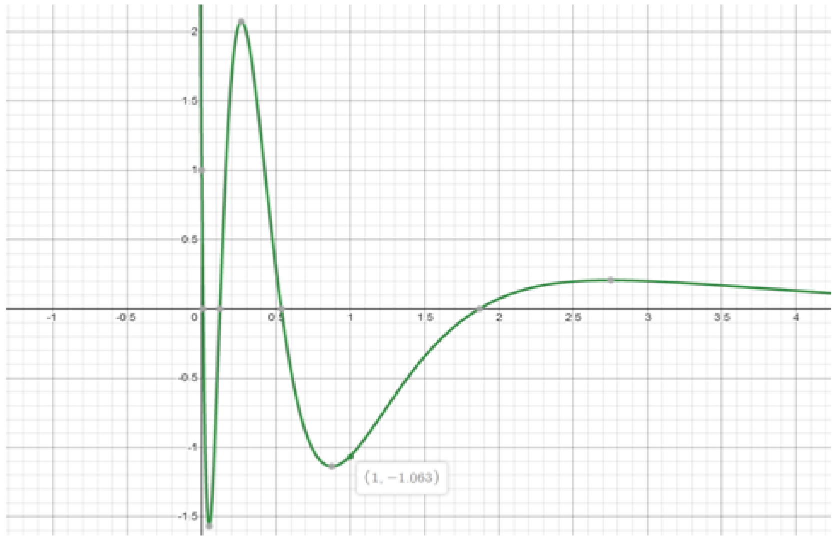

FIGURE 8. Graph of
$$S_7(x) = \frac{1 - 120x + 1192x^2 - 2416x^3 + 1192x^4 - 120x^5 + x^6}{(1+x)^8}$$
near $x = 1$

We notice from Figure 8 that $S_7(1) \approx -1.0625 = -\dfrac{17}{16}$. Hence by equation (8.2) we get

$$\zeta(-7) = \frac{-S_7(1)}{255} = \frac{-\left(-\frac{17}{16}\right)}{255} = \frac{1}{240}.$$

Thus
$$\zeta(-7) = 1^7 + 2^7 + 3^7 + 4^7 + \cdots = \frac{1}{240} = -\frac{B_8}{8},$$
which agrees with equation (7.6) when $m = 4$.

In general, we find that the graph of $S_k(x)$ pass through (1,0) if k is even. Hence by equation (8.2) we have $\zeta(-k) = \dfrac{S_k(1)}{1 - 2^{k+1}} = 0$ whenever k is even. That is, the Riemann zeta function vanishes at all negative even integer values. These values are called trivial zeros of the Riemann zeta function.

In the case if k is odd, $\zeta(-k)$ is identified through Bernoulli Numbers as proved in equation (7.6). The values of $\zeta(-k)$ gives an idea of how $S_k(x)$ would behave near $x = 1$.

9. CONCLUSION

In this paper, I had proved the general result of Ramanujan summation through equations (7.5) and (7.6) using very different approach. The advantage of this method is that it never require complex function theory which is usually employed by many researchers in dealing with Riemann zeta function. In fact, the truth established in equation (7.5) that trivial zeros of zeta function are negative even integers, is viewed as analytic continuation of Euler zeta function to extend to Riemann zeta function. This paper has completely avoided the usual analytic continuation theory and had obtained same results just by using real valued functions and basic ideas. This is one of the significant aspects of this paper. I had made use of the Desmos Graphing Software tool to create graphs presented in Figures 1 to 8 of this paper.

The other significant part of this paper is to provide the Geometric meaning by conveying the fact that the behavior of Riemann zeta function for negative integer values depends on the behavior of $S_k(x)$ near $x = 1$. I am very much sure, that Srinivasa Ramanujan would himself have got similar Geometric insights and could have arrived at the values $\zeta(-1), \zeta(-2), \zeta(-3), \cdots$ which he mentioned consistently in his letters to seek patronage for his work, but never seem to have disclosed any information that he was actually dealing with Riemann zeta function. At the same time, his thought process would be much more wonderful compared to how I discussed these concepts. He could have obtained these results effortlessly and elegantly. Nevertheless, I am happy that I had obtained two of his notebook jotting results by introducing novel approach using elementary concepts and providing some hint to the meaning of the answers obtained.

I dedicate this paper in memory of the Genius Srinivasa Ramanujan commemorating his Centenary Remembrance Year.

10. APPLICATIONS AND SCOPE

It is well known in mathematical world, that the Riemann zeta function has profound applications both in mathematical sub-disciplines and other areas of science like Theoretical Physics, particularly in String Theory and Atomic Physics. The quest for locating the non-trivial zeros of Riemann zeta function has been the holy grail of mathematics today. Even after 160 years we couldn't either prove or disprove Riemann Hypothesis.

In this paper, a simple and novel way is presented to understand the behavior of Riemann zeta function at negative integer values. Similarly, we can try to construct nice functions of complex variables to understand the non-trivial zeros and general behavior of the zeta function. That will be a very big step towards developing mathematics to greater horizon.

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