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PACKING COLORING ON SUBDIVISION-VERTEX AND SUBDIVISION-EDGE JOIN OF CYCLE C_M WITH PATH P_N

K. RAJALAKSHMI[1], M. VENKATACHALAM, M. BARANI, AND DAFIK

ABSTRACT. The packing chromatic number χ_ρ of a graph G is the smallest integer k for which there exists a mapping π from $V(G)$ to $\{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i + 1$. In this paper, the authors find the packing chromatic number of subdivision vertex join of cycle graph with path graph and subdivision edge join of cycle graph with path graph.

1. INTRODUCTION

A set of connected graphs are contemplated in this paper, which are undirected, loops less and without multiple edges. Let $G = (V, E)$ be a graph. No two adjacent vertices receive the same color as the vertex coloring of graph G is the assignment of colors to the vertices of G.

Let $G(p, q)$ [7] be a graph with $p = |V|$ and $q = |E|$ correspondingly indicate the count of vertices and edges of a graph G.

A packing k- coloring of a graph G [1,3,12] is a mapping π from $V(G)$ to $\{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i + 1$. The packing chromatic number χ_ρ of a graph G is the smallest integer k for which G has packing k-coloring.

[1]*corresponding author*

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Mr. Goddard et al. [5] have justified that the packing coloring problem is NP-complete for general graphs. Fiala and Golovach [4] has confirmed that NP-complete is even for trees.

By inserting a new vertex of degree a subdivision graph $S(G)$ [11,6] of the graph G is obtained. From G of degree 2 on each edge of G. For $k \geq 1$, the k-th subdivision graph $S_k(G)$ is acquired. By connecting each vertex of $V(G_1)$ with every vertex of $V(G_2)$ a subdivision-vertex join [8] of two vertex disjoint graphs G_1 and G_2 is represented by $G_1\dot{V}G_2$, has been procured from $S(G_1)$ and (G_2).

Let C_m and P_n denote the cycle and path graph with m and n vertices, respectively. By definition of subdivision vertex join of graphs [2, 9, 13] we subdivide each edge of the cycle graph and join each vertex of the cycle graph with every vertex of the path graph [10, 11, 14].

Throughout this paper, $\{v_k : 1 \leq k \leq m\}$,$\{u_k : 1 \leq k \leq m\}$ and $\{s_p : 1 \leq p \leq n\}$ denote the vertices of cycle, the subdivided vertices of the cycle and the vertices of the path, respectively. The total number of vertices of the subdivision vertex join graph is $2m + \dot{n}$.

2. Packing coloring of subdivision-vertex join and subdivision-edge join graph

Theorem 2.1. *$\chi_\rho[C_m\dot{v}P_n]$ is packing chromatic number of the subdivision-vertex join of a cycle graph C_m and a path graph P_n for $m \geq 3$ and $n \geq 2$. Then*

$$\chi_\rho(C_m\dot{v}P_n) = \begin{cases} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd} \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even} \end{cases}$$

Proof. Let us assume $V(C_m\dot{v}P_n) = \{v_k, u_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$. For $1 \leq i \leq m-1$

- Each edge v_kv_{k+1} is subdivided by u_i of $C_m\dot{v}P_n$

For $1 \leq p \leq m$

- Each edge v_1v_m subdivided by u_m of $C_m\dot{v}P_n$

Case (i): m is odd or even, n is odd.

We assume $\chi_\rho[C_m\dot{v}P_n] < \frac{2m+n+1}{2}$ to get $\chi_\rho[C_m\dot{v}P_n] \geq \frac{2m+n+1}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n-1}{2})$ colors for each valid vertex in $C_m\dot{v}P_n$. As per the definition, we have two rules for subdivision-vertex join graph of cycle and path graph. The rules that $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We ramaining of $(\frac{2m+n-3}{2})$ colors after select $(\frac{2m+n-1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-3}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two statement of $\chi_\rho[C_m\dot{v}P_n] < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then we accept the statemet of $\chi_\rho[C_m\dot{v}P_n] \geq \frac{2m+n+1}{2}$. We get the upper bound of packing chromatic number $\chi_\rho[C_m\dot{v}P_n] \leq \frac{2m+n+1}{2}$ has to be calculated as follows.

The function of color $c : V[C_m\dot{v}P_n] \to \{c_1, c_2, ...c_{\frac{2m+n+1}{2}}\}$ is defined by,

$$
\begin{aligned}
c(u_k) &= c_1 && \text{for } 1 \leq k \leq m\,; \\
c(s_{2p-1}) &= c_1 && \text{for } 1 \leq p \leq n\,; \\
c(v_k) &= c_{k+1} && \text{for } 1 \leq k \leq m\,; \\
c(s_{2p}) &= c_{m+p+1} && \text{for } 1 \leq p \leq n\,.
\end{aligned}
$$

Therefore, $\chi_\rho[C_m\dot{v}P_n] \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m\dot{v}P_n] = \frac{2m+n+1}{2}$.

Case (ii) : m is odd or even, n is even.

We assume $\chi_\rho[C_m\dot{v}P_n] < \frac{2m+n+2}{2}$ to get $\chi_\rho[C_m\dot{v}P_n] \geq \frac{2m+n+2}{2}$ as lower bound of the packing chromatic number.We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m\dot{v}P_n$. As per the definition, we have two rules for subdivision-vertex join graph of cycle and path graph. The rules that $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two statement of $\chi_\rho[C_m\dot{v}P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then we accept the statemet of $\chi_\rho[C_m\dot{v}P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound of packing chromatic number $\chi_\rho[C_m\dot{v}P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

The function of color $c : V[C_m\dot{v}P_n] \to \{c_1, c_2, ...c_{\frac{2m+n+2}{2}}\}$ is defined by,

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\begin{aligned}
c(u_k) &= c_1 && \text{for } 1 \leq k \leq m\,; \\
c(s_{2p-1}) &= c_1 && \text{for } 1 \leq p \leq n\,; \\
c(v_k) &= c_{k+1} && \text{for } 1 \leq k \leq m\,; \\
c(s_{2p}) &= c_{m+p+1} && \text{for } 1 \leq p \leq n\,.
\end{aligned}
$$

Therefore, $\chi_\rho[C_m\dot{v}P_n] \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m\dot{v}P_n] = \frac{2m+n+2}{2}$. □

Theorem 2.2. *$\chi_\rho[C_m\underline{v}P_n]$ is packing chromatic number of the subdivision-edge join of a cycle graph C_m and a path graph P_n for $m \geq 3$ and $n \geq 2$. Then*

$$\chi_\rho(C_m\underline{v}P_n) = \begin{cases} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd}; \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even}. \end{cases}$$

Proof. Let us assume $V(C_m\underline{v}P_n) = \{v_k, u_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$
For $1 \leq k \leq m-1$

- Each edge v_kv_{k+1} is subdivided by u_k of $C_m\underline{v}P_n$.

For $1 \leq p \leq m$

- Each edge v_1v_m subdivided by u_m of $C_m\underline{v}P_n$.

Case (i) : m is odd or even, n is odd.

We assume $\chi_\rho[C_m\underline{v}P_n] < \frac{2m+n+1}{2}$ to get $\chi_\rho[C_m\underline{v}P_n] \geq \frac{2m+n+1}{2}$, as lower bound of the packing chromatic number. We choose $(\frac{2m+n-1}{2})$ colors for each valid vertex in $C_m\underline{v}P_n$. As per the definition, we have two rules for subdivision-edge join graph of cycle and path graph. The rules are $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We ramaining of $(\frac{2m+n-3}{2})$ colors after select $(\frac{2m+n-1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-3}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two vertices of color i are at distance at least $i+1$ apart and $d(v_k, v_{k+1}) = 2$. The statement $\chi_\rho[C_m\underline{v}P_n] < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_\rho[C_m\underline{v}P_n] \geq \frac{2m+n+1}{2}$. We get the upper bound of packing chromatic number, $\chi_\rho[C_m\underline{v}P_n] \leq \frac{2m+n+1}{2}$ has to be calculated as follows.

The function $c : V[C_m\underline{v}P_n] \to \{c_1, c_2, ...c_{\frac{2m+n+1}{2}}\}$ defined by,

$$\begin{aligned} c(v_k) &= c_1 && \text{for } 1 \leq k \leq m \\ c(s_{2p-1}) &= c_1 && \text{for } 1 \leq p \leq n \\ c(u_k) &= c_{k+1} && \text{for } 1 \leq k \leq m \\ c(s_{2p}) &= c_{m+p+1} && \text{for } 1 \leq p \leq n \end{aligned}$$

Therefore, $\chi_\rho[C_m\underline{v}P_n] \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m\underline{v}P_n] = \frac{2m+n+1}{2}$.

Case (ii) : m is odd or even, n is even.

We assume $\chi_\rho[C_m\underline{v}P_n] < \frac{2m+n+2}{2}$ to get $\chi_\rho[C_m\underline{v}P_n] \geq \frac{2m+n+2}{2}$, as lower bound of the packing chromatic number. We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m\underline{v}P_n$. As per the definition, we have two rules for subdivision-edge join graph of cycle and path graph. The rules are $c(u_k) = c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We ramaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two vertices of color i are at distance at least $i+1$ apart and $d(v_k, v_{k+1}) = 2$. The statement $\chi_\rho[C_m\underline{v}P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the statement of $\chi_\rho[C_m\underline{v}P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound of packing chromatic number, $\chi_\rho[C_m\underline{v}P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

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c(u_k) &= c_{k+1} && \text{for } 1 \leq k \leq m\,;\\
c(s_{2p}) &= c_{m+p+1} && \text{for } 1 \leq p \leq n\,.
\end{aligned}$$

Therefore, $\chi_\rho[C_m\underline{v}P_n] \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m\underline{v}P_n] = \frac{2m+n+2}{2}$. □

Theorem 2.3. *$\chi_\rho[C_m vP_n]$ is packing chromatic number of the join of a cycle graph C_m and a path graph P_n for $m \geq 3$ and $n \geq 2$. Then*

$$\chi_\rho(C_m vP_n) = \begin{cases} \frac{2m+n+1}{2} & \text{if } m \text{ is odd or even, } n \text{ is odd} \\ \frac{2m+n+2}{2} & \text{if } m \text{ is odd or even, } n \text{ is even} \end{cases}$$

Proof. Let us assume $V(C_m vP_n) = \{v_k, s_p : 1 \leq k \leq m, 1 \leq p \leq n\}$. For $1 \leq k \leq m-1$

- Each edge $v_k v_{k+1}$ is subdivided by u_k of $C_m vP_n$.

For $1 \leq p \leq m$

- Each edge $v_1 v_m$ subdivided by u_m of $C_m vP_n$.

Case (i) : m is odd or even, n is odd.

We assume $\chi_\rho[C_m v P_n] < \frac{2m+n+1}{2}$ to get $\chi_\rho[C_m v P_n] \geq \frac{2m+n+1}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n-1}{2})$ colors for each valid vertex in $C_m v P_n$. As per the definition, we have two rules for join graph of cycle and path graph. The rules are $c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-3}{2})$ colors after select $(\frac{2m+n+1}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-3}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two vertices of color i are at distance atleast $i+1$ apart and $d(v_k, v_{k+1}) = 2$, the statement $\chi_\rho[C_m v P_n] < \frac{2m+n+1}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the satement of $\chi_\rho[C_m v P_n] \geq \frac{2m+n+1}{2}$. We get the upper bound for the packing chromatic number, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+1}{2}$ has to be culated as follows.

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Therefore, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+1}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m v P_n] = \frac{2m+n+1}{2}$.

Case (ii) : m is odd or even, n is even.

We assume $\chi_\rho[C_m v P_n] < \frac{2m+n+2}{2}$ to get $\chi_\rho[C_m v P_n] \geq \frac{2m+n+2}{2}$ as lower bound of the packing chromatic number. We choose $(\frac{2m+n}{2})$ colors for each valid vertex in $C_m v P_n$. As per the definition, we have two rules for join graph of cycle and path graph. The rules are $c(s_{2p-1}) = c_1$ and $d(v_k, s_p) = 1, d(v_k, v_{k+1}) = 2$. We remaining of $(\frac{2m+n-2}{2})$ colors after select $(\frac{2m+n}{2})$ colors. We get $c(v_k) \neq c(v_{k+1})$ and $(\frac{2m+n-2}{2})$ colors are required for each v_k and v_{k+1}, as per the definition of packing coloring that two vertices of color i are at distance atleast $i+1$ apart and $d(v_k, v_{k+1}) = 2$, the statement $\chi_\rho[C_m v P_n] < \frac{2m+n+2}{2}$ is wrong because it is a contrary value compared to the expected output. Then, we accept the satement of $\chi_\rho[C_m v P_n] \geq \frac{2m+n+2}{2}$. We get the upper bound for the packing chromatic number, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+2}{2}$ has to be calculated as follows.

The function of color $c : V[C_m v P_n] \to \{c_1, c_2, ...c_{\frac{2m+n+2}{2}}\}$ defined by,

$$\begin{aligned} c(s_{2p-1}) &= c_1 && \text{for } 1 \leq p \leq n\,; \\ c(v_k) &= c_{k+1} && \text{for } 1 \leq k \leq m\,; \\ c(s_{2p}) &= c_{m+p+1} && \text{for } 1 \leq p \leq n\,. \end{aligned}$$

Therefore, $\chi_\rho[C_m v P_n] \leq \frac{2m+n+2}{2}$. It is not difficult to prove that this coloring is a packing coloring. Hence, $\chi_\rho[C_m v P_n] = \frac{2m+n+2}{2}$. □

At the end of this paper we can conclude that for any cycle and path graph $m \geq 3$ and $n \geq 2$, we have:

$$\chi_\rho(C_m v P_n) = \chi_\rho(C_m \dot{v} P_n) = \chi_\rho(C_m \underline{v} P_n)$$

$$= \begin{cases} \frac{2m+n+1}{2} & if\ m\ is\ odd\ or\ even,\ n\ is\ odd\,, \\ \frac{2m+n+2}{2} & if\ m\ is\ odd\ or\ even,\ n\ is\ even\,. \end{cases}$$

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DEPARTMENT OF SCIENCE AND HUMANITIES
SRI KRISHNA COLLEGE OF ENGINEERING AND TECHNOLOGY
COIMBATORE-641008, TAMILNADU, INDIA
E-mail address: rajalakshmikandhasamy@gmail.com

DEPARTMENT OF MATHEMATICS
KONGUNADU ARTS AND SCIENCE COLLEGE
COIMBATORE - 641 029, TAMILNADU, INDIA
E-mail address: venkatmaths@gmail.com

DEPARTMENT OF MATHEMATICS
KONGUNADU ARTS AND SCIENCE COLLEGE
COIMBATORE - 641 029, TAMILNADU, INDIA
E-mail address: baranibe2013@gmail.com

CGANT UNIVERSITY OF JEMBER
MATHEMATICS EDUCATION DEPARTMENT
UNIVERSITY OF JEMBER, INDONESIA
E-mail address: d.dafik@unej.ac.id