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SPLITTING ON CLASSES OF HELM GRAPHS AND ITS EQUITABLE TOTAL CHROMATIC NUMBER

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ABSTRACT. The equitable total coloring of a graph G is the different colors used to color all the vertices and edges of G, in the order that adjacent vertices and edges are assigned with least different k-colors and can be partitioned into colors sets which differ by maximum one. The minimum of k-colors required is known as the equitable total chromatic number. In this paper the splitting graph of Helm and Closed Helm graph is constructed and its equitable total chromatic number is acquired.

1. INTRODUCTION

In graphs various types of colorings have been introduced and studied over decades of time. It studies the importance in allocation of resources based on time, energy, efficiency and availability etc. The equitable condition in coloring makes the proper scheduling of resources in the optimal way. It started with the idea of equitize colors on vertex, edge and then total coloring of graphs respectively.

The notion of equitable total coloring in splitting of graphs is a recent approach. In 1973, Meyer first presented the concept of equitable coloring for the vertices of graph. In 1994, Hung-Lin Fu [4] first presented the idea of equitable total coloring as well as its chromatic number and also conjectured

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that $\chi''_{=}(G) \leq \Delta + 2$. In 1980, Sampathkumar and Walikar [3] introduced the formation of splitting graph of graphs.

In this paper, first the construction of splitting graph is done for Helm graph H_n and Closed Helm graph CH_n and then its equitable total coloring is determined. Many real time applications in networks, assignment and scheduling problems can be optimally solved with the help of graph colorings. This optimization can be made equally distributed with respect to time and availability of resources by the method of equitability in coloring.

2. Prelimnaries

In this section the basic terminologies required for this paper are discussed.

Definition 2.1. *The total coloring of a graph G is the proper mixture of vertex and edge coloring in the condition that no two adjacent vertices and edges are assigned the same color. The least number of colors needed is called the total chromatic number of G and is denoted by* $\chi''(G)$.

Definition 2.2. *[5, 6] If the set of vertices of a graph G can be partitioned into* k*-classes* $T_1, T_2, \ldots, T_k$ *such that each* T_i *is an independent set and the condition* $||T_i| - |T_j|| \leq 1$ *holds for every* $i \neq j$*, then* G *is said to be equitably* k*- total colorable. The minimum of such* k *is called the equitable total chromatic number of* G *and is denoted by* $\chi''_{=}(G)$.

Definition 2.3. *[8] For any integer* $n \geq 4$*, the wheel graph* W_n *is the* n*-vertex graph obtained by joining a vertex* v *to each of the* $n-1$ *vertices* $\{v_1, v_2, \ldots, v_{n-1}\}$ *of the cycle graph* C_{n-1}.

Definition 2.4. *[7, 8] The Helm graph* H_n *is the graph obtained from a Wheel graph* W_n *by adjoining a pendant edge to each vertex of the* $n-1$ *cycle in* W_n.

Definition 2.5. *[7] A closed helm* CH_n *is the graph obtained by taking a helm* H_n *and adding edges between the pendant vertices.*

Definition 2.6. *[3] For each point* v *of a graph* G*, take a new point* v'*. Join* v' *to all points of* G *adjacent to* v*. The graph* $S(G)$ *thus obtained is called the Splitting graph of* G.

Conjecture 2.1. *[ETCC] [4] For every graph* G*,* $\chi''_{=}(G) \leq \Delta(G) + 2$.

Conjecture 2.2. *[4] For every graph G, G has an equitable total k-coloring for each,* $k \geq max\{\chi''(G), \Delta(G)+2\}$.

The additional graph theory terminologies defined in this paper are found in [1,2].

3. EQUITABLE TOTAL COLORING ON SPLITTING OF HELM GRAPH FAMILIES

In this section, the equitable total chromatic number on splitting of H_n and CH_n are investigated.

Theorem 3.1. *For $n \geq 6$, the equitable total chromatic number of Splitting of Helm graph is* $\chi''_{=}[S(H_n)] = 2n-1$.

Proof. The Helm graph H_n consists of $2n-1$ vertices and $3(n-1)$ edges. Let the vertex set be

$$V\left(H_n\right) = \{v\} \cup \{v_i : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n-1\},$$

where v is a vertex on the hub, $v_i (1 \leq i \leq n-1)$ is the vertices on the $n-1$ cycle and $u_i (1 \leq i \leq n-1)$ is the pendant vertices. The edge sets are classified as

$$\begin{aligned} E\left(H_n\right) = &\{vv_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \\ &\cup \{v_{n-1}v_1\} \cup \{v_i u_i : 1 \leq i \leq n-1\}. \end{aligned}$$

By the construction of splitting graph of Helm graph it expands as a new graph with $4n-2$ vertices and $8(n-1)$ edges. Its vertex and edge sets are as follows:

$$\begin{aligned} V\left[S\left(H_n\right)\right] = &\{v\} \cup \{v_i : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n-1\} \\ &\cup \{v'\} \quad \cup \{v'_i : 1 \leq i \leq n-1\} \cup \{u'_i : 1 \leq i \leq n-1\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &E[S\left(H_n\right)] \\ &\quad = \{vv_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \\ &\quad \cup \{v_{n-1}v_1\} \cup \{v_i u_i : 1 \leq i \leq n-1\} \cup \{vv'_i : 1 \leq i \leq n-1\} \\ &\quad \cup \{v' v_i : 1 \leq i \leq n-1\} \cup \{v_i v'_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1} v'_1\} \\ &\quad \cup \{v_1 v'_{n-1}\} \cup \{v_i v'_{i-1} : 2 \leq i \leq n-1\} \cup \{v_i u'_i : 1 \leq i \leq n-1\}. \end{aligned} \tag{3.2}$$

The total coloring on the vertices and edges of the splitted helm graph is defined as a function $f : T \longrightarrow C$ where $T = V[S(H_n)] \cup E[S(H_n)]$ and $C = \{1, 2, \ldots, 2n-1\}$. The function f applied on the elements yields the coloring to the appropriate vertices and edges which is categorized in the following tables. The vertex coloring is sorted as follows:

TABLE 1

Vertices	Colors	Vertices	Colors
v	1	u_{n-1}	1
v_1	n	v'	1
v_i	$i, 2 \leq i \leq n-1$	v'_i	$i+2, 1 \leq i \leq n-2$
u_1	1	v'_{n-1}	2
u_i	$i+3, 2 \leq i \leq n-2$	u'_i	$i+n, 1 \leq i \leq n-1$

Similarly the coloring on edges is summarized as follows:

TABLE 2

Edges	Colors	Edges	Colors
vv_i	$i+1, 1 \leq i \leq n-1$	$v_2v'_3$	n
v_iv_{i+1}	$i+3(modn), 1 \leq i \leq n-2$	$v_iv'_{i+1}$	$i-1, 3 \leq i \leq n-2$
$v_{n-1}v_1$	3	$v_iv'_{i-1}$	$i+n+2, 2 \leq i \leq n-3$
v_1u_1	$2n-1$	$v_{n-2}v'_{n-3}$	$n+1$
v_iu_i	$i+n-1, 2 \leq i \leq n-1$	$v_{n-1}v'_{n-2}$	$n+2$
vv'_i	$i+n, 1 \leq i \leq n-1$	$v_{n-1}v'_1$	2
$v'v_i$	$i+n, 1 \leq i \leq n-1$	$v_iu'_i$	$i+n+1, 1 \leq i \leq n-2$
$v_1v'_2$	$n+3$	$v_{n-1}u'_{n-1}$	$n-2$

By the above process, all the vertices and edges of splitting graph of Helm graph is colored with $2n-1$ colors. While coloring if the value $modn = 0$, it is replaced with n. The colors applied can be classified into independent sets $T_1, T_2, \ldots, T_{2n-1}$ such that $||T_i| - |T_j|| \leq 1$, $i \neq j$ and therefore it becomes equitable. For example, consider the case $n = 6$ (see Fig. 1) it is evident that it is equitably total colored with $2n-1$. Hence for any splitting of Helm graph $\chi''_{=}[S(H_n)] = 2n-1$. □

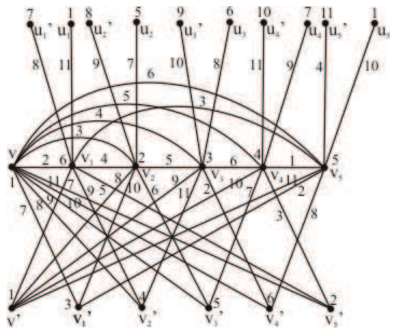

FIGURE 1. Graph of $S(H_6)$

Theorem 3.2. *For $n \geq 6$, the equitable total chromatic number of splitting of Closed Helm graph is $\chi''_{=}[S(CH_n)] = 2n-1$.*

Proof. The proof is similar to Theorem 3.1. The closed helm graph CH_n consists of $2n-1$ vertices and $4(n-1)$ edges. By definition 2.5, CH_n is obtained from helm by adding edges between the pendant vertices. So the vertex set is same as that of H_n and let it be

$$V(CH_n) = \{v\} \cup \{v_i : 1 \leq i \leq n-1\} \cup \{u_i : 1 \leq i \leq n-1\},$$

where v is a vertex on the central point. But the edge set of CH_n is added with pendant edges $u_iu_{i+1} : 1 \leq i \leq n-2$ and $u_{n-1}u_1$ which is generalised as

$$E(CH_n) = \{vv_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1}v_1\} \\ \cup \{v_iu_i : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-2\} \cup \{u_{n-1}u_1\}.$$

Thus the formation of splitting graph on CH_n extends as a graph with $4n-2$ vertices and $11(n-1)$ edges. The vertex and edge set remains identical as in (3.1) and (3.2). Further the edge set in (3.2) is included with $3(n-1)$ edges which is grouped as

$$E[S(CH_n)] = \{vv_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_{n-1}v_1\} \\ \cup \{v_iu_i : 1 \leq i \leq n-1\} \cup \{vv'_i : 1 \leq i \leq n-1\} \cup \{v'v_i : 1 \leq i \leq n-1\} \\ \cup \{v_iv'_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v'_{n-1}\} \cup \{v_iv'_{i-1} : 2 \leq i \leq n-1\} \cup \{v_{n-1}v'_1\} \\ \cup \{v_iu'_i : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-2\} \cup \{u_{n-1}u_1\} \cup \{u_1u'_{n-1}\} \\ \cup \{u_iu'_{i+1} : 1 \leq i \leq n-2\} \cup \{u'_iu_{i+1} : 1 \leq i \leq n-2\} \cup \{u'_1u_{n-1}\}.$$

TABLE 3

Pendant Vertices	Colors
u_1	1
u_i	$i+3, 2 \leq i \leq n-1$
u'_1	1
u'_2	$n+1$
u'_i	$i+n, 3 \leq i \leq n-1$

TABLE 4

Pendant Edges	Colors
$u_i u_{i+1}$	$i+2, 1 \leq i \leq n-3$
$u_{n-2} u_{n-1}$	$2n-1$
$u_{n-1} u_1$	2
$u'_i u_{i+1}$	$i+1, 1 \leq i \leq n-3$
$u'_{n-2} u_{n-1}$	1
$u'_1 u_{n-1}$	$2n-3$
$u_i u'_{i+1}$	$i+n-1, 1 \leq i \leq n-3$
$u_{n-2} u'_{n-1}$	1
$u_1 u'_{n-1}$	$2n-2$

The coloring of both vertices and edges follows the equivalent pattern of Theorem 3.1 except for few elements on the pendant and its coloring is exposed in the Tables.

The remaining elements follow coloring in Table 1 and Table 2. It is clear that all the elements of the graph is total colored with $2n-1$ colors and satisfies the equitable condition. For example, consider the case $n = 6$ (see Fig. 2) it is evident that it is equitably total colored with $2n-1$. By this method of coloring for any CH_n with $n \geq 6$ is optimally total colored with equitability. Therefore it is determined that $\chi''_{=}[S(CH_n)] = 2n-1$. □

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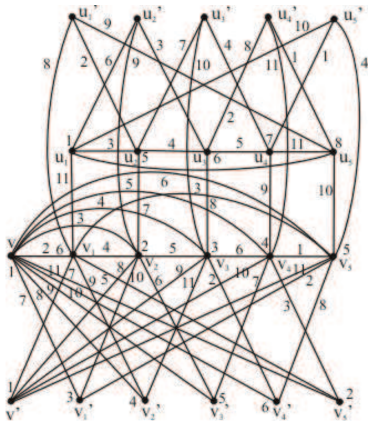

FIGURE 2. Graph of $S(CH_6)$

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