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ON b-CHROMATIC NUMBER OF THETA GRAPH FAMILIES

M. VINITHA[1], M. VENKATACHALAM AND DAFIK

ABSTRACT. In this paper, we investigate the b-chromatic number for the theta graph $\theta(s_1, s_2, \cdots, s_n)$, middle graph of theta graph $M(G)$, total graph of theta graph $T(G)$, line graph of theta graph $L(G)$ and the central graph of theta graph $C(G)$.

1. INTRODUCTION

The b-coloring is the maximal integer k such that G have b-coloring by k colors.

The b-chromatic number of G is the largest positive integer k, it is a proper coloring with the additional property that each color class contains a color dominating vertex (a vertex that has a neighbour in all other color classes, [7].

The b-chromatic number was found by Irving and Manlove in the year 1999. It is denoted as $\varphi(G)$ [7]. Irving and Manlove introduced b-chromatic number by considering the proper coloring that are minimal with respect to the partial order defines on the set of all partitions of vertices $V(G)$. They also proved that determining of $\varphi(G)$ is NP-hard in general polynomial for trees.

Irving and Manlove [7] have shown shown a result for upper bound of $\varphi(G)$,

$$\varphi(G) \leq \Delta(G) + 1.$$

[1]*corresponding author*

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Effaintin and Kheddouci studied [3–5] the b-chromatic number for the complete caterpillars, the powers of paths, cycles, and complete k-ary trees.

Kouider and Maheo [8] gave some lower and upper bounds for the b-chromatic number of the chartesian product of two graphs.

Here, we consider the b-chromatic number of graphs derived from Theta graph.

2. Preliminaries

Definition 2.1. *The generalized theta graph [1] $\theta(s_1, s_2, \cdots, s_n)$ consists of a pair of end vertices joined by n internally disjoint paths of lengths ≥ 1, where $s_1, s_2, \cdots, s_n$ denote the number of internal vertices in the paths. The end vertices are North pole (N) and South pole (S). A path between the North pole and south pole are the longitude and is denoted as L.*

In this paper we denote the end vertices as X and Y.

Definition 2.2. *The Line graph [6] of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.*

Definition 2.3. *Let G be a graph with vertex set and edge set $V(G)$ and $E(G)$. The Middle graph [2, 10] of G, denoted by $M(G)$ is defined as follows.*

The vertex set is adjacent of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

- *x, y are in $E(G)$ and x, y are adjacent in G.*
- *x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.*

Definition 2.4. *Let G be a graph with vertex set and edge set $V(G)$ and $E(G)$. The Total graph [6] of G, denoted by $T(G)$ is defined in the following ways. The vertex set $T(G)$ is $V(G) \cup E(G)$. Two vertices x,y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:*

- *x, y are in $V(G)$ and x is adjacent to y in G.*
- *x, y are adjacent in G.*
- *x is in $V(G)$, y is in $E(G)$, and x y are incident in G.*

Definition 2.5. *The central graph [9] of G, is denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.*

3. Main results

Theorem 3.1. *Let $G = \theta(s_1, s_2, \cdots, s_n)$ be the generalized theta graph with longitudes L_1, L_2, L_3, L_4 respectively. Then the b-chromatic number of theta graph is $\varphi(G) = 3$.*

Proof. Let $\{t_1, t_2, \ldots, t_n\}$, $\{u_1, u_2, \ldots, u_n\}$, $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$, be, respectively, the vertices of the longitude. Assign the color,

- c_1 to t_1, c_2 to t_2, c_3 to t_3 in L_1.
- c_1 to u_1, c_2 to u_2, c_3 to u_3 in L_2.
- c_1 to v_1, c_2 to v_2, c_3 to v_3 in L_3.
- c_1 to w_1, c_2 to w_2, c_3 to w_3 in L_4.

For the remaining vertices of t_i,u_i, v_i, w_i $(4 \leq i \leq n)$ and also for the end vertices X and Y assign the existing colors without affecting the conditions of proper coloring.

Suppose, we assume that $\varphi(G) = 4$. The maximum degree vertices are X and Y.

Remaining vertices are of degree 2. All 2 degree vertices, overrulled the b-coloring conditions. It contradicts our assumption. Hence, $\varphi(G) = 3$. □

Theorem 3.2. *Let $G = \theta(s_1, s_2, \cdots, s_n)$ be the generalized theta graph with longitudes L_1, L_2, L_3, L_4. Then the b-chromatic number of middle graph of theta graph is $\varphi(M(G)) = 6$.*

Proof. Let the vertices of the longitude L_1 from X to Y be $\{t_1, t_2, \cdots, t_n\}$, the longitude L_2 is $\{u_1, u_2, \cdots, u_n\}$, the longitude L_3 is $\{v_1, v_2, \cdots, v_n\}$, and the longitude L_4 is $\{w_1, w_2, \cdots, w_n\}$.

Let,

$$V(G) = \{t_1, t_2, \cdots, t_n\} \cup \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{w_1, w_2, \cdots, w_n\}$$

and

$$E(G) = \{a'_i : 1 \leq i \leq n, b'_i : 1 \leq i \leq n, c'_i : 1 \leq i \leq n, d'_i : 1 \leq i \leq n\},$$

where

- a'_1 be the edge corresponding to Xt_1, a'_{i+1} are the edges corresponding to $t_i t_{i+1}$ $(1 \leq i \leq n-1)$ and a'_{n+1} be the edge corresponding to Yt_n;

- b'_1 be the edge corresponding to Xu_1, b'_{i+1} are the edges corresponding to u_iu_{i+1} $(1 \leq i \leq n-1)$ and b'_{n+1} be the edge corresponding to Yu_n;
- c'_1 be the edge corresponding to Xv_1, c'_{i+1} are the edges corresponding to v_iv_{i+1} $(1 \leq i \leq n-1)$ and c'_{n+1} be the edge corresponding to Yv_n;
- d'_1 be the edge corresponding to Xw_1, d'_{i+1} are the edges corresponding to w_iw_{i+1} $(1 \leq i \leq n-1)$ and d'_{n+1} be the edge corresponding to Yw_n.

By the definition of middle graph,

$$\begin{aligned} V[M(G)] = &\{t_i : 1 \leq i \leq n\} \cup \{t'_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \\ &\cup \{u'_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \\ &\cup \{w_i : 1 \leq i \leq n\} \cup \{w'_i : 1 \leq i \leq n\}, \end{aligned}$$

where,

- t'_1 be the sub divided vertices of Xt_1, t'_{i+1} are the sub divided vertices of t_it_{i+1} $(1 \leq i \leq n-1)$ and t'_{n+1} be the sub divided vertices of Yt_n;
- u'_1 be the sub divided vertices of Xu_1, u'_{i+1} are the sub divided vertices of u_iu_{i+1} $(1 \leq i \leq n-1)$ and u'_{n+1} be the sub divided vertices of Yu_n;
- v'_1 be the sub divided vertices of Xv_1, v'_{i+1} are the sub divided vertices of v_iv_{i+1} $(1 \leq i \leq n-1)$ and v'_{n+1} be the sub divided vertices of Yv_n;
- w'_1 be the sub divided vertices of Xw_1, w'_{i+1} are the sub divided vertices of w_iw_{i+1} $(1 \leq i \leq n-1)$, and w'_{n+1} be the sub divided vertices of Yw_n.
- t_i, u_i, v_i, w_i , $(1 \leq i \leq n)$ are the vertices respectively.

Consider the following 6-coloring $(c_1, c_2, c_3, c_4, c_5, c_6)$ of $M(G)$.

Assign c_1 to t_1, c_2 to t'_1, c_3 to X, c_4 to w'_1, c_5 to v'_1, c_6 to u'_1, c_1 to u_1, c_1 to v_1, c_1 to w_1. For $(2 \leq i \leq n)$ t_i, u_i, v_i, w_i, t'_i, u'_i, v'_i, w'_i and for end vertex Y. Assign the existing colors without affecting the conditions of proper coloring.

Suppose we assume $\varphi(M(G)) = 6$.

Here, the maximal degree is 6. i.e, $\Delta(G) = 6$. This maximal degree occurs only at X and Y. But, the remaining vertices are of degree 4, they are not connected to all the 6-color.

This cannot satisfy b-coloring conditions. This contradicts our assumption. Hence, $\varphi(M(G)) = 6$. □

Theorem 3.3. *Let $G = \theta(s_1, s_2, \cdots, s_n)$ be the generalized theta graph with longitudes L_1, L_2, L_3, L_4. Then the b-chromatic number of total graph of theta graph is $\varphi(T(G)) = 6$.*

Proof. Consider the coloring of $M(G)$ which has been proved in theorem 2. By the definition of Total graph and by theorem 2 of middle graph, we say that this satisfies all the conditions of b-coloring of $T(G)$. Hence, $\varphi(T(G)) = 6$. $\square$

Theorem 3.4. *Let $G = \theta(s_1, s_2, \cdots, s_n)$ be the generalized theta graph with longitudes L_1, L_2, L_3, L_4. Then the b-chromatic number of the line graph of theta graph is $\varphi(L(G)) = 5$.*

Proof. Let the vertices of the longitude L_1 from X to Y be $\{t_1, t_2, \cdots, t_n\}$, the longitude L_2 is $\{u_1, u_2, \cdots, u_n\}$, the longitude L_3 is $\{v_1, v_2, \cdots, v_n\}$, and the longitude L_4 is $\{w_1, w_2, \cdots, w_n\}$.

Let

$$V(G) = \{t_1, t_2, \cdots, t_n\} \cup \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{w_1, w_2, \cdots, w_n\}$$

and $E(G) = \{e_i, e_i', e_i'', e_i''', 1 \leq i \leq n\}$, where

- e_1 be the edge corresponding to Xt_1, e_{i+1} are the edges corresponding to t_it_{i+1} $(1 \leq i \leq n-1)$ and e_{n+1} be the edge corresponding to Yt_n;
- e_1' be the edge corresponding to Xu_1, e_{i+1}' are the edges corresponding to u_iu_{i+1} $(i \leq i \leq n-1)$and e_{n+1}' be the edge corresponding to Yu_n;
- e_1'' be the edge corresponding to Xv_1, e_{i+1}'' are the edges corresponding to v_iv_{i+1} $(1 \leq i \leq n-1)$ and e_{n+1}'' be the edge corresponding to Yv_n;
- e_1''' be the edge corresponding to Xw_1, e_{i+1}''' are the edges corresponding to w_iw_{i+1} $(1 \leq i \leq n-1)$, and e_{n+1}''' be the edge corresponding to Yw_n.

By the definition of line graph, the $E(G)$ is converted into $V[L(G)]$ i.e., $V[L(G)] = \{e_i \cup e_i' \cup e_i'' \cup e_i''', 1 \leq i \leq n\}$.

Consider the following 5-coloring $(1, 2, 3, 4, 5)$ of the line of theta graph.

In $L(G)$, the maximum degree occurs in the vertex e_1 by the adjacency of e_2, e_1', e_1'', e_1'''.

By the definition of b-chromatic number, we assign c_1 to e_1, c_2 to e_2, c_3 to e_1', c_4 to e_1'', c_5 to e_1'''. For the remaining vertices of e_i $(3 \leq i \leq n)$ and e_i', e_i'', e_i''' $(2 \leq i \leq n)$, assign the existing colors without affecting the conditions of proper coloring.

Here, the minimum degree is 2. This shows that this is b-coloring, $\varphi(L(G)) \geq 5$. Since, $\Delta(L(G)) = 4$, we know that, $\varphi(G) \leq \Delta(G) + 1$ and we have, $\varphi(L(G)) \leq 5$. Hence, $\varphi(L(G)) = 5$. $\square$

Theorem 3.5. *Let $G = \theta\{s_1, s_2, \cdots, s_n\}$ be the generalized theta graph with longitudes L_1, L_2, L_3, L_4. Then the b-chromatic number of central graph of theta graph is $\varphi(C(G)) = m + 2$, where $m = n_1 + n_2 + n_3 + n_4 + 2$.*

Proof. Let the vertices of the longitude L_1 from X to Y be $\{t_1, t_2, \cdots, t_{n_1}\}$, the longitude L_2 is $\{u_1, u_2, \cdots, u_{n_2}\}$, the longitude L_3 is $\{v_1, v_2, \cdots, v_{n_3}\}$, and the longitude L_4 is $\{w_1, w_2, \cdots, w_{n_4}\}$.

We have, $m = n_1 + n_2 + n_3 + n_4 + 2$.

Let,

$$V(G) = \{t_1, t_2, \cdots, t_{n_1}\} \cup \{u_1, u_2, \cdots, u_{n_2}\} \cup \{v_1, v_2, \cdots, v_{n_3}\} \cup \{w_1, w_2, \cdots, w_{n_4}\}$$

and $E(G) = \{a'_i : 1 \leq i \leq n_1, b'_i : 1 \leq i \leq n_2, c'_i : 1 \leq i \leq n_3, d'_i : 1 \leq i \leq n_4\}$, where

- a'_1 be the edge corresponding to Xt_1, a'_{i+1} are the edges corresponding to $t_i t_{i+1}$ $(1 \leq i \leq n_1 - 1)$, and a'_{n_1+1} be the edge corresponding to Yt_{n_1};
- b'_1 be the edge corresponding to Xu_1, b'_{i+1} are the edges corresponding to $u_i u_{i+1}$ $(1 \leq i \leq n_2 - 1)$, and b'_{n_2+1} be the edge corresponding to Yu_{n_2};
- c'_1 be the edge corresponding to Xv_1, c'_{i+1} are the edges corresponding to $v_i v_{i+1}$ $(1 \leq i \leq n_3 - 1)$, and c'_{n_3+1} be the edge corresponding Yv_{n_3};
- d'_1 be the edge corresponding to Xw_1, d'_{i+1} are the edges corresponding to $w_i w_{i+1}$ $(1 \leq i \leq n_4 - 1)$, and d'_{n_4+1} be the edge corresponding to Yw_{n_4}.

By the definition of central graph, the edge joining the vertices $V(G)$ are subdivided by the new vertices as follows,

$$\begin{aligned} V[C(G)] = \{&t_i(1 \leq i \leq n_1) \cup t'_i(1 \leq i \leq n_1) \cup u_i(1 \leq i \leq n_2) \\ &\cup u'_i(1 \leq i \leq n_2) \cup v_i(1 \leq i \leq n_3) \cup v'_i(1 \leq i \leq n_3) \\ &\cup w_i(1 \leq i \leq n_4) \cup w'_i(1 \leq i \leq n_4)\}, \end{aligned}$$

where,

- t'_1 be the sub divided vertices of Xt_1, t'_{i+1} are the sub divided vertices of $t_i t_{i+1}$, $(1 \leq i \leq n_1 - 1)$, and t'_{n+1} be the sub divided vertices of Yt_{n_1};
- u'_1 be the sub divided vertices of Xu_1, u'_{i+1} are the sub divided vertices of $u_i u_{i+1}$, $(1 \leq i \leq n_2 - 1)$, and u'_{n+1} be the sub divided vertices of Yu_{n_2};
- v'_1 be the sub divided vertices of Xv_1, v'_{i+1} are the sub divided vertices of $v_i v_{i+1}$, $(1 \leq i \leq n_3 - 1)$, and v'_{n+1} be the sub divided vertices of Yv_{n_3};

- w'_1 be the sub divided vertices of Xw_1, w'_{i+1} are the sub divided vertices of w_iw_{i+1}, $(1 \leq i \leq n_4 - 1)$, and w'_{n+1} be the sub divided vertices of Yw_{n_4};
- $t_i(1 \leq i \leq n_1)$, $u_i(1 \leq i \leq n_2)$, $v_i(1 \leq i \leq n_3)$, $w_i(1 \leq i \leq n_4)$ are the vertices respectively.

Consider the following $(m+2)$ coloring as follows.

Assign, c_1 to t_1 in L_1, c_2 to u_1 in L_2, c_3 to v_1 in L_3, c_4 to w_1 in L_4. Repeat this procedure for all other vertices $(ie., t_{n_1}, u_{n_2}, v_{n_3}, w_{n_4})$ by increasing the color from L_1 to L_4. Also for the end vertices, assign c_{n_4+1} to X and c_{n_4+2} to Y.

For the remaining vertices of $t'_i(1 \leq i \leq n_1)$, $u'_i(1 \leq i \leq n_2)$, $v'_i(1 \leq i \leq n_3)$, $w'_i(1 \leq i \leq n_4)$ assign the existing colors without affecting the condition of the proper coloring.

Here, the maximal degree is $\Delta(C(G)) = m + 1$. This satisfies all the conditions of b-coloring. Therefore, $\varphi(G) \geq m + 2$. From $\varphi(G) \leq \Delta(G) + 1$ we have, $\varphi(C(G)) \leq m + 2$.

Hence, $\varphi(C(G)) = m + 2$. □

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PG AND RESEARCH DEPARTMENT OF MATHEMATICS
KONGUNADU ARTS AND SCIENCE COLLEGE
COIMBATORE-641029
E-mail address: muruganvinitha5@gmail.com

PG AND RESEARCH DEPARTMENT OF MATHEMATICS
KONGUNADU ARTS AND SCIENCE COLLEGE
COIMBATORE-641029
E-mail address: venkatmaths@gmail.com

CGANT - UNIVERSITY OF JEMBER
MATHEMATICS EDUCATION DEPARTMENT
UNIVERSITY OF JEMBER, INDONESIA
E-mail address: d.dafik@unej.ac.id