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ON SOME TYPES OF CONTINUITY FOR MULTIFUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT. This article deals with the concepts of upper and lower $\alpha(\star)$-continuous multifunctions. Some characterizations of upper and lower $\alpha(\star)$-continuous multifunctions are investigated. The relationships between upper and lower $\alpha(\star)$ continuous multifunctions and the other types of continuity for multifunctions are established.

1. INTRODUCTION

The field of mathematical science called topology is concerned with all questions directly or indirectly related to continuity. Continuity is an important concept for the study and investigation in topological spaces. This concept has been extended to the setting multifunctions and has been generalized by weaker forms of open sets. In 1965, Njåstad [18] introduced a weak form of open sets called α-sets. Mashhour et al. [17] defined a function to be α-continuous if the inverse image of each open set is an α-set and obtained several characterizations of such functions. Noiri [20] investigated the relationships between α-continuous functions and several known functions, for example, almost continuous functions, η-continuous functions, δ-continuous functions or irresolute functions. In [21], the present author introduced the concept of almost α-continuity in topological spaces as a generalization of α-continuity

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and almost continuity. Neubrunn [19] introduced the notion of upper (resp. lower) α-continuous multifunctions. These multifunctions are further investigated by the present authors [24]. In 1996, Popa and Noiri [23] introduced the notion of upper (resp. lower) almost α-continuous multifunctions and investigated several characterizations and some basic properties concerning upper (resp. lower) almost α-continuous multifunctions. Some characterizations of weakly α-continuous multifunctions are investigated in [7], [23] and [22].

Topological ideals have played an important role in topology. Kuratowski [16] and Vaidyanathswamy [25] introduced and studied the concept of ideals in topological spaces. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1990, Janković and Hamlett [15] introduced the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [3] studied the concepts of α-$\mathscr{I}$-continuity and α-$\mathscr{I}$-openness in ideal topological spaces and obtained several characterizations of these functions. Hatir and Noiri [14] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [2], the present authors introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. In 2005, Hatir and Noiri [12] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuity. Moreover, the present authors [11] introduced and investigated the notions of strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions.

The article is organized as follows. In Section 3, we introduce the notions of upper and lower $\alpha(\star)$-continuous multifunctions and investigate some characterizations of such multifunctions. In Section 4, is devoted to introducing and studying upper and lower almost $\alpha(\star)$-continuous multifunctions. In the last section, we present the concepts of upper and lower weakly $\alpha(\star)$-continuous multifunctions. Moreover, several interesting characterizations of upper and lower weakly $\alpha(\star)$-continuous multifunctions are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will denoted by Cl(A) and Int(A), respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ implies $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$. A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [16], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ. Observe additionally that Cl$^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by the base $\mathscr{B}(\mathscr{I}, \tau) = \{U - I_0 \mid U \in \tau$ and $I_0 \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I}, \tau)$ is not always a topology [25]. A subset A is said to be $\star$-closed [15] if $A^\star \subseteq A$. The complement of a $\star$-closed set is said to be $\star$-open. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by Int$^\star(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Definition 2.1. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *semi*-$\mathscr{I}$-open* [10] *if $A \subseteq Cl(Int^\star(A))$;*
(2) *semi*-$\mathscr{I}$-closed* [10] *if its complement is semi*-$\mathscr{I}$-open;*
(3) *semi-$\mathscr{I}$-open* [14] *if $A \subseteq Cl^\star(Int(A))$;*
(4) *semi-$\mathscr{I}$-closed* [8] *if its complement is semi-$\mathscr{I}$-open;*
(5) *α-$\mathscr{I}$-open* [14] *if $A \subseteq Int(Cl^\star(Int(A)))$;*
(6) *α-$\mathscr{I}$-closed if its complement is α-$\mathscr{I}$-open.*

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}$-closed (resp. semi*-$\mathscr{I}$-closed) sets containing A is called the semi-$\mathscr{I}$-closure [8] (resp. semi*-$\mathscr{I}$-closure [8]) of A and is denoted by $s\text{Cl}_{\mathscr{I}}(A)$ (resp. $s^\star\text{Cl}_{\mathscr{I}}(A)$). The union of all semi-$\mathscr{I}$-open (resp. semi*-$\mathscr{I}$-open) sets contained in A is called the semi-$\mathscr{I}$-interior (resp. semi*-$\mathscr{I}$-interior) of A and is denoted by $s\text{Int}_{\mathscr{I}}(A)$ (resp. $s^\star\text{Int}_{\mathscr{I}}(A)$).

Lemma 2.1. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *If A is an open set, then $s^\star Cl_{\mathscr{I}}(A) = Int(Cl^\star(A))$.*
(2) *If A is a $\star$-open set, then $sCl_{\mathscr{I}}(A) = Int^\star(Cl(A))$.*

Proof. (1) Suppose that A is an open set. Then $A \subseteq \text{Int}(\text{Cl}^\star(A))$ and by Lemma 13(1) of [8], we have $s^\star\text{Cl}_{\mathscr{I}}(A) = A \cup \text{Int}(\text{Cl}^\star(A)) = \text{Int}(\text{Cl}^\star(A))$.

(2) Suppose that A is a $\star$-open set. Then, we have $A \subseteq \text{Int}^\star(\text{Cl}(A))$ and by Lemma 13(2) of [8], $s\text{Cl}_{\mathscr{I}}(A) = A \cup \text{Int}^\star(\text{Cl}(A)) = \text{Int}^\star(\text{Cl}(A))$. □

Proposition 2.1. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $\{A_\gamma \mid \gamma \in \Gamma\}$ be a family of subsets of X. If A_γ is α-$\mathscr{I}$-closed for each $\gamma \in \Gamma$, then $\bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\mathscr{I}$-closed.*

Proof. Suppose that A_γ is α-$\mathscr{I}$-closed for each $\gamma \in \Gamma$. Then, we have $X - A_\gamma$ is α-$\mathscr{I}$-open for each $\gamma \in \Gamma$ and by Proposition 3.2(2) of [3], $\bigcup_{\gamma\in\Gamma}(X - A_\gamma) = X - \bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\mathscr{I}$-open and so $\bigcap_{\gamma\in\Gamma} A_\gamma$ is α-$\mathscr{I}$-closed. □

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\mathscr{I}$-closed sets containing A is called the α-$\mathscr{I}$-closure of A and is denoted by $\alpha\text{Cl}_{\mathscr{I}}(A)$. The α-$\mathscr{I}$-interior of A is defined by the union of all α-$\mathscr{I}$-open sets contained in A and is denoted by $\alpha\text{Int}_{\mathscr{I}}(A)$.

Proposition 2.2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are hold:*

(1) *$\alpha Cl_{\mathscr{I}}(A)$ is α-$\mathscr{I}$-closed.*
(2) *A is α-$\mathscr{I}$-closed if and only if $A = \alpha Cl_{\mathscr{I}}(A)$.*

Proof. (1) Follows from Proposition 2.1.

(2) Follows from (1). □

Lemma 2.2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) *A is α-$\mathscr{I}$-open in X.*
(2) *$G \subseteq A \subseteq Int(Cl^\star(G))$ for some open set G.*
(3) *$G \subseteq A \subseteq s^\star Cl_{\mathscr{I}}(G)$ for some open set G.*
(4) *$A \subseteq s^\star Cl_{\mathscr{I}}(Int(A))$.*

Proof. $(1) \Rightarrow (2)$: Suppose that A is an α-$\mathscr{I}$-open set. Then, we have $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$. Put $G = \mathrm{Int}(A)$, then G is an open set such that $G \subseteq A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(G))$.

$(2) \Rightarrow (3)$: This follows from Lemma 2.1(1).

$(3) \Rightarrow (4)$: Suppose that $G \subseteq A \subseteq s^\star\mathrm{Cl}_{\mathscr{I}}(G)$ for some open set G. Then, we have $G \subseteq \mathrm{Int}(A)$ and hence $A \subseteq s^\star\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}(A))$.

$(4) \Rightarrow (1)$: Suppose that $A \subseteq s^\star\mathrm{Cl}_{\mathscr{I}}(\mathrm{Int}(A))$. Since $\mathrm{Int}(A)$ is open in X and by Lemma 2.1(1), we have $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$. This shows that A is α-$\mathscr{I}$-open in X. □

Lemma 2.3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *A is α-$\mathscr{I}$-closed in X if and only if $s^\star Int_{\mathscr{I}}(Cl(A)) \subseteq A$.*
(2) *$s^\star Int_{\mathscr{I}}(Cl(A)) = Cl(Int^\star(Cl(A)))$.*
(3) *$\alpha Cl_{\mathscr{I}}(A) = A \cup Cl(Int^\star(Cl(A)))$.*
(4) *$\alpha Int_{\mathscr{I}}(A) = A \cap Int(Cl^\star(Int(A)))$.*

Proof. (1) Follows from Lemma 2.2.

(2) Follows from Lemma 13(1) of [8].

(3) We observe that

$$\begin{aligned}\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A \cup \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A)))))) &\subseteq \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A \cup (\mathrm{Cl}(A))))) \\ &\subseteq \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A))) \\ &\subseteq A \cup \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A))).\end{aligned}$$

Thus, $A \cup \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A)))$ is α-$\mathscr{I}$-closed and hence $\alpha\mathrm{Cl}_{\mathscr{I}}(A) \subseteq A \cup \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A)))$. On the other hand, since $\alpha\mathrm{Cl}_{\mathscr{I}}(A)$ is α-$\mathscr{I}$-closed, we have

$$\mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A))) \subseteq \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(\alpha_{\mathscr{I}}\mathrm{Cl}(A)))) \subseteq \alpha\mathrm{Cl}_{\mathscr{I}}(A)$$

and hence $A \cup \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(A))) \subseteq \alpha\mathrm{Cl}_{\mathscr{I}}(A)$. Consequently, we obtain $\alpha\mathrm{Cl}_{\mathscr{I}}(A) = A \cup \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(A)))$.

(4) Since $\alpha\mathrm{Int}_{\mathscr{I}}(A)$ is α-$\mathscr{I}$-open, we have

$$\alpha\mathrm{Int}_{\mathscr{I}}(A) \subseteq \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(\alpha\mathrm{Int}_{\mathscr{I}}(A)))) \subseteq \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A))).$$

Therefore, $\alpha\mathrm{Int}_{\mathscr{I}}(A) \subseteq A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A)))$. On the other hand, we have

$$\begin{aligned} A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A))) &\subseteq \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A))) \\ &= \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A))))) \\ &= \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A)))))). \end{aligned}$$

Thus, $A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A)))$ is α-$\mathscr{I}$-open and so $A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A))) \subseteq \alpha\mathrm{Int}_{\mathscr{I}}(A)$. Consequently, we obtain $\alpha\mathrm{Int}_{\mathscr{I}}(A) = A \cap \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(A)))$. □

3. On upper and lower $\alpha(\star)$-continuous multifunctions

In this section, we introduce the notions of upper and lower $\alpha(\star)$-continuous multifunctions and investigate several characterizations of such multifunctions.

Definition 3.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper $\alpha(\star)$-continuous at a point x of X if for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq V$;*
(2) *lower $\alpha(\star)$-continuous at a point x of X if for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$;*
(3) *upper (resp. lower) $\alpha(\star)$-continuous if F is upper (resp. lower) $\alpha(\star)$-continuous at each point of X.*

Theorem 3.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper $\alpha(\star)$-continuous at $x \in X$;*
(2) *$x \in s^{\star}Cl_{\mathscr{I}}(Int(F^{+}(V)))$ for every α-$\mathscr{J}$-open set V of Y containing $F(x)$;*
(3) *$x \in \alpha Int_{\mathscr{I}}(F^{+}(V))$ for every α-$\mathscr{J}$-open set V of Y containing $F(x)$.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq V$; hence

$$x \in U \subseteq F^+(V).$$

Since U is α-$\mathscr{I}$-open, by Lemma 2.2, we have $x \in U \subseteq s^\star\text{Cl}_\mathscr{I}(\text{Int}(U)) \subseteq s^\star\text{Cl}_\mathscr{I}(\text{Int}(F^+(V)))$.

$(2) \Rightarrow (3)$: Let V be any $\star$-open set of Y containing $F(x)$. Then by (2), we have $x \in s^\star\text{Cl}_\mathscr{I}(\text{Int}(F^+(V)))$ and by Lemma 2.1(1), $x \in \text{Int}(\text{Cl}^\star(\text{Int}(F^+(V))))$. Thus, $x \in \alpha\text{Int}_\mathscr{I}(F^+(V))$ by Lemma 2.3(4).

$(3) \Rightarrow (1)$: Let V be any $\star$-open set of Y containing $F(x)$. By (3), we have $x \in \alpha\text{Int}_\mathscr{I}(F^+(V))$ and so there exists an α-$\mathscr{I}$-open set U of X containing x such that $U \subseteq F^+(V)$; hence $F(U) \subseteq V$. This shows that F is upper $\alpha(\star)$-continuous at x. □

Theorem 3.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower $\alpha(\star)$-continuous at $x \in X$;*
(2) *$x \in s^\star Cl_\mathscr{I}(Int(F^-(V)))$ for every α-$\mathscr{J}$-open set V of Y such that*

$$F(x) \cap V \neq \emptyset;$$

(3) *$x \in \alpha Int_\mathscr{I}(F^-(V))$ for every α-$\mathscr{J}$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.2. *A subset N of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-neighbourhood (resp. α-$\mathscr{I}$-neighbourhood) of $x \in X$ if there exists a $\star$-open (resp. α-$\mathscr{I}$-open) set V of X such that $x \in V \subseteq N$.*

The following results give some characterizations of upper and lower $\alpha(\star)$-continuous multifunctions.

Theorem 3.3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper $\alpha(\star)$-continuous;*
(2) *$F^+(V)$ is α-$\mathscr{I}$-open in X for every $\star$-open set V of Y;*
(3) *$F^-(K)$ is α-$\mathscr{I}$-closed in X for every $\star$-closed set K of Y;*
(4) *$s^\star Int_\mathscr{I}(Cl^\star(F^-(B))) \subseteq F^-(Cl^\star(B))$ for every subset B of Y;*
(5) *$\alpha Cl_\mathscr{I}(F^-((B))) \subseteq F^-(Cl^\star(B))$ for every subset B of Y;*

(6) *for each* $x \in X$ *and each* $\star$*-neighbourhood* V *of* $F(x)$*,* $F^+(V)$ *is an* α*-*$\mathscr{I}$*-neighbourhood of* x*;*

(7) *for each* $x \in X$ *and each* $\star$*-neighbourhood* V *of* $F(x)$*, there exists an* α*-*$\mathscr{I}$*-neighbourhood* U *of* x *such that* $F(U) \subseteq V$*.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. Since F is upper $\alpha(\star)$-continuous at x, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^+(V)$. By Lemma 2.2, we have $x \in U \subseteq s^\star\text{Cl}_{\mathscr{I}}(\text{Int}(U)) \subseteq s^\star\text{Cl}_{\mathscr{I}}(\text{Int}(F^+(V)))$. Consequently, we obtain $F^+(V) \subseteq s^\star\text{Cl}_{\mathscr{I}}(\text{Int}(F^+(V)))$. It follows from Lemma 2.2 that $F^+(V)$ is α-$\mathscr{I}$-open in X.

$(2) \Leftrightarrow (3)$: This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

$(3) \Rightarrow (4)$: Let B be any subset of Y. Then $\text{Cl}^\star(B)$ is $\star$-closed in Y and by (3), $F^-(\text{Cl}^\star(B))$ is α-$\mathscr{I}$-closed in X. By Lemma 2.3(1), we have

$$\begin{aligned} s^\star\text{Int}_{\mathscr{I}}(\text{Cl}(F^-(B))) &\subseteq s^\star\text{Int}_{\mathscr{I}}(\text{Cl}(F^-(\text{Cl}^\star(B)))) \\ &\subseteq F^-(\text{Cl}^\star(B)). \end{aligned}$$

$(4) \Rightarrow (5)$: Let B be any subset of Y. By (4) and Lemma 2.3,

$$\begin{aligned} \alpha\text{Cl}_{\mathscr{I}}(F^-(B)) &= F^-(B) \cup s^\star\text{Int}_{\mathscr{I}}(\text{Cl}(F^-(B))) \\ &\subseteq F^-(\text{Cl}^\star(B)). \end{aligned}$$

$(5) \Rightarrow (3)$: Let K be any $\star$-closed set of Y. By (5), we have

$$\alpha\text{Cl}_{\mathscr{I}}(F^-(K)) \subseteq F^-(\text{Cl}^\star(K)) = F^-(K).$$

This shows that $F^-(K)$ is α-$\mathscr{I}$-closed in X.

$(2) \Rightarrow (6)$: Let $x \in X$ and V be a $\star$-neighbourhood of $F(x)$. Then, there exists a $\star$-open set G of Y such that $F(x) \subseteq G \subseteq V$. Thus, $x \in F^+(G) \subseteq F^+(V)$. By (2), $F^+(G)$ is α-$\mathscr{I}$-open and hence $F^+(V)$ is an α-$\mathscr{I}$-neighbourhood of x.

$(6) \Rightarrow (7)$: Let $x \in X$ and V be a $\star$-neighbourhood of $F(x)$. By (6), we have $F^+(V)$ is an α-$\mathscr{I}$-neighbourhood of x. Put $U = F^+(V)$, then U is an α-$\mathscr{I}$-neighbourhood of x such that $F(U) \subseteq V$.

$(7) \Rightarrow (1)$: Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Then V is a $\star$-neighbourhood of $F(x)$ and so there exists an α-$\mathscr{I}$-neighbourhood U of x such that $F(U) \subseteq V$. Since U is an α-$\mathscr{I}$-neighbourhood of x, there exists an α-$\mathscr{I}$-open set G of X such that $x \in G \subseteq U$; hence $F(G) \subseteq V$. This shows that F is upper $\alpha(\star)$-continuous. $\square$

Theorem 3.4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower $\alpha(\star)$-continuous;*
(2) *$F^{-}(V)$ is α-$\mathscr{I}$-open in X for every $\star$-open set V of Y;*
(3) *$F^{+}(K)$ is α-$\mathscr{I}$-closed in X for every $\star$-closed set K of Y;*
(4) *$s^{\star}Int_{\mathscr{I}}(Cl^{\star}(F^{+}(B))) \subseteq F^{+}(Cl^{\star}(B))$ for every subset B of Y;*
(5) *$\alpha Cl_{\mathscr{I}}(F^{+}((B))) \subseteq F^{+}(Cl^{\star}(B))$ for every subset B of Y;*
(6) *$F(\alpha Cl_{\mathscr{I}}(A)) \subseteq Cl^{\star}(F(A))$ for any subset A of X;*
(7) *$F(s^{\star}Int_{\mathscr{I}}(Cl(A))) \subseteq Cl^{\star}(F(A))$ for any subset A of X;*
(8) *$F(Cl(Int^{\star}(Cl(A)))) \subseteq Cl^{\star}(F(A))$ for any subset A of X.*

Proof. The proofs except for the following are similar to the proof of Theorem 3.3.

$(5) \Rightarrow (6)$: Let A be any subset of X. Since $A \subseteq F^{+}(F(A))$, we have $\alpha \mathrm{Cl}_{\mathscr{I}}(A) \subseteq \alpha \mathrm{Cl}_{\mathscr{I}}(F^{+}(F(A))) \subseteq F^{+}(\mathrm{Cl}^{\star}(F(A)))$ and hence

$$F(\alpha \mathrm{Cl}_{\mathscr{I}}(A)) \subseteq \mathrm{Cl}^{\star}(F(A)).$$

$(6) \Rightarrow (7)$: Let A be any subset of X. By (6) and Lemma 2.3,

$$\begin{aligned} F(s^{\star}\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}(A))) &= F(\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(A)))) \\ &\subseteq F(A \cup \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(A)))) \\ &= F(\alpha \mathrm{Cl}_{\mathscr{I}}(A)) \\ &\subseteq \mathrm{Cl}^{\star}(F(A)). \end{aligned}$$

$(7) \Rightarrow (8)$: Let A be any subset of X. By (7) and Lemma 2.3(2), we have $F(\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(A)))) = F(s^{\star}\mathrm{Int}_{\mathscr{I}}(\mathrm{Cl}(A))) \subseteq \mathrm{Cl}^{\star}(F(A))$.

$(8) \Rightarrow (1)$: Let $x \in X$ and V be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. Then, we have $x \in F^{-}(V)$. We shall show that $F^{-}(V)$ is α-$\mathscr{I}$-open in X. By the hypothesis, $F(\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{+}(Y - V))))) \subseteq \mathrm{Cl}^{\star}(F(F^{+}(Y - V))) \subseteq Y - V$ and hence $\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{+}(Y - V)))) \subseteq F^{+}(Y - V) = X - F^{-}(V)$. Consequently, we obtain $F^{-}(V) \subseteq \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{-}(V))))$. This shows that $F^{-}(V)$ is α-$\mathscr{I}$-open in X. Put $U = F^{-}(V)$, then U is an α-$\mathscr{I}$-open set of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. Therefore, F is lower $\alpha(\star)$-continuous. $\square$

Definition 3.3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be $\alpha(\star)$-continuous if for every $\star$-open set V of Y, $f^{-1}(V)$ is α-$\mathscr{I}$-open in X.*

Corollary 3.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $\alpha(\star)$-continuous;*
(2) *$f^{-1}(K)$ is α-$\mathscr{I}$-closed in X for every $\star$-closed set K of Y;*
(3) *$s^\star Int_{\mathscr{I}}(Cl^\star(f^{-1}(B))) \subseteq f^{-1}(Cl^\star(B))$ for any subset B of Y;*
(4) *$\alpha Cl_{\mathscr{I}}(f^{-1}(B)) \subseteq f^{-1}(Cl^\star(B))$ for any subset B of Y;*
(5) *for each $x \in X$ and each $\star$-neighbourhood V of $f(x)$, $f^{-1}(V)$ is an α-$\mathscr{I}$-neighbourhood of x;*
(6) *for each $x \in X$ and each $\star$-neighbourhood V of $f(x)$, there exists an α-$\mathscr{I}$-neighbourhood U of x such that $f(U) \subseteq V$;*
(7) *$f(\alpha Cl_{\mathscr{I}}(A)) \subseteq Cl^\star(f(A))$ for every subset A of X;*
(8) *$f(s^\star Int_{\mathscr{I}}(Cl^\star(A))) \subseteq Cl^\star(f(A))$ for every subset A of X;*
(9) *$f(Cl(Int^\star(Cl(A)))) \subseteq Cl^\star(f(A))$ for every subset A of X.*

Definition 3.4. *A family $\mathscr{U}$ of subsets of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\star$-locally finite if every $x \in X$ has a $\star$-neighbourhood which intersects only finitely many elements of $\mathscr{U}$.*

Definition 3.5. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *$\star$-paracompact if every cover of A by $\star$-open sets of X is refined by a cover of A which consists of $\star$-open sets of X and is $\star$-locally finite in X;*
(2) *$\star$-regular if for each $x \in A$ and each $\star$-open set U of X containing x, there exists a $\star$-open set V of X such that $x \in V \subseteq Cl(V) \subseteq U$.*

Lemma 3.1. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If A is a $\star$-regular $\star$-paracompact set of X and each $\star$-open set U containing A, then there exists a $\star$-open set V such that $A \subseteq V \subseteq Cl(V) \subseteq U$.*

Proof. For each $x \in A$, there exists a $\star$-open set W_x such that $x \in W_x \subseteq \mathrm{Cl}(W_x) \subseteq U$. Now, the family $\mathscr{W} = \{W_x \mid x \in A\}$ is a $\star$-open covering of A and so there exists a $\star$-locally finite family of $\star$-open sets $\mathscr{V} = \cup\{V_\alpha \mid \alpha \in \nabla\}$ which refines $\mathscr{W}$ and covers A. Consequently, we obtain $A \subseteq \mathscr{V} \subseteq \mathrm{Cl}(\mathscr{V}) = \mathrm{Cl}(\cup_{\alpha\in\nabla} V_\alpha) = \cup_{\alpha\in\nabla}\mathrm{Cl}(V_\alpha) \subseteq U$. □

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called *punctually $\star$-paracompact* (resp. *punctually $\star$-regular*) if for each $x \in X$, $F(x)$ is $\star$-paracompact (resp. $\star$-regular).

By $\alpha^\star\mathrm{Cl}(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, we shall denote a multifunction defined as follows: $[\alpha^\star\mathrm{Cl}(F)](x) = \alpha\mathrm{Cl}_{\mathscr{J}}(F(x))$ for each $x \in X$.

Lemma 3.2. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is punctually $\star$-regular and punctually $\star$-paracompact, then $[\alpha^\star Cl(F)]^+(V) = F^+(V)$ for every $\star$-open set V of Y.*

Proof. Let V be any $\star$-open set of Y and $x \in [\alpha^\star\mathrm{Cl}(F)]^+(V)$. Then $\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$ and hence $F(x) \subseteq V$. Thus, $x \in F^+(V)$. Therefore, $[\alpha^\star\mathrm{Cl}(F)]^+(V) \subseteq F^+(V)$. On the other hand, let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. Since $F(x)$ is punctually $\star$-regular and punctually $\star$-paracompact, by Lemma 3.1, there exists a $\star$-open set G such that $F(x) \subseteq G \subseteq \mathrm{Cl}(G) \subseteq V$; hence $\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq \mathrm{Cl}(G) \subseteq V$. This shows that $x \in [\alpha^\star\mathrm{Cl}(F)]^+(V)$ and hence $F^+(V) \subseteq [\alpha^\star\mathrm{Cl}(F)]^+(V)$. Consequently, we obtain $[\alpha^\star\mathrm{Cl}(F)]^+(V) = F^+(V)$. □

Theorem 3.5. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ be punctually $\star$-regular and punctually $\star$-paracompact. Then F is upper $\alpha(\star)$-continuous if and only if*

$$\alpha^\star Cl(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$$

is upper $\alpha(\star)$-continuous.

Proof. Suppose that F is upper $\alpha(\star)$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$. By Lemma 3.2, we have

$$x \in [\alpha^\star\mathrm{Cl}(F)]^+(V) = F^+(V).$$

Since F is upper $\alpha(\star)$-continuous, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is punctually $\star$-regular and punctually $\star$-paracompact for each $z \in U$, by Lemma 3.1, there exists a $\star$-open set G such that $F(z) \subseteq G \subseteq \mathrm{Cl}(G) \subseteq V$. Consequently, we obtain $\alpha\mathrm{Cl}_{\mathscr{J}}(F(z)) \subseteq \mathrm{Cl}(G) \subseteq V$ and hence $\alpha\mathrm{Cl}_{\mathscr{J}}(F(U)) \subseteq V$. Thus, $\alpha^\star\mathrm{Cl}(F)$ is upper $\alpha(\star)$-continuous.

Conversely, suppose that $\alpha^\star\mathrm{Cl}(F)$ is upper $\alpha(\star)$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. By Lemma 3.2, we have $x \in F^+(V) = [\alpha^\star\mathrm{Cl}(F)]^+(V)$ and hence $\alpha\mathrm{Cl}_{\mathscr{J}}(F(x)) \subseteq V$. Since $\alpha^\star\mathrm{Cl}(F)$ is upper $\alpha(\star)$-continuous, there exists an α-$\mathscr{I}$-open set U of X containing x such that $\alpha\text{Cl}_{\mathscr{J}}(F(U)) \subseteq V$; hence $F(U) \subseteq V$. This shows that F is upper $\alpha(\star)$-continuous. □

Lemma 3.3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, it follows that for each $\star$-open set V of Y $[\alpha^\star\text{Cl}(F)]^-(V) = F^-(V)$.*

Proof. Suppose that V is any $\star$-open set of Y. Let $x \in [\alpha^\star\text{Cl}(F)]^-(V)$. Then, we have $\alpha\text{Cl}_{\mathscr{J}}(F(x)) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$. Thus, $x \in F^-(V)$. This shows that $[\alpha^\star\text{Cl}(F)]^-(V) \subseteq F^-(V)$. On the other hand, let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq \alpha\text{Cl}_{\mathscr{J}}(F(x)) \cap V$ and hence $x \in [\alpha^\star\text{Cl}(F)]^-(V)$. Therefore, $F^-(V) \subseteq [\alpha^\star\text{Cl}(F)]^-(V)$. Consequently, we obtain $[\alpha^\star\text{Cl}(F)]^-(V) = F^-(V)$. □

Theorem 3.6. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $\alpha(\star)$-continuous if and only if $\alpha^\star\text{Cl}(F) : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $\alpha(\star)$-continuous.*

Proof. By utilizing Lemma 3.3, this can be proved similarly to that of Theorem 3.5. □

4. On upper and lower almost $\alpha(\star)$-continuous multifunctions

In this section, we introduce the concepts of upper and lower almost $\alpha(\star)$-continuous multifunctions. Moreover, several characterizations of upper and lower almost $\alpha(\star)$-continuous multifunctions are investigated.

Definition 4.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper almost $\alpha(\star)$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq \text{Int}^\star\text{Cl}(V)$;*

(2) *lower almost $\alpha(\star)$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(z) \cap \text{Int}^\star(\text{Cl}(V)) \neq \emptyset$ for every $z \in U$;*

(3) *upper (resp. lower) almost $\alpha(\star)$-continuous if F is upper (resp. lower) almost $\alpha(\star)$-continuous at each point of X.*

Theorem 4.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper almost $\alpha(\star)$-continuous at $x \in X$;*

(2) *for any $\star$-open set V of Y containing $F(x)$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$;*

(3) $x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$ *for every $\star$-open set V of Y containing $F(x)$;*

(4) $x \in \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$ *for every $\star$-open set V of Y containing $F(x)$.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\mathscr{I}$-open set U containing x such that $F(U) \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}(V))$ and by Lemma 2.1(2), we have $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$.

(2) $\Rightarrow$ (3): Let V be any $\star$-open set of Y containing $F(x)$. By (2), there exists an α-$\mathscr{I}$-open set U containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$ and hence $U \subseteq F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))$. Consequently, we obtain $x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$.

(3) $\Rightarrow$ (4): Let V be any $\star$-open set of Y containing $F(x)$. Then by (3), we have $x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$ and by Lemma 2.3(4),

$$x \in \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))))).$$

(4) $\Rightarrow$ (1): Let V be any $\star$-open set of Y containing $F(x)$. By (4), we have

$$x \in \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$$

and by Lemma 2.3(4), $x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$. Therefore, there exists an α-$\mathscr{I}$-open set U of X containing x such that $U \subseteq F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))$; hence $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$. Since V is $\star$-open, by Lemma 2.1(2), $F(U) \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}(V))$. This shows that F is upper almost $\alpha(\star)$-continuous at x. □

Theorem 4.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower almost $\alpha(\star)$-continuous at $x \in X$;*

(2) *for any $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(z) \cap s\mathrm{Cl}_{\mathscr{J}}(V) \neq \emptyset$;*

(3) $x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{-}(s\mathrm{Cl}_{\mathscr{J}}(V)))$ *for every $\star$-open set V of Y such that*

$$F(x) \cap V \neq \emptyset;$$

(4) $x \in \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{-}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$ *for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 4.1. □

Definition 4.2. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) $R^\star$-$\mathscr{I}$-*open if* $A = \mathit{Int}^\star(\mathit{Cl}(A))$;
(2) $R^\star$-$\mathscr{I}$-*closed if its complement is* $R^\star$-$\mathscr{I}$-*open.*

The following results give some characterizations of upper and lower almost $\alpha(\star)$-continuous multifunctions.

Theorem 4.3. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, *the following properties are equivalent:*

(1) F *is upper almost* $\alpha(\star)$-*continuous;*
(2) *for each* $x \in X$ *and each* $\star$-*open set* V *of* Y *containing* $F(x)$, *there exists an* α-$\mathscr{I}$-*open set* U *of* X *containing* x *such that* $F(U) \subseteq s\mathit{Cl}_{\mathscr{J}}(V)$;
(3) *for each* $x \in X$ *and each* $R^\star$-$\mathscr{J}$-*open set* V *of* Y *containing* $F(x)$, *there exists an* α-$\mathscr{I}$-*open set* U *of* X *containing* x *such that* $F(U) \subseteq V$;
(4) $F^+(V)$ *is* α-$\mathscr{I}$-*open in* X *for every* $R^\star$-$\mathscr{J}$-*open set* V *of* Y;
(5) $F^-(K)$ *is* α-$\mathscr{I}$-*closed in* X *for every* $R^\star$-$\mathscr{J}$-*closed set* K *of* Y;
(6) $F^+(V) \subseteq \alpha\mathit{Int}_{\mathscr{I}}(F^+(s\mathit{Cl}_{\mathscr{J}}(V)))$ *for every* $\star$-*open set* V *of* Y;
(7) $\alpha\mathit{Cl}_{\mathscr{I}}(F^-(s\mathit{Int}_{\mathscr{J}}(K))) \subseteq F^-(K)$ *for every* $\star$-*closed set* K *of* Y;
(8) $\alpha\mathit{Cl}_{\mathscr{I}}(F^-(\mathit{Cl}^\star(\mathit{Int}(K)))) \subseteq F^-(K)$ *for every* $\star$-*closed set* K *of* Y;
(9) $\alpha\mathit{Cl}_{\mathscr{I}}(F^-(\mathit{Cl}^\star(\mathit{Int}(\mathit{Cl}^\star(B))))) \subseteq F^-(\mathit{Cl}^\star(B))$ *for every subset* B *of* Y;
(10) $\mathit{Cl}(\mathit{Int}^\star(\,\mathit{Cl}(F^-(\mathit{Cl}^\star(\mathit{Int}(K)))))) \subseteq F^-(K)$ *for every* $\star$-*closed set* K *of* Y;
(11) $\mathit{Cl}(\mathit{Int}^\star(\,\mathit{Cl}(F^-(s\mathit{Int}_{\mathscr{J}}(K))))) \subseteq F^-(K)$ *for every* $\star$-*closed set* K *of* Y;
(12) $F^+(V) \subseteq \mathit{Int}(\mathit{Cl}^\star(\mathit{Int}(F^+(s\mathit{Cl}_{\mathscr{J}}(V)))))$ *for every* $\star$-*open set* V *of* Y.

Proof. $(1) \Rightarrow (2)$: The proof follows from Theorem 4.1.

$(2) \Rightarrow (3)$: The proof is obvious.

$(3) \Rightarrow (4)$: Let V be any $R^\star$-$\mathscr{J}$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and by (3), there exists an α-$\mathscr{I}$-open set U_x of X containing x such that $F(U_x) \subseteq V$. Therefore, we have $x \in U_x \subseteq F^+(V)$ and so $F^+(V) = \cup_{x \in F^+(V)} U_x$ is α-$\mathscr{I}$-open in X.

$(4) \Rightarrow (5)$: This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

$(5) \Rightarrow (6)$: Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then, we have $F(x) \subseteq V \subseteq s\mathrm{Cl}_{\mathscr{J}}(V)$ and hence

$$x \in F^+(s\mathrm{Cl}_{\mathscr{J}}(V)) = X - F^-(Y - s\mathrm{Cl}_{\mathscr{J}}(V)).$$

Since $Y - s\mathrm{Cl}_{\mathscr{J}}(V)$ is $R^{\star}$-$\mathscr{J}$-closed in Y and by (5), $F^{-}(Y - s\mathrm{Cl}_{\mathscr{J}}(V))$ is α-$\mathscr{I}$-closed in X. This shows that $F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))$ is α-$\mathscr{I}$-open in X and hence

$$x \in \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))).$$

Consequently, we obtain $F^{+}(V) \subseteq \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))$.

$(6) \Rightarrow (7)$: Let K be any $\star$-closed set of Y. Then $Y - K$ is $\star$-open and by (6), we have

$$\begin{aligned} X - F^{-}(K) &= F^{+}(Y - K) \\ &\subseteq \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(Y - K))) \\ &= \alpha\mathrm{Int}_{\mathscr{I}}(F^{+}(Y - s\mathrm{Int}_{\mathscr{J}}(K))) \\ &= \alpha\mathrm{Int}_{\mathscr{I}}(X - F^{-}(s\mathrm{Int}_{\mathscr{J}}(K))) \\ &= X - \alpha\mathrm{Cl}_{\mathscr{I}}(F^{-}(s\mathrm{Int}_{\mathscr{J}}(K))) \end{aligned}$$

and hence $\alpha\mathrm{Cl}_{\mathscr{I}}(F^{-}(s\mathrm{Int}_{\mathscr{J}}(K))) \subseteq F^{-}(K)$.

$(7) \Rightarrow (8)$: The proof is obvious since $s\mathrm{Int}_{\mathscr{J}}(K) = \mathrm{Cl}^{\star}(\mathrm{Int}(K))$ for every $\star$-closed set K of Y.

$(8) \Rightarrow (9)$: The proof is obvious.

$(9) \Rightarrow (10)$: It follows from Lemma 2.3(3) that $\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(B))) \subseteq \alpha\mathrm{Cl}_{\mathscr{J}}(B)$ for every subset B of Y. Thus, for every $\star$-closed set K of Y, we have

$$\begin{aligned} \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}(K)))))) &\subseteq \alpha\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}(K)))) \\ &= \alpha\mathrm{Cl}_{\mathscr{I}}(F^{-}(\mathrm{Cl}^{\star}(\mathrm{Int}(\mathrm{Cl}^{\star}(K))))) \\ &\subseteq F^{-}(\mathrm{Cl}^{\star}(K)) \\ &= F^{-}(K). \end{aligned}$$

$(10) \Rightarrow (11)$: The proof is obvious since $s\mathrm{Int}_{\mathscr{J}}(K) = \mathrm{Cl}^{\star}(\mathrm{Int}(K))$ for every $\star$-closed set K of Y.

$(11) \Rightarrow (12)$: Let V be any $\star$-open set of Y. Then $Y - V$ is $\star$-closed in Y and by (11), $\mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{-}(s\mathrm{Int}_{\mathscr{J}}(Y-V))))) \subseteq F^{-}(Y-V) = X - F^{+}(V)$. Moreover, we have

$$\begin{aligned} \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{-}(s\mathrm{Int}_{\mathscr{J}}(Y - V))))) &= \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(F^{-}(Y - s\mathrm{Cl}_{\mathscr{J}}(V))))) \\ &= \mathrm{Cl}(\mathrm{Int}^{\star}(\mathrm{Cl}(X - F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))))) \\ &= X - \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V))))). \end{aligned}$$

Consequently, we obtain $F^{+}(V) \subseteq \mathrm{Int}(\mathrm{Cl}^{\star}(\mathrm{Int}(F^{+}(s\mathrm{Cl}_{\mathscr{J}}(V)))))$.

$(12) \Rightarrow (1)$: Let $x \in X$ and V be any $\star$-open set of Y containing $F(x)$. By (12), we have $x \in F^+(V) \subseteq \text{Int}(\text{Cl}^\star(\text{Int}(F^+(s\text{Cl}_{\mathscr{J}}(V)))))$ and hence F is upper almost $\alpha(\star)$-continuous at x by Theorem 4.1. This shows that F is upper almost $\alpha(\star)$-continuous. □

Remark 4.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication hold:*

$$\textit{upper } \alpha(\star)\textit{-continuity} \Rightarrow \textit{upper almost } \alpha(\star)\textit{-continuity.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1, 2\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{3\}\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{c\}$ and $F(2) = \{a\}$ and $F(3) = \{a, b\}$. Then F is upper almost $\alpha(\star)$-continuous but F is not upper $\alpha(\star)$-continuous.*

Theorem 4.4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower almost $\alpha(\star)$-continuous;*
(2) *for each $x \in X$ and each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $U \subseteq F^-(sCl_{\mathscr{J}}(V))$;*
(3) *for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $U \subseteq F^-(V)$;*
(4) *$F^-(V)$ is α-$\mathscr{I}$-open in X for every $R^\star$-$\mathscr{J}$-open set V of Y;*
(5) *$F^+(K)$ is α-$\mathscr{I}$-closed in X for every $R^\star$-$\mathscr{J}$-closed set K of Y;*
(6) *$F^-(V) \subseteq \alpha Int_{\mathscr{I}}(F^-(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y;*
(7) *$\alpha Cl_{\mathscr{I}}(F^+(sInt_{\mathscr{J}}(K))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(8) *$\alpha Cl_{\mathscr{I}}(F^+(Cl^\star(Int(K)))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(9) *$\alpha Cl_{\mathscr{I}}(F^+(Cl^\star(Int(Cl^\star(B))))) \subseteq F^+(Cl^\star(B))$ for every subset B of Y;*
(10) *$Cl(Int^\star(Cl(F^+(Cl^\star(Int(K)))))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(11) *$Cl(Int^\star(Cl(F^+(sInt_{\mathscr{J}}(K))))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(12) *$F^-(V) \subseteq Int(Cl^\star(Int(F^-(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y.*

Proof. The proof is similar to that of Theorem 4.3. □

Definition 4.3. *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *$pre^{\star}_{\mathscr{I}}$-open* [9] *if $A \subseteq Int^{\star}(Cl(A))$;*
(2) *$pre^{\star}_{\mathscr{I}}$-closed* [9] *if its complement is $pre^{\star}_{\mathscr{I}}$-open;*
(3) *strong β-$\mathscr{I}$-open* [13] *if $A \subseteq Cl^{\star}(Int(Cl^{\star}(A)))$;*
(4) *strong β-$\mathscr{I}$-closed* [11] *if its complement is strong β-$\mathscr{I}$-open.*

Theorem 4.5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper almost $\alpha(\star)$-continuous;*
(2) *$\alpha Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^{\star}(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\alpha Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^{\star}(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$F^+(V) \subseteq \alpha Int_{\mathscr{I}}(F^+(Int^{\star}(Cl(V))))$ for every $pre^{\star}_{\mathscr{J}}$-open set V of Y.*

Proof. $(1) \Rightarrow (2)$: Let V be any strong β-$\mathscr{J}$-open set of Y. Then $\mathrm{Cl}^{\star}(V)$ is $R^{\star}$-$\mathscr{J}$-closed in Y. By (1) and Theorem 4.3, we have $F^-(\mathrm{Cl}^{\star}(V))$ is α-$\mathscr{I}$-closed in X and hence $\alpha\mathrm{Cl}_{\mathscr{I}}(F^-(V)) \subseteq \alpha\mathrm{Cl}_{\mathscr{I}}(F^-(\mathrm{Cl}^{\star}(V))) = F^-(\mathrm{Cl}^{\star}(V))$.

$(2) \Rightarrow (3)$: This is obvious since every semi-$\mathscr{J}$-open set is strong β-$\mathscr{J}$-open.

$(3) \Rightarrow (1)$: Let K be any $R^{\star}$-$\mathscr{J}$-closed set of Y. Then K is semi-$\mathscr{J}$-open in Y and by (3), we have $\alpha\mathrm{Cl}_{\mathscr{I}}(F^-(K)) \subseteq F^-(\mathrm{Cl}^{\star}(K)) = F^-(K)$. Therefore, $F^-(K)$ is α-$\mathscr{I}$-closed in X and hence F is upper almost $\alpha(\star)$-continuous by Theorem 4.3.

$(1) \Rightarrow (4)$: Let V be any $\mathrm{pre}^{\star}_{\mathscr{J}}$-open set of Y. Then, we have $\mathrm{Int}^{\star}(\mathrm{Cl}(V))$ is $R^{\star}$-$\mathscr{J}$-open in Y. By (1) and Theorem 4.3, $F^+(\mathrm{Int}^{\star}(\mathrm{Cl}(V)))$ is α-$\mathscr{I}$-open in X. Consequently, we obtain $F^+(V) \subseteq F^+(\mathrm{Int}^{\star}(\mathrm{Cl}(V))) = \alpha\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Int}^{\star}(\mathrm{Cl}(V))))$.

$(4) \Rightarrow (1)$: Let V be any $R^{\star}$-$\mathscr{J}$-open set of Y. Then V is $\mathrm{pre}^{\star}_{\mathscr{J}}$-open in Y and by (4), we have $F^+(V) \subseteq \alpha\mathrm{Int}_{\mathscr{I}}(F^+(\mathrm{Int}^{\star}(\mathrm{Cl}(V)))) = \alpha\mathrm{Int}_{\mathscr{I}}(F^+(V))$. This shows that $F^+(V)$ is α-$\mathscr{I}$-open in X. It follows from Theorem 4.3 that F is upper almost $\alpha(\star)$-continuous. □

Theorem 4.6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower almost $\alpha(\star)$-continuous;*
(2) *$\alpha Cl_{\mathscr{I}}(F^+(V)) \subseteq F^+(Cl^{\star}(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\alpha Cl_{\mathscr{I}}(F^+(V)) \subseteq F^+(Cl^{\star}(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$F^-(V) \subseteq \alpha Int_{\mathscr{I}}(F^-(Int^{\star}(Cl(V))))$ for every $pre^{\star}_{\mathscr{J}}$-open set V of Y.*

Proof. The proof is similar to that of Theorem 4.5. □

Definition 4.4. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be almost $\alpha(\star)$-continuous if $f^{-1}(V)$ is α-$\mathscr{I}$-open in X for every $R^\star$-$\mathscr{J}$-open set V of Y.*

Corollary 4.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\alpha(\star)$-continuous;*
(2) *for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq s_{\mathscr{J}}Cl(V)$;*
(3) *for each $x \in X$ and each $R^\star$-$\mathscr{J}$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq V$;*
(4) *for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq Int^\star(Cl(V))$;*
(5) *$f^{-1}(K)$ is α-$\mathscr{I}$-closed in X for every $R^\star$-$\mathscr{J}$-closed set K of Y;*
(6) *$f^{-1}(V) \subseteq \alpha Int_{\mathscr{I}}(f^{-1}(sCl_{\mathscr{J}}(V)))$ for every $\star$-open set V of Y;*
(7) *$\alpha Cl_{\mathscr{I}}(f^{-1}(sInt_{\mathscr{J}}(K))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(8) *$\alpha Cl_{\mathscr{I}}(f^{-1}(Cl^\star(Int(K)))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(9) *$\alpha Cl_{\mathscr{I}}(f^{-1}(Cl^\star(Int(Cl^\star(B))))) \subseteq f^{-1}(Cl^\star(B))$ for every subset B of Y;*
(10) *$Cl(Int^\star(Cl(f^{-1}(Cl^\star(Int(K)))))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(11) *$Cl(Int^\star(Cl(f^{-1}(sInt_{\mathscr{J}}(K))))) \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(12) *$f^{-1}(V) \subseteq Int(Cl^\star(Int(f^{-1}(sCl_{\mathscr{J}}(V)))))$ for every $\star$-open set V of Y.*

Corollary 4.2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\alpha(\star)$-continuous;*
(2) *$\alpha Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every strong β-$\mathscr{J}$-open set V of Y;*
(3) *$\alpha Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every semi-$\mathscr{J}$-open set V of Y;*
(4) *$f^{-1}(V) \subseteq \alpha Int_{\mathscr{I}}(f^{-1}(Int^\star(Cl(V))))$ for every $pre^\star_{\mathscr{J}}$-open set V of Y.*

5. On upper and lower weakly $\alpha(\star)$-continuous multifunctions

We begin this section by introducing the concepts of upper and lower weakly $\alpha(\star)$-continuous multifunctions.

Definition 5.1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) *upper weakly $\alpha(\star)$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \subseteq V$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq Cl^\star(V)$;*

(2) *lower weakly $\alpha(\star)$-continuous at a point $x \in X$ if for each $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(z) \cap Cl^\star(V) \neq \emptyset$ for every $z \in U$;*

(3) *upper (resp. lower) weakly $\alpha(\star)$-continuous if F is upper (resp. lower) weakly $\alpha(\star)$-continuous at each point of X.*

Theorem 5.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper weakly $\alpha(\star)$-continuous at $x \in X$;*

(2) *$x \in \alpha Int_{\mathscr{I}}(F^+(Cl^\star(V)))$ for every $\star$-open set V of Y containing $F(x)$;*

(3) *$x \in Int(Cl^\star(Int(F^+(Cl^\star(V)))))$ for every $\star$-open set V of Y containing $F(x)$.*

Proof. $(1) \Rightarrow (2)$: Let V be any $\star$-open set of Y containing $F(x)$. Then, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq \text{Cl}^\star(V)$; hence $U \subseteq F^+(\text{Cl}^\star(V))$. Consequently, we obtain $x \in \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V)))$.

$(2) \Rightarrow (3)$: Let V be any $\star$-open set of Y containing $F(x)$. Then by (2), we have $x \in \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V)))$ and by Lemma 2.3(4),

$$x \in \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(V))))).$$

$(3) \Rightarrow (1)$: Let V be any $\star$-open set of Y containing $F(x)$. By (3), we have

$$x \in \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(V)))))$$

and by Lemma 2.3(4), $x \in \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V)))$. Therefore, there exists an α-$\mathscr{I}$-open set U of X containing x such that $U \subseteq F^+(\text{Cl}^\star(V))$; hence $F(U) \subseteq \text{Cl}^\star(V)$. This shows that F is upper weakly $\alpha(\star)$-continuous at x. □

Theorem 5.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower weakly $\alpha(\star)$-continuous at $x \in X$;*

(2) *$x \in \alpha Int_{\mathscr{I}}(F^-(Cl^\star(V)))$ for every $\star$-open set V of Y such that*

$$F(x) \cap V \neq \emptyset;$$

(3) $x \in$ *Int*(*Cl**(*Int*(F^-(*Cl*$^\star(V)$)))) *for every $\star$-open set V of Y such that $F(x) \cap V \neq \emptyset$.*

Proof. The proof is similar to that of Theorem 5.1. □

Definition 5.2. [6] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *R-$\mathscr{I}^\star$-open if $A =$ Int*(Cl$^\star(A)$);*
(2) *R-$\mathscr{I}^\star$-closed if its complement is R-$\mathscr{I}^\star$-open.*

Definition 5.3. [5] *A point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a $\star_\theta$-cluster point of A if Cl$^\star(U) \cap A \neq \emptyset$ for every $\star$-open set U of X containing x. The set of all $\star_\theta$-cluster points of A is called the $\star_\theta$-closure of A and is denoted by $\star_\theta$Cl(A).*

Definition 5.4. [5] *A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:*

(1) *$\star_\theta$-closed if $\star_\theta$Cl$(A) = A$;*
(2) *$\star_\theta$-open if its complement is $\star_\theta$-closed.*

Lemma 5.1. [5] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *If A is $\star$-open in X, then Cl$^\star(A) = \star_\theta$Cl(A).*
(2) *$\star_\theta$Cl(A) is $\star$-closed in X.*

The following results give some characterizations of upper and lower weakly $\alpha(\star)$-continuous multifunctions.

Theorem 5.3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is upper weakly $\alpha(\star)$-continuous;*
(2) *$F^+(V) \subseteq$ Int(Cl*(Int(F^+(Cl$^\star(V)$)))) for every $\star$-open set V of Y;*
(3) *Cl(Int*(Cl(F^-(Int$^\star(K)$)))) $\subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(4) *$\alpha Cl_{\mathscr{I}}(F^-$(Int$^\star(K))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;*
(5) *$\alpha Cl_{\mathscr{I}}(F^-$(Int*(Cl$^\star(B)))) \subseteq F^-$(Cl$^\star(B)$) for every subset B of Y;*
(6) *F^+(Int$^\star(B)) \subseteq \alpha Int_{\mathscr{I}}(F^+$(Cl*(Int$^\star(B)$))) for every subset B of Y;*
(7) *$F^+(V) \subseteq \alpha Int_{\mathscr{I}}(F^+$(Cl$^\star(V)$)) for every $\star$-open set V of Y;*
(8) *$\alpha Cl_{\mathscr{I}}(F^-$(Int$^\star(K))) \subseteq F^-(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;*

(9) $\alpha Cl_{\mathscr{I}}(F^-(V)) \subseteq F^-(Cl^\star(V))$ *for every* $\star$*-open set* V *of* Y*;*
(10) $\alpha Cl_{\mathscr{I}}(F^-(Int^\star(\star_\theta Cl(B)))) \subseteq F^-(\star_\theta Cl(B))$ *for every subset* B *of* Y.

Proof. $(1) \Rightarrow (2)$: Let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$ and there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq \text{Cl}^\star(V)$; hence $U \subseteq F^+(\text{Cl}^\star(V))$ and so $x \in U \subseteq \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(V)))))$. This shows that $F^+(V) \subseteq \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(V)))))$.

$(2) \Rightarrow (3)$: Let K be any $\star$-closed set of Y. Then $Y - K$ is a $\star$-open set in Y and by (2), we have

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(Y - K))))) \\ &= \text{Int}(\text{Cl}^\star(\text{Int}(F^+(Y - \text{Int}^\star(K))))) \\ &= \text{Int}(\text{Cl}^\star(\text{Int}(X - F^-(\text{Int}^\star(K))))) \\ &= \text{Int}(\text{Cl}^\star(X - \text{Cl}(F^-(\text{Int}^\star(K))))) \\ &= \text{Int}(X - \text{Int}^\star(\text{Cl}(F^-(\text{Int}^\star(K))))) \\ &= X - \text{Cl}(\text{Int}^\star(\text{Cl}(F^-(\text{Int}^\star(K))))) \end{aligned}$$

and hence $\text{Cl}(\text{Int}^\star(\text{Cl}(F^-(\text{Int}^\star(K))))) \subseteq F^-(K)$.

$(3) \Rightarrow (4)$: Let K be any $\star$-closed set of Y. By (3), we have

$$\text{Cl}(\text{Int}^\star(\text{Cl}(F^-(\text{Int}^\star(K))))) \subseteq F^-(K)$$

and hence $\alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(K))) \subseteq F^-(K)$ by Lemma 2.3(3).

$(4) \Rightarrow (5)$: Let B be any subset of Y. Then $\text{Cl}^\star(B)$ is $\star$-closed in Y and by (4), we have $\alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(\text{Cl}^\star(B)))) \subseteq F^-(\text{Cl}^\star(B))$.

$(5) \Rightarrow (6)$: Let B be any subset of Y. By (5),

$$\begin{aligned} F^+(\text{Int}^\star(B)) &= X - F^-(\text{Cl}^\star(Y - B)) \\ &\subseteq X - \alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(\text{Cl}^\star(Y - B)))) \\ &= X - \alpha\text{Cl}_{\mathscr{I}}(F^-(Y - \text{Cl}^\star(\text{Int}^\star(B)))) \\ &= X - \alpha\text{Cl}_{\mathscr{I}}(X - F^+(\text{Cl}^\star(\text{Int}^\star(B)))) \\ &= \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(\text{Int}^\star(B)))). \end{aligned}$$

$(6) \Rightarrow (7)$: The proof is obvious.

$(7) \Rightarrow (1)$: Let $x \in X$ and V be any $\star$-open set of Y containing $F(x)$. It follows from Lemma 2.3(4) that

$$x \in F^+(V) \subseteq \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V))) \subseteq \text{Int}(\text{Cl}^\star(\text{Int}(F^+(\text{Cl}^\star(V)))))$$

and hence F is upper weakly $\alpha(\star)$-continuous at x by Theorem 5.1. This shows that F is upper weakly $\alpha(\star)$-continuous.

$(4) \Rightarrow (8)$: The proof is obvious.

$(8) \Rightarrow (9)$: Let V be any $\star$-open set of Y. Then, we have $\text{Cl}^\star(V)$ is R-$\mathscr{J}^\star$-closed in Y and by (8), $\alpha\text{Cl}_{\mathscr{I}}(F^-(V)) \subseteq \alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(\text{Cl}^\star(V)))) \subseteq F^-(\text{Cl}^\star(V))$.

$(9) \Rightarrow (7)$: Let V be any $\star$-open set of Y. By (9), we have

$$\begin{aligned} X - \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V))) &= \alpha\text{Cl}_{\mathscr{I}}(X - F^+(\text{Cl}^\star(V))) \\ &= \alpha\text{Cl}_{\mathscr{I}}(F^-(Y - \text{Cl}^\star(V))) \\ &\subseteq F^-(\text{Cl}^\star(Y - \text{Cl}^\star(V))) \\ &= X - F^+(\text{Int}^\star(\text{Cl}^\star(V))) \end{aligned}$$

and hence $F^+(V) \subseteq F^+(\text{Int}^\star(\text{Cl}^\star(V))) \subseteq \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V)))$.

$(9) \Rightarrow (10)$: Let B be any subset of Y. Then $\text{Int}^\star(\star_\theta\text{Cl}(B))$ is $\star$-open in Y. By (9) and Lemma 5.1(1),

$$\begin{aligned} \alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(\star_\theta\text{Cl}(B)))) &\subseteq F^-(\text{Cl}^\star(\text{Int}^\star(\star_\theta\text{Cl}(B)))) \\ &\subseteq F^-(\star_\theta\text{Cl}(\text{Int}^\star(\star_\theta\text{Cl}(B)))) \\ &\subseteq F^-(\star_\theta\text{Cl}(B)). \end{aligned}$$

$(10) \Rightarrow (8)$: Let K be any R-$\mathscr{J}^\star$-closed set of Y. Then by (10) and Lemma 5.1(1), we have

$$\begin{aligned} \alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(K))) &= \alpha\text{Cl}_{\mathscr{I}}(F^-(\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(K))))) \\ &= \alpha\text{Cl}_{\mathscr{I}}(\text{Int}^\star(\star_\theta\text{Cl}(\text{Int}^\star(K)))) \\ &\subseteq F^-(\star_\theta\text{Cl}(\text{Int}^\star(K))) \\ &= F^-(\text{Cl}^\star(\text{Int}^\star(V))) \\ &= F^-(K). \end{aligned}$$

□

Remark 5.1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication hold:*

upper almost $\alpha(\star)$-continuity $\Rightarrow$ upper weakly $\alpha(\star)$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 2. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, X\}$ and an ideal $\mathscr{I} = \{\emptyset\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{c\}\}$. Define $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ as follows: $F(1) = \{a\}$, $F(2) = \{b\}$ and $F(3) = \{a, c\}$. Then F is upper weakly $\alpha(\star)$-continuous but F is not upper almost $\alpha(\star)$-continuous.*

Theorem 5.4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is lower weakly $\alpha(\star)$-continuous;*
(2) *$F^-(V) \subseteq Int(Cl^\star(Int(F^-(Cl^\star(V)))))$ for every $\star$-open set V of Y;*
(3) *$Cl(Int^\star(Cl(F^+(Int^\star(K))))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(4) *$\alpha Cl_{\mathscr{I}}(F^+(Int^\star(K))) \subseteq F^+(K)$ for every $\star$-closed set K of Y;*
(5) *$\alpha Cl_{\mathscr{I}}(F^+(Int^\star(Cl^\star(B)))) \subseteq F^+(Cl^\star(B))$ for every subset B of Y;*
(6) *$F^-(Int^\star(B)) \subseteq \alpha Int_{\mathscr{I}}(F^-(Cl^\star(Int^\star(B))))$ for every subset B of Y;*
(7) *$F^-(V) \subseteq \alpha Int_{\mathscr{I}}(F^-(Cl^\star(V)))$ for every $\star$-open set V of Y;*
(8) *$\alpha Cl_{\mathscr{I}}(F^+(Int^\star(K))) \subseteq F^+(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;*
(9) *$\alpha Cl_{\mathscr{I}}(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set V of Y;*
(10) *$\alpha Cl_{\mathscr{I}}(F^+(Int^\star(\star_\theta Cl(B)))) \subseteq F^+(\star_\theta Cl(B))$ for every subset B of Y.*

Proof. The proof is similar to that of Theorem 5.3. □

Definition 5.5. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be weakly $\alpha(\star)$-continuous if for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}$-open set U of X containing x such that $f(U) \subseteq Cl^\star(V)$.*

Corollary 5.1. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\alpha(\star)$-continuous;*
(2) *$f^{-1}(V) \subseteq \alpha Int_{\mathscr{I}}(f^{-1}(Cl^\star(V)))$ for every $\star$-open set V of Y;*
(3) *$\alpha Cl_{\mathscr{I}}(f^{-1}(Int^\star(K))) \subseteq f^{-1}(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;*
(4) *$\alpha Cl_{\mathscr{I}}(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every $\star$-open set V of Y;*
(5) *$\alpha Cl_{\mathscr{I}}(f^{-1}(Int^\star(\star_\theta Cl(B)))) \subseteq f^{-1}(\star_\theta Cl(B))$ for every subset B of Y;*
(6) *$Cl(Int^\star(Cl(f^{-1}(V)))) \subseteq f^{-1}(Cl^\star(V))$ for every $\star$-open set V of Y;*

(7) $f^{-1}(V) \subseteq Int(Cl^{\star}(Int(f^{-1}(Cl^{\star}(V)))))$ *for every* $\star$*-open set* V *of* Y*;*
(8) $f(Cl(Int^{\star}(Cl(A)))) \subseteq \star_{\theta}Cl(f(A))$ *for every subset* A *of* X*;*
(9) $Cl(Int^{\star}(Cl(f^{-1}(B)))) \subseteq f^{-1}(\star_{\theta}Cl(B))$ *for every subset* B *of* Y*.*

Theorem 5.5. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *such that*

$$\alpha Int_{\mathscr{I}}(f^{-1}(Cl^{\star}(V))) \subseteq \alpha Int_{\mathscr{I}}(f^{-1}(V))$$

for every $\star$*-open set* V *of* Y*, the following properties are equivalent:*

(1) f *is* $\alpha(\star)$*-continuous;*
(2) f *is almost* $\alpha(\star)$*-continuous;*
(3) f *is weakly* $\alpha(\star)$*-continuous.*

Proof. We prove only the implication $(3) \Rightarrow (1)$. Suppose that f is weakly $\alpha(\star)$-continuous. Let V be any $\star$-open set of Y. Since f is weakly $\alpha(\star)$-continuous, by Corollary 5.1, we have $f^{-1}(V) \subseteq \alpha\text{Int}_{\mathscr{I}}(f^{-1}(\text{Cl}^{\star}(V)))$ and hence

$$f^{-1}(V) \subseteq \alpha\text{Int}_{\mathscr{I}}(f^{-1}(\text{Cl}^{\star}(V))) \subseteq \alpha\text{Int}_{\mathscr{I}}(f^{-1}(V)).$$

Thus, $f^{-1}(V)$ is α-$\mathscr{I}$-open in X. This shows that f is $\alpha(\star)$-continuous. □

Definition 5.6. *A subset* A *of an ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be:*

(1) $\star(\alpha)$*-open if* $A \subseteq Int^{\star}(Cl^{\star}(Int^{\star}(A)))$*;*
(2) $\star(\alpha)$*-closed if its complement is* $\star(\alpha)$*-open.*

Definition 5.7. *A subset* A *of an ideal topological space* $(X, \tau, \mathscr{I})$ *is said to be* $\star$*-clopen if* A *is both* $\star$*-open and* $\star$*-closed.*

Lemma 5.2. *Let* $(X, \tau, \mathscr{I})$ *be an ideal topological space. If* V *is both* $\star(\alpha)$*-open and* $\star(\alpha)$*-closed in* X*, then* V *is* $\star$*-clopen.*

Proof. Let V be $\star(\alpha)$-open and $\star(\alpha)$-closed. Then $V \subseteq \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(V)))$ and $\text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(V))) \subseteq V$. Thus, $\text{Cl}^{\star}(V) = \text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(V)))) = \text{Cl}^{\star}(\text{Int}^{\star}(V))$ and hence $\text{Cl}^{\star}(V) \subseteq \text{Cl}^{\star}(\text{Int}^{\star}(\text{Cl}^{\star}(V))) \subseteq V$. This shows that V is $\star$-closed. Since $V \subseteq \text{Int}^{\star}(\text{Cl}^{\star}(\text{Int}^{\star}(V))) \subseteq \text{Int}^{\star}(\text{Cl}^{\star}(V)) = \text{Int}^{\star}(V)$, we have V is $\star$-open. Consequently, we obtain V is $\star$-clopen. □

Lemma 5.3. *If a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is upper weakly* $\alpha(\star)$*-continuous and lower weakly* $\alpha(\star)$*-continuous, then* $F^{+}(V)$ *is* $\star$*-clopen in* X *for every* $\star$*-clopen set* V *of* Y*.*

Proof. Let V be any $\star$-clopen set of Y. It follows from Theorem 5.3 that $F^+(V) \subseteq \alpha\text{Int}_{\mathscr{I}}(F^+(\text{Cl}^\star(V))) = \alpha\text{Int}_{\mathscr{I}}(F^+(V))$. This shows that $F^+(V)$ is α-$\mathscr{I}$-open and so $F^+(V)$ is $\star(\alpha)$-open in X. Furthermore, since V is $\star$-open, it follows from Theorem 5.4 that $\alpha\text{Cl}_{\mathscr{I}}(F^+(V)) \subseteq F^+(\text{Cl}^\star(V)) = F^+(V)$. Thus, $F^+(V)$ is $\star(\alpha)$-closed and hence $F^+(V)$ is $(\star)\alpha$-closed. Therefore, it follows from Lemma 5.2, $F^+(V)$ is $\star$-clopen in X. □

Definition 5.8. [6] *An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-$\mathscr{I}$-connected if X cannot be written as the union of two nonempty disjoint $\star$-open sets.*

Theorem 5.6. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is an upper $\alpha(\star)$-continuous and lower $\alpha(\star)$-continuous surjective multifunction such that $F(x)$ is $\star$-$\mathscr{J}$-connected for each $x \in X$ and $(X, \tau, \mathscr{I})$ is $\star$-$\mathscr{I}$-connected, then $(Y, \sigma, \mathscr{J})$ is $\star$-$\mathscr{J}$-connected.*

Proof. Suppose that $(Y, \sigma, \mathscr{J})$ is not $\star$-$\mathscr{J}$-connected. There exist nonempty $\star$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\star$-$\mathscr{J}$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and so $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Consequently, we obtain $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$, $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$, $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$. By Lemma 5.3, $F^+(U)$ and $F^+(V)$ are $\star$-clopen. Therefore, $(X, \tau, \mathscr{I})$ is not $\star$-$\mathscr{I}$-connected. □

Theorem 5.7. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is a $\star$-regular $\star$-paracompact set for each $x \in X$, the following properties are equivalent:*

(1) *F is upper $\alpha(\star)$-continuous;*
(2) *F is upper almost $\alpha(\star)$-continuous;*
(3) *F is upper weakly $\alpha(\star)$-continuous.*

Proof. We prove only the implication $(3) \Rightarrow (1)$. Suppose that F is upper weakly $\alpha(\star)$-continuous. Let $x \in X$ and V be any $\star$-open set of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\star$-regular $\star$-paracompact, by Lemma 3.1, there exists a $\star$-open set G such that $F(x) \subseteq G \subseteq \text{Cl}(G) \subseteq V$. Since F is upper weakly $\alpha(\star)$-continuous, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(U) \subseteq \text{Cl}^\star(G)$ and hence $F(U) \subseteq \text{Cl}^\star(G) \subseteq \text{Cl}(G) \subseteq V$. This shows that F is upper $\alpha(\star)$-continuous. □

Theorem 5.8. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that $F(x)$ is $\star$-open in Y for each $x \in X$, the following properties are equivalent:*

(1) *F is lower $\alpha(\star)$-continuous;*
(2) *F is lower almost $\alpha(\star)$-continuous;*
(3) *F is lower weakly $\alpha(\star)$-continuous.*

Proof. We prove the only implication $(3) \Rightarrow (1)$. Suppose that F is lower weakly $\alpha(\star)$-continuous. Let $x \in X$ and V be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. Since F is lower weakly $\alpha(\star)$-continuous, there exists an α-$\mathscr{I}$-open set U of X containing x such that $F(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$ for each $z \in U$. Since $F(z)$ is $\star$-open, we have $F(z) \cap V \neq \emptyset$ for each $z \in U$ and hence F is lower $\alpha(\star)$-continuous. □

6. Conclusion

The concepts of openness and continuity are fundamental with respect to the investigation of general topology. The study of openness and continuity have been found to be useful in computer science and digital topology. Continuity of functions and multifunctions in topological spaces have been researched by many mathematicians. Several investigations related to open sets and various forms of continuity types have been introduced. This article is dealing with the concepts of upper and lower $\alpha(\star)$-continuous multifunctions. Some characterizations of upper and lower $\alpha(\star)$-continuous multifunctions are obtained. Moreover, the relationships between upper and lower $\alpha(\star)$-continuous multifunctions and the other types of continuity for multifunctions are explored. The ideas and results of this article may motivate further research.

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