Advances in Mathematics: Scientific Journal **9** (2020), no.3, 909–919
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.3.16

EQUITABLE COLORING OF DERIVED GRAPH OF GRAPHS

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ABSTRACT. A proper vertex coloring of the graph is assigning colors to the vertices such that the adjacent vertices are assigned different colors. A proper vertex coloring is said to be equitable if the cardinality of each color classes differ in size by at most one. The derived graph of a simple graph G, is the graph having the same set of vertices as G in which two vertices are adjacent if and only if the corresponding edges are adjacent in G. In this paper we discuss the equitable chromatic number of the derived graph of path, cycle, wheel graph, helm graph, sunlet graph, friendship graph and tadpole graph.

1. INTRODUCTION

All graphs considered in this paper are simple, undirected graphs. One of the most important types of graph coloring is vertex coloring [3] in which no two adjacent vertices are assigned the same color. The least number of colors used for such a coloring is called chromatic number. The set of vertices with the same color is termed as color class. If the coloring is assigned to the graph in such a way that the number of vertices in any two color classes differ in size by atmost one, then such a coloring is called equitable coloring. The notion of equitable coloring was introduced by Meyer in 1973 [7]. The most important practical application of this equitable coloring is dividing a system with binary conflict relations into equal or almost equal subsystems [1].

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2010 *Mathematics Subject Classification.* 05C15, 05C76.
Key words and phrases. equitable coloring, derived graph, path, cycle, wheel graph, helm graph, sunlet graph, friendship graph and tadpole graph.

One of the most interesting operation applied to the graphs in graph theory is the derived graph. On applying this operation, a new graph is obtained from a graph. The derived graph concept has been rediscovered by many names: line graph [8], adjoint [6], interchange graph [9], edge-to-vertex dual [10] etc., The derived graph of a graph is the one in which two vertices are adjacent iff their distance in the original graph is two. The distance between two vertices is the length of the shortest path between those two vertices [2]. The n^{th} derived graph of a graph G is the graph in which two vertices are adjacent iff their distance is n in G. The $1st$ derived graph is the graph itself. If mentioned as derived graph we mean as 2^{nd} derived graph.

2. PRELIMINARIES

A graph G is said to be equitably k-colorable if its vertices can be partitioned into k classes $V_1, V_2, \dots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every (i, j). The smallest integer k for which G is equitably k-colorable is known as the equitable chromatic number of G and denoted by $\chi_=(G)$, see [1]. Let G be a simple graph with vertex set $V(G)$. The derived graph of G, see [11], is the graph with the same vertex set $V(G)$, in which two vertices are adjacent if and only if their distance in G is two and it is denoted by G^+. The derived graph G^+ of a graph G, see [5], is also defined as that graph having the edges of G as its vertices, with two vertices being adjacent if and only if the corresponding edges are adjacent in G. For any integer $m \geq 4$, the wheel graph W_m is the m vertex graph obtained by joining a vertex v to each of the $m-1$ vertices $\{v_1, v_2, \dots v_{m-1}\}$ of the cycle graph C_{m-1}. The m sunlet graph on $2m$ vertices is obtained by attaching m pendant edges to the cycle C_m and is denoted by S_m, see [4]. The graph obtained from an m-wheel graph by adjoining a pendant edge at each node of the cycle is termed as the Helm graph H_m, see [4]. The graph obtained by joining a cycle C_m to a path P_n with a bridge is called a (m, n)-tadpole graph and is denoted by $T_{m,n}$.

In this paper, we determine the equitable chromatic number of derived graph of path $\chi_=(G^+(P_n))$, cycle $\chi_=(G^+(C_m))$, wheel graph $\chi_=(G^+(W_m))$, helm graph $\chi_=(G^+(H_m))$, sunlet graph $\chi_=(G^+(S_m))$, friendship graph $\chi_=(G^+(F_m))$,tadpole graph $\chi_=(G^+(T_{m,n}))$.

3. Equitable coloring of derived graph of Path

Theorem 3.1. *The Equitable chromatic number of derived graph of path $G^+(P_m)$, where m is any positive integer and $m \geq 3$ is:*

$$\chi_=(G^+(P_m)) = \begin{cases} 3 & \text{if } m = 2i,\ i = 3, 5, 7, 9, \ldots \\ 2 & \text{otherwise}. \end{cases}$$

Proof. Let $V(P_m) = V(G^+(P_m)) = \{v_1, v_2, v_3, \cdots, v_m\}$. By the definition of derived graph, the adjacent vertices with distance two in P_m are adjacent in $G^+(P_m)$. Hence in the derived graph of P_m, each v_i is adjacent to $v_{i+2}(1 \leq i \leq m-2)$ and each v_i is non-adjacent with $v_{i+1}(1 \leq i \leq m-1)$. Since each v_i and v_{i+2} $(1 \leq i \leq m-2)$ are adjacent, a minimum of two colors must be used. The assigning of colors is done by partitioning the color classes into the following two cases:

Case 1: When m is even:

$$\begin{aligned} V_1 &= \{v_1, v_2, v_5, v_6, \cdots, v_{m-3}, v_{m-2}\} \\ V_2 &= \{v_3, v_4, v_7, v_8, \cdots, v_{m-1}, v_m\} \\ |V_1| &= |V_2| = \frac{m}{2}. \end{aligned}$$

Case 2: When m is odd:

$$\begin{aligned} V_1 &= \{v_1, v_2, v_5, v_6, v_9, v_{10}, \cdots, \} \\ V_2 &= \{v_3, v_4, v_7, v_8, v_{11}, v_{12}, \cdots, \}. \end{aligned}$$

Here $|V_1| - |V_2|$ is either 0 or 1.

Equitable coloring is satisfied on both the cases.

The above partitioning does not hold good when $m = 2i, i = 3, 5, 7, 9, \ldots$

In this case equitability is not satisfied when two colors are used. Hence an additional color 3 is used. Colors $1, 2, 3$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m$ such that the equitability is satisfied. □

4. Equitable coloring of derived graph of Cycle

Theorem 4.1. *The Equitable chromatic number of derived graph of cycle $G^+(C_m)$, where $'m'$ is any positive integer and $m \geq 4$ is*

$$\chi_=(G^+(C_m)) = \begin{cases} 2 & \textit{if } m \textit{ is even} \\ 3 & \textit{if } m \textit{ is odd}. \end{cases}$$

Proof. Let $V(C_m) = V(G^+(C_m)) = \{v_i : 1 \leq i \leq m\}$. The neighbouring vertices of the cycle graph are non-adjacent in its derived graph. Since each vertex is adjacent to atleast one vertex in the derived graph of cycle, a minimum of two colors is needed. The assigning of colors is done by the following two cases:

Case 1: When m is even.

The partition of $V(G^+(C(P_m))$ is as follows:

$$\begin{aligned} V_1 &= \{v_1, v_2, v_5, v_6, \cdots, v_{m-3}, v_{m-2}\} \\ V_2 &= \{v_3, v_4, v_7, v_8, \cdots, v_{m-1}, v_m\} \\ |V_1| &= |V_2| = \frac{m}{2}. \end{aligned}$$

Thus a maximum of two colors is sufficient satisfying equitability.

Case 2: When m is odd.

If the above partitioning is done when the number of vertices is odd, proper vertex coloring is not satisfied because the adjacent vertices v_2 and v_m will have the same color. So an additional color is used. The assigning of colors is done by three ways:

(i) $m \ mod \ 3 = 0$.

Colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $\{v_1, v_2, v_3, \cdots, v_m\}$ such that each color appears $\frac{m}{3}$ times satisfying equitability.

(ii) $m \ mod \ 3 = 1$.

Colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $\{v_1, v_2, v_3, \cdots, v_{m-1}\}$ and the vertex v_m is assigned the color 1. The cardinality of each of the color classes is either equal or differs by one satisfying equitability.

(iii) $m \ mod \ 3 = 2$.

Colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $\{v_1, v_2, v_3, \cdots, v_{m-2}\}$. The vertices v_{m-1} and v_m are assigned the colors 3 and 1, respectively. The cardinality of each of the color classes differ in size by atmost one satisfying equitability. Thus, when the number of vertices is odd, equatability is satisfied with three colors.

□

5. Equitable coloring of derived graph of Wheel graph

Theorem 5.1. *The Equitable chromatic number of derived graph of wheel graph* $(G^+(W_m))$, $m \geq 5$ *and* m *being any positive integer is*

$$\chi_=(G^+(W_m) = \begin{cases} 2 & \textit{if } m-1 \textit{ is divisible by } 4 \\ 3 & \textit{otherwise} \end{cases}.$$

Proof. Let $V(W_m) = V(G^+(W_m)) = \{v\} \cup \{v_i : 1 \leq i \leq m-1\}$. The vertices with degree two in the wheel graph are the adjacent vertices in its derived graph. Since each vertex is adjacent to atleast one vertex, a minimum of two colors are required. The assigning of colors is done by the following cases:

Case 1: When $(m-1)$ is divisible by 4.

Since $(m-1)$ is divisible by 4, the colors $\{1, 1, 2, 2\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-1$ such that both the colors appears $\frac{m-1}{2}$ times each. Since v is an isolated vertex, color 1 or 2 can be assigned which satisfies the equitability. Hence, $\chi_=(G^+(W_m) = 2$.

Case 2: When $(m-1)$ is not divisible by 4.

In this case, proper vertex coloring is not satisfied when two colors are used. Hence an additional color is used. The assigning of colors is done by the following three ways:

(i) $(m-1)\ mod\ 3 = 0$.

The colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-1$ such that each color is assigned $\frac{m-1}{3}$ times. The isolated vertex v is assigned color 1. Thus color 1 is assigned $\frac{m-1}{3} + 1$ times whereas colors 2 and 3 are assigned $\frac{m-1}{3}$ times.

(ii) $(m-1)\ mod\ 3 = 1$.

The colors $\{1,2,3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-2$ such that each color is assigned $\frac{m-2}{3}$ times. The vertex v_{m-1} is assigned color 3 satisfying proper vertex coloring and the isolated vertex v is assigned color 1. Thus colors 1 and 3 are assigned $\frac{m-2}{3}+1$ times whereas colors 2 is assigned $\frac{m-2}{3}$ times.
(iii) $(m-1)\ mod\ 3 = 2$.

The colors $\{1,2,3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-3$ such that each color is assigned $\frac{m-3}{3}$ times. The remaining two vertices v_{m-2} and v_{m-1} are assigned colors 3 and 1, respectively and the isolated vertex v is assigned color 2. Thus all the colors are assigned $\frac{m}{3}$ times. Equitable coloring is thus satisfied in (i), (ii) and (iii).

Therefore, $\chi_{=}(G^{+}(W_m) = 3$.

□

6. Equitable coloring of Derived graph of Helm graph

Theorem 6.1. *The Equitable chromatic number of derived graph of helm graph* $(G^{+}(H_m))$*,* $m > 3$ *and* $'m'$ *being any positive integer is*

$$\chi_{=}(G^{+}(H_m)) = 4\,.$$

Proof. Let $V(H_m) = V(G^{+}(H_m)) = \{v\} \cup \{v_i : 1 \leq i \leq m\} \cup \{u_i : 1 \leq i \leq m\}$. By the definition of derived graph, each vertex with degree two in the helm graph is adjacent in its derived graph.

The vertex v is adjacent to each of the vertices $u_i : 1 \leq i \leq m$. Since a clique of order 3 is formed for each u_i, a minimum of 3 colors is required. To satisfy equitability an additional color 4 is used. The assigning of colors is done by the following cases:

Case 1: $m\ mod\ 4 = 0$.

Let the vertex v be assigned color 1. The vertices $v_i : 1 \leq i \leq m$ are assigned the colors $\{1,1,2,2\}$ consecutively in the same order such that the colors 1 and 2 appears $\frac{m}{2}$ times each. Similarly the vertices $u_i : 1 \leq i \leq m$ are assigned the colors $\{3,3,4,4\}$ consecutively in the same order such that both the colors 3 and 4 are assigned $\frac{m}{2}$ times each. Thus, color 1 is assigned $\frac{m}{2}+1$ times and the colors 2,3,4 are assigned $\frac{m}{2}$ times each.

Case 2: $m \ mod \ 4 = 1$.

Let the vertex v be assigned color 1. The colors $\{1,1,2,2\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \le i \le m-1$ such that each color appears $\frac{m-1}{2}$ times and the vertex v_m is assigned color 3 satisfying proper vertex coloring. Since u_1 forms a clique with v_2 *and* v_m, colors 1 and 3 are forbidden. Hence we assign color 2 to the vertex u_1. Now the remaining $m-1$ vertices are assigned colors 3 and 4 equally. Color 3 is assigned to the vertices $\left\{u_2, u_3, \cdots, u_{\frac{m-1}{2}+1}\right\}$ and color 4 is assigned to the vertices $\left\{u_{\frac{m-1}{2}+2}, u_{\frac{m-1}{2}+3}, \cdots, u_m\right\}$ each appearing $\frac{m-1}{2}$ times. Thus, colors 1,2,3 are assigned $\frac{m+1}{2}$ times each and color 4 is assigned $\frac{m-1}{2}$ times.

Case 3: $m \ mod \ 4 = 2$.

Let the vertex v be assigned color 1. The colors $\{1,1,2,2\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \le i \le m-2$. The remaining two vertices v_{m-1} and v_m are both assigned color 3. The vertex u_1 is assigned color 2. The remaining $m-1$ vertices are assigned colors 3 and 4 equally. Color 3 is assigned to the vertices $\left\{u_2, u_3, \cdots, u_{\frac{m}{2}}\right\}$ and color 4 is assigned to the vertices $\left\{u_{\frac{m}{2}+1}, u_{\frac{m}{2}+2}, \cdots, u_m\right\}$.

Thus colors 1,2,4 are assigned $\frac{m}{2}$ times each and color 3 is assigned $\frac{m}{2}+1$ times.

Case 4: $m \ mod \ 4 = 3$.

The vertex v is assigned color 1. The vertices $\{v_1, v_2, \cdots, v_{m-3}\}$ are assigned the colors $\{1,1,2,2\}$ consecutively in the same order. The remaining three vertices v_{m-2}, v_{m-1} *and* v_m are assigned color 1,3,3, respectively. Color 2 is assigned to the vertex u_1. The vertices $\left\{u_2, u_3, \cdots, u_{\frac{m-1}{2}}\right\}$ are assigned color 3 and the vertices $\left\{u_{\frac{m-1}{2}+1}, u_{\frac{m-1}{2}+2}, \cdots, u_m\right\}$ are assigned color 4. Thus, colors 1,3,4 are assigned $\dfrac{m+1}{2}$ times each and color 2 is assigned $\frac{m-1}{2}$ times which satisfies equitable coloring. Thus, in all the above cases the color classes differ in size by atmost one satisfying equitability.

Therefore, $\chi_{=}(G^{+}(H_m) = 4$. □

7. Equitable coloring of Derived graph of Sunlet graph

Theorem 7.1. *The Equitable chromatic number of derived graph of sunlet graph* $(G^{+}(S_m))$, $m \geq 3$ *and* m *being any positive integer is*

$$\chi_{=}(G^{+}(S_m)) = 3\,.$$

Proof. Let $V(S_m) = V(G^{+}(S_m)) = \{v_i : 1 \leq i \leq m\} \cup \{u_i : 1 \leq i \leq m\}$ where $v_i' s$ are the vertices of the cycle and $u_i' s$ are the vertices of the pendant edges attached to v_i. Each $u_i' s$ form a clique of order 3 with the vertices of v_i. Hence a minimum of 3 colors is needed.

The assigning of colors to the vertices $v_i : 1 \leq i \leq m$ is done as follows:

(i) If $m \ mod \ 3 = 0$, the colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m$.

(ii) If $m \ mod \ 3 = 1$, the colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-1$ and the vertex v_m is assigned color 1.

(iii) If $m \ mod \ 3 = 2$, the colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $v_i : 1 \leq i \leq m-2$. The vertices v_{m-1} and v_m are assigned colors 3 and 1, respectively. Each u_i $(2 \leq i \leq m-2)$ forms a clique with the vertices v_i and v_{i+2} $(1 \leq i \leq m-2)$. The vertex u_m forms a clique with v_{m-1} and v_1. The vertex u_1 forms a clique with v_m and v_2. Each u_i $(1 \leq i \leq m)$ is assigned a color from the set $\{1, 2, 3\}$ satisfying the clique and proper vertex coloring. The above assigning of colors is done in such a way that the color classes differ in size by atmost one which satisfies equitable coloring.

Therefore, $\chi_{=}(G^{+}(S_m) = 3$. □

8. Equitable coloring of Derived graph of Tadpole graph

Theorem 8.1. *The Equitable chromatic number of derived graph of Tadpole graph* $(G^+(T_{m,n}))$*,* $m > 3$ *and* m *being any positive integer is*

$$\chi_=((G^+(T_{m,n})) = 3\,.$$

Proof. Let $V(T_{m,n}) = V(G^+(T_{m,n})) = \{v_i : 1 \le i \le m\} \cup \{p_j : 1 \le j \le n\}$ where $v_i' s$ are the vertices of the cycle and $p_j' s$ are the vertices of the path. Let the path be connected to the cycle by the vertex v_1.

By the definition of derived graph, the vertices v_2, p_1 and v_m are adjacent to each other and hence forms a clique. Thus a minimum of three colors are required.

Let the assigning of colors be done by the following three cases:

Case 1: $m\ mod\ 3 = 0\,.$

Let the colors $\{1, 2, 3\}$ be assigned consecutively in the same order to the vertices $\{v_i : 1 \le i \le m\}$ such that each of the three colors are assigned $\frac{m}{3}$ times. Now to the vertices $\{p_j : 1 \le j \le n\}$, the colors $\{1, 2, 3\}$ are assigned consecutively in the same order such that the size is either 0 or 1 for each color class.

Case 2: $m\ mod\ 3 = 1\,.$

The colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $\{v_i : 1 \le i \le m - 1\}$ and the vertex v_m is assigned color 1. Thus the colors 2 and 3 are assigned $\dfrac{m-1}{3}$ times and color 1 is assigned $\dfrac{m-1}{3} + 1$ times. The vertices $\{p_j : 1 \le j \le n\}$ are assigned the colors $\{3, 2, 1\}$ consecutively in the same order. Thus the three color classes differ in size by atmost one.

Case 3: $m\ mod\ 3 = 2\,.$

The colors $\{1, 2, 3\}$ are assigned consecutively in the same order to the vertices $\{v_i : 1 \le i \le m - 2\}$. The vertices $v_m - 1$ and v_m are assigned the colors 3 and 2, respectively. Now for the vertices p_1 and p_2 the colors 3 and 2 are assigned, respectively. For the remaining vertices $\{p_j : 3 \le j \le n\}$ the colors $\{2, 1, 3\}$ are assigned consecutively in the same order such that the three color classes differ in size by atmost one. Thus, in all the three cases equitable 3-coloring is satisfied.

Therefore, $\chi_=((G^+(T_{m,n})) = 3$. □

9. EQUITABLE COLORING OF DERIVED GRAPH OF FRIENDSHIP GRAPH

Theorem 9.1. *The Equitable chromatic number of derived graph of friendship graph* $(G^+(F_m))$*,* $m \geq 2$ *and* $'m'$ *being any positive integer is*

$$\chi_=(G^+(F_m)) = m\,.$$

Proof. Let $V(F_m) = V(G^+(F_m)) = \{v\} \cup \{v_i : 1 \leq i \leq 2m\}$ where v is the common vertex and $v_i : 1 \leq i \leq 2m$ are the outer vertices.

By the definition of derived graph, two vertices are adjacent if their distance is two in the original graph. Since v is a common vertex, each vertices $v_i : 1 \leq i \leq 2m - 1$ is adjacent to all other vertices except $v_{i+1} : 1 \leq i \leq 2m - 1$. Thus the only non-adjacent pair of vertices are $v_i \; and \; v_{i+1} (1 \leq i \leq 2m - 1)$. Hence these two vertices can be assigned the same color. Since the total number of outer vertices is $2m$ and two adjacent vertices are given the same color, the total number of colors required are $\frac{2m}{2} = m$. The common vertex v is assigned color 1. Thus, colors $2, 3, \cdots, m$ appears two times each and color 1 appears three times satisfying equitability.

Therefore, $\chi_=(G^+(F_m)) = m$. □

ACKNOWLEDGMENT

We are gratefully acknowledged for a wonderful support from the CGANT - University of Jember, Indonesia and Kongunadu Arts and Science College, Coimbatore, India.

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