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rr-SEQUENCE PRODUCT GRAPHS

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ABSTRACT. Let $G(V(G), E(G))$ be a connected graph. A radial radio labeling, f, of G is an assignment of positive integers to the vertices such that $d(u, v) + |f(u) - f(v)| \geq 1 + rad(G)$, for any two distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ and $rad(G)$ denote the distance between u and v and the radius of the graph G, respectively. The span of a radial radio labeling f is the largest integer in the range of f and is denoted by $span\ f$. The radial radio number of G, $rr(G)$, is the minimum span taken over all radial radio labelings of G. The sequence $(\mu_1(v))_{v \in V(G)}$ arranged in decreasing order is called the $(\mu_1(v)) - rr$ sequence of G, where $\mu_1(v)$ is the radial radio number of the induced subgraph induced by the neighbors of v together with v in G. In this paper, we determine the $(\mu_1(v)) - rr$ sequence of the vertex corona, edge corona, neighborhood corona and Cartesian product of any two simple connected graphs. Also, we determine the $(\mu_1(v)) - rr$ sequence of the $k-$ subdivision of a graph.

1. INTRODUCTION

In this paper, by a graph, we mean only a simple, connected, undirected and finite graph. For basic notations and terminology, we follow [3]. Let $G = (V(G), E(G))$ be a simple connected graph. Let $deg_G(v)$ denote the degree of a vertex v in G. The *distance* between two vertices u and v, the *eccentricity* of a vertex u, the *radius* of G and the *diameter* of G are respectively denoted

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by $d(u,v)$, $e(u)$, $rad(G)$ and $diam(G)$. For more notations and terminology on distance in graphs, one can refer [4].

For a subset S of $V(G)$, let $< S >$ denote the induced subgraph of G induced by S. A *clique* C is a subset of $V(G)$ with maximum number of vertices such that $< C >$ is complete. The *clique number* of a graph G, denoted by $\omega(G)$ or ω, is the number of vertices in a clique of G.

Let G_1 be a graph with n_1 vertices and m_1 edges and let G_2 be a graph with n_2 vertices and m_2 edges. Then $G_1 \times G_2$ denotes the *Cartesian product* of G_1 and G_2 and $G_1 + G_2$ denotes the *join* of G_1 and G_2. The *vertex corona*, $G_1 \circ G_2$, is the graph obtained from one copy of G_1 and n_1 copies of G_2 by joining each vertex in the $i-th$ copy of G_2 to the $i-th$ vertex of G_1. The *edge corona*, $G_1 \diamond G_2$, of G_1 and G_2 is obtained by taking one copy of G_1 and m_1 copies of G_2 and then joining two end vertices of the $i-th$ edge to every vertex in the $i-th$ copy of G_2. The *neighborhood corona*, $G_1 \star G_2$ is the graph obtained by taking n_1 copies of G_2 and for each i, making all vertices in the $i-$th copy of G_2 adjacent with the neighbors of v_i. The $k-$ *subdivision of a graph* G is obtained by subdividing each edge of G, k times and is denoted by $S_k(G)$.

For any vertex $v \in V(G)$, the *neighborhood* $N(v)$ is the set of all vertices adjacent to v. Denote $N_\lambda[v] = \{w : d(v,w) \leq \lambda\}$. Then $N_0[v] = \{v\}$. For simplicity, we write $N_1[v]$ as $N[v]$ or $N_G[v]$.

In 1960's Rosa[8] introduced the concept of graph labeling. A *graph labeling* is an assignment of numbers to the vertices or edges or both, satisfying some constraints. Rosa named the labeling introduced by him as $\beta - valuation$ and later on it becomes a very famous interesting graph labeling called *graceful labeling*, which is the origin for many graph labeling problems. Motivated by lot many real life applications, mathematicians introduced various labeling concepts[6].

The problem of assigning frequencies to the channels for the FM radio stations is known as *Frequency Assignment Problem* . This problem was studied by W. K. Hale[7]. In a telecommunication system, the assignment of channels to FM radio stations play a vital role. Motivated by the frequency assignment problem, Chartrand et al.[5] introduced the concept of radio labeling. For a given k, $1 \leq k \leq diam(G)$, a *radio k- coloring*, f, is an assignment of positive integers to the vertices satisfying the following condition: $d(u,v) + |f(u) - f(v)| \geq 1 + k$, for any two distinct vertices $u, v \in V(G)$.

Whenever, $diam(G) = k$, the radio k- coloring is called a *radio labeling* of G. The *span* of a radio labeling f is the largest integer in the range of f and is denoted by $span\ f$. The *radio number* of G is the minimum span taken over all radio labelings of G and is denoted by $rn(G)$.

In a similar way, K. M. Kathiresan and S. Vimalajenifer introduced the concept of radial radio labeling. A *radial radio labeling* f of G is a function $f : V(G) \to \{1, 2, ...\}$ satisfying the condition, $d(u, v) + |f(u) - f(v)| \geq 1 + rad(G)$, for any two distinct vertices $u, v \in V(G)$. This condition is obtained by taking $k = rad(G)$, in the radio $k-$ coloring. The *span* of a radial radio labeling f is the largest integer in the range of f and is denoted by $span\ f$. The *radial radio number* is the minimum span taken over all radial radio labelings of G and is denoted by $rr(G)$. That is, $rr(G) = \min_f \max_{v \in V} f(v)$, where the minimum runs over all radial radio labelings of G. A radial radio labeling, f of G, is said to be an $rr(G)-$*radial radio labeling* if $span\ f = rr(G)$.

Let f be a radial radio labeling of a graph G and let C be a clique in G. Then the minimum label in C under f is denoted by $m_f(C)$. That is, $m_f(C) = \min_{v \in C} f(v)$. Also the maximum of all such $m_f(C)$, where the maximum runs over all cliques in G is denoted by $Cmm_f(G)$.

The concept $(\mu_\lambda(v))-rr$ sequence of a graph, has been introduced by Selvam Avadayappan et al., see [1]. For a vertex $v \in V(G)$, let $\mu_\lambda(v)$ denote the radial radio number of the induced subgraph induced by $N_\lambda[v]$. That is, $\mu_\lambda(v) = rr(< N_\lambda[v] >)$. Then the sequence $(\mu_\lambda(v))_{v \in V(G)}$ arranged in decreasing order is called the $(\mu_\lambda(v)) - rr$ *sequence* of the graph G.

A graph G is said to be $\lambda - rr$ *regular* if the $(\mu_1(v)) - rr$ sequence of G is $(\lambda, \lambda, ..., \lambda)$, that is, $rr(< N[u] >) = \lambda$, for any vertex u in G. In this case, λ is called the $rr-$ *constant* of G. A $\lambda - rr$ regular graph G is said to be $\lambda - rr$ *highly regular* if $rr(G) = \lambda$. A graph G on n vertices is said to be rr *irregular*, if it is not $\lambda - rr$ regular, for any λ, $2 \leq \lambda \leq n$. For further details on the regularity of the $(\mu_1(v)) - rr$ sequence of graph, one can refer [1] and [2]. Throughout this paper, let $\delta^1_{rr}(G) = \min_{v \in V(G)} \mu_1(v)$ and $\Delta^1_{rr}(G) = \max_{v \in V(G)} \mu_1(v)$.

Now, we state the following results which have been proved earlier.

Lemma 1.1. *Let G be a simple connected graph on n vertices with radius $rad(G)$. Then $\chi(G) \leq rr(G) \leq (n - 1)(rad(G) - 1) + \chi(G)$.*

Theorem 1.1. *[2] Let f be a radial radio labeling of a graph G. Then $span\ f \geq Cmm_f(G) + (\omega(G) - 1)(rad(G))$.*

In this paper, we present some facts on the radial radio number and the $(\mu_1(v)) - rr$ sequence of G and we determine the $(\mu_1(v)) - rr$ sequence of the $k-$subdivision of G, where $k \geq 1$. Also, we establish the $(\mu_1(v)) - rr$ sequences of $G \times H$, $G \circ H$, $G \diamond H$ and $G \star H$.

2. Some Basic Results

From Lemma 1.1, we can easily observe that, for any simple connected graph G, $rr(G) = \chi(G)$ if and only if $rad(G) = 1$. Therefore, if $rad(G) \geq 2$, then $rr(G) > \chi(G)$.

Fact 1. *If H is a subgraph of G with $rad(G) = rad(H) = 1$, then $rr(H) \leq rr(G)$.*

For, if f is a $rr(G)-$radial radio labeling of G, then for any two distinct vertices $u, v \in V(G)$, we have $|f(u) - f(v)| \geq 2 - d_G(u,v) \geq 2 - d_H(u,v)$ (since $d_H(u,v) \geq d_G(u,v)$) and so f is a radial radio labeling of H. Also, for any $rr(H)-$radial radio labeling g of H, we have $g(V(H)) \subseteq f(V(G))$. Thus $span\ g \leq span\ f$ and hence $rr(H) \leq rr(G)$.

Fact 2. *If $rad(G) = 1$, then $\Delta^1_{rr}(G) = \chi(G)$.*

For, let v be a full vertex of G. Then $rr(< N[v] >) = rr(G) = \chi(G)$, since $rad(G) = 1$. By Fact 2.1, we have $rr(< N[u] >) \leq rr(G) = \chi(G)$, for all $u \in V(G)$ so that $\Delta^1_{rr}(G) = \chi(G)$.

Remark 2.1. *The converse of Fact 2 is not true. For, consider the graph shown in Figure 1.*

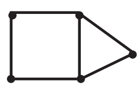

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FIGURE 1

It is clear that, the $(\mu_1(v)) - rr$ sequence of G is $(3,3,3,2,2)$ and hence $\Delta^1_{rr}(G) = 3$. But $rad(G) \neq 1$.

Note 1. *Theorem 1.1 insists that, if* $rad(G) \geq 2$*, then* $rr(G) > \omega(G)$*. For, let* f *be a radial radio labeling of* G*. Then* $span\ f \geq Cmm_f(G) + (\omega(G) - 1)(rad(G)) \geq 1 + (\omega(G) - 1)2 = 2\omega(G) - 1 > \omega(G)$*.*

Fact 3. *If* G *is* $\lambda - rr$ *highly regular, then* $rad(G) = 1$*.*

For, since G is $\lambda - rr$ highly regular, $rr(G) = rr(< N[u] >) = \lambda$ for all $u \in V(G)$. We now claim that, $rad(G) = 1$. On contrary, assume that, $rad(G) \geq 2$. Then by Note 1, $rr(G) > \omega(G)$. Since $\omega(G)$ is the clique number of G, there exists a vertex $v \in K_{\omega(G)}$ such that $rr(< N[v] >) = \omega(G)$. This implies that, $rr(G) \neq rr(< N[v] >)$, which is a contradiction. Hence $rad(G) = 1$.

Remark 2.2. *The converse of Fact 3 is not true. For, consider the graph shown in Figure 2.*

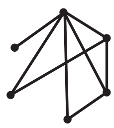

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FIGURE 2

Here, the $(\mu_1(v)) - rr$ *sequence of* G *is* $(4, 4, 4, 4, 3, 2)$ *and* $rad(G) = 1$*.*

3. $(\mu_1(v)) - rr$ SEQUENCE OF PRODUCT OF GRAPHS

The following Theorem gives the radial radio number of the induced subgraph induced by $N_{G\times H}[(u, x)]$ in the Cartesian product of G and H.

Theorem 3.1. *The radial radio number of the induced subgraph induced by* $N_{G\times H}[(u, x)]$*, where* $u \in V(G)$ *and* $x \in V(H)$ *is* $rr(< N_{G\times H}[(u, x)] >) = \max\{rr(< N_G[u] >), rr(< N_H[x] >)\}$*.*

Proof. Here, $N_{G\times H}[(u, x)] = N_G(u) \bigcup N_H(x) \bigcup \{u = x\}$ and
$E(< N_{G\times}[(u, x)] >) = E(< N_G[u] >) \bigcup E(< N_H[x] >)$. We have, for any radial radio labeling, f of $< N_{G\times}[(u, x)] >$, the restricted functions $f|_{N_G[u]}$ and $f|_{N_H[x]}$ are the radial radio labelings of $< N_G[u] >$ and $< N_H[x] >$, respectively. This forces that, $rr(< N_{G\times H}[(u, x)] >) = \max\{rr(< N_G[u] >), rr(< N_H[x] >)\}$. $\square$

Next Theorem gives the $(\mu_1(v)) - rr$ sequence of the $k-$subdivision graph of G.

Theorem 3.2. *Let G be any simple connected graph on n vertices and m edges. Let $(\lambda_1, \lambda_2, ..., \lambda_n)$ be the $(\mu_1(v))-rr$ sequence of G. Then the $(\mu_1(v))-rr$ sequence of the $k-$ subdivision graph of G, $S_k(G)$ is $(\underbrace{2, 2, 2, ..., 2}_{n+mk\ times})$, $k \geq 1$.*

Proof. Here, $|V(S_k(G))| = n + km$ and $|E(S_k(G))| = (k+1)m$. We note that, for any vertex $u \in V(G)$, $< N_{S_k(G)}[u] > \cong K_{1,t}$, where $t = |N_{S_k(G)}(u)|$ and $t \geq 1$. Since $rr(K_{1,t}) = 2$, $rr(< N_{S_k(G)}[u] >) = 2$, for all $u \in V(G)$. Also, we have $< N_1[v] > \cong K_{1,2}$ and so $rr(< N_1[v] >) = rr(K_{1,2}) = 2$, for every $v \in V(S_k(G)) - V(G)$. This gives that , the $(\mu_1(v)) - rr$ sequence of $S_k(G)$ is $(\underbrace{2, 2, 2, ..., 2}_{n+mk\ times})$. □

Fact 4. *If G is a connected graph on n vertices, then $rr(G + K_m) = \chi(G) + m$, where $m \geq 1$.*

For, since $rad(G + K_m) = 1$, $rr(G + K_m) = \chi(G + K_m) = \chi(G) + m$.

Fact 5. *If $rad(G) = 1$, then $rr(G + K_m) = rr(G) + m$, where $m \geq 1$.*

For, since $rad(G) = 1$, $rr(G) = \chi(G)$. By Fact 4, $rr(G + K_m) = \chi(G) + m = rr(G) + m$.

To prove the next two theorems, let us take G be a graph on n_1 vertices and m_1 edges and let H be a graph on n_2 vertices and m_2 edges. Assume that, $(\lambda'_1, \lambda'_2, ..., \lambda'_{n_1})$ and $(\lambda''_1, \lambda''_2, ..., \lambda''_{n_2})$ are the $(\mu_1(v)) - rr$ sequences of G and H, respectively.

Theorem 3.3. *The $(\mu_1(v)) - rr$ sequence of $G \circ H$ is given as follows:*

(1) *If $\chi(H) + 1 \geq \lambda'_1$, then*
$$(\underbrace{\chi(H) + 1, \chi(H) + 1, ..., \chi(H) + 1}_{n_1\ times}, \underbrace{\lambda''_i + 1, \lambda''_i + 1, ..., \lambda''_i + 1}_{n_1\ times})_{i=1}^{n_1}.$$

(2) *If $\lambda'_k > \chi(H) + 1$, for some k, where $1 \leq k \leq n_1$, then*
$$(\lambda'_1, \lambda'_2, ..., \lambda'_k, \underbrace{\chi(H) + 1, \chi(H) + 1, ..., \chi(H) + 1}_{n_1-k\ times}, \underbrace{\lambda''_i + 1, \lambda''_i + 1, ..., \lambda''_i + 1}_{n_1\ times})_{i=1}^{n_1}.$$

Proof. Let $V(G) = \{v_1, v_2, ..., v_{n_1}\}$ such that $rr(< N_G[v_i] >) = \lambda'_i$, where $1 \leq i \leq n_1$. Let $H_j \cong H$, $1 \leq j \leq n_1$ and let $V(H_j) = \{u_i^j : 1 \leq i \leq n_2\}$, $1 \leq j \leq n_1$. We have, $rr(< N_{H_j}[u_i^j] >) = \lambda''_i$.
Then $V(G \circ H) = V(G) \bigcup \{\bigcup_{j=1}^{n_1} V(H_j)\}$ and
$E(G \circ H) = E(G) \bigcup \{\bigcup_{j=1}^{n_1} E(H_j)\} \bigcup \{v_i u_j^i : 1 \leq i \leq n_1 \; and \; 1 \leq j \leq n_2\}$. Now, we shall find the $(\mu_1(v)) - rr$ sequence of $G \circ H$.

Case 1: Consider the vertex u_i^j, where $1 \leq i \leq n_2$ and $1 \leq j \leq n_1$.

Here $< N_{G \circ H}[u_i^j] > \cong < N_H[u_i^j] > +K_1$, for all i, j, where $1 \leq i \leq n_2$ and $1 \leq j \leq n_1$. By Fact 5, we have $rr(< N_H[u_i^j] > +K_1) = rr(< N_H[u_i^j] >) + 1 = \lambda''_i + 1$. Therefore, $rr(< N_{G \circ H}[u_i^j] >) = \lambda''_i + 1$, for all i, j, where $1 \leq i \leq n_2$ and $1 \leq j \leq n_1$.

Case 2: Consider the vertex v_i, where $1 \leq i \leq n_1$.

We have $N_{G \circ H}[v_i] = N_G[v_i] \bigcup V(H_i)$. Since $d_{< N_{G \circ H}[v_i] >}(u, u_i^j) = 2$, for every $u \in N_G(v_i)$, we can reuse the labels of the vertices of $N_G(v_i)$ to the vertices of H_i. Also, any radial radio labeling of $< N_{G \circ H}[v_i] >$ is a radial radio labeling of $< N_G[v_i] >$ and $H_i + K_1$. Thus $rr(< N_{G \circ H}[v_i] >) = \max\{rr(< N_G[v_i] >), rr(H + K_1)\} = \max\{\lambda'_i, \chi(H) + 1\}$. From the above two cases, we obtain the required result. □

Theorem 3.4. *The $(\mu_1(v)) - rr$ sequence of $G \diamond H$ is given as follows:*

(1) *If $\chi(H) + 2 \geq \lambda'_1$, then*
$$(\underbrace{\chi(H) + 2, \chi(H) + 2, ..., \chi(H) + 2}_{n_1 \; times}, \underbrace{\lambda''_i + 2, \lambda''_i + 2, ..., \lambda''_i + 2}_{m_1 \; times})_{i=1}^{n_2}.$$

(2) *If $\lambda'_k > \chi(H) + 2$, for some k, where $1 \leq k \leq n_1$, then*
$$(\lambda'_1, \lambda'_2, ..., \lambda'_k, \underbrace{\chi(H) + 2, \chi(H) + 2, ..., \chi(H) + 2}_{n_1 - k \; times},$$
$$\underbrace{\lambda''_i + 2, \lambda''_i + 2, ..., \lambda''_i + 2}_{n_1 \; times})_{i=1}^{n_2}.$$

Proof. Let $V(G) = \{v_1, v_2, ..., v_{n_1}\}$ such that $rr(< N_G[v_i] >) = \lambda'_i$, where $1 \leq i \leq n_1$ and let $E(G) = \{e_1, e_2, ..., e_{m_1}\}$. For each j, $1 \leq j \leq n_2$ let $H_j \cong H$. Let $V(H_j) = \{u_i^j : 1 \leq i \leq n_2\}$, $1 \leq j \leq n_1$. We have $rr(< N_{H_j}[u_i^j] >) = \lambda''_i$, where $1 \leq j \leq m_1$ and $1 \leq i \leq n_2$.
Then $V(G \diamond H) = V(G) \bigcup \{\bigcup_{j=1}^{n_1} V(H_j)\}$. Now, we shall find the $(\mu_1(v)) - rr$

sequence of $G \diamond H$.

Case 1: Consider the vertex u_i^j, where $1 \leq i \leq n_2$ and $1 \leq j \leq m_1$.

Here, we have $< N_{G\diamond H}[u_i^j] > \cong < N_H[u_i^j] > +K_2$. By Fact 5, $rr(< N_{G\diamond H}[u_i^j] >) = rr(< N_H[u_i^j] > +K_2) = rr(< N_H[u_i^j] >)+2 = \lambda_i'' + 2$. Thus $rr(< N_{G\diamond H}[u_i^j] >) = \lambda_i'' + 2$, for all i, j, where $1 \leq i \leq n_2$ and $1 \leq j \leq m_1$.

Case 2: Consider the vertex v_i, where $1 \leq i \leq n_1$.

We note that, $< N_{G\diamond H}[v_i] >$ contains $deg_G(v_i)$ copies of H and the distance between any two vertices of two copies of H is 2. Therefore, we can reuse the labels of the vertices of H_i to H_j, where $1 \leq i \neq j \leq k$. Also, any radial radio labeling of $< N_{G\diamond H}[v_i] >$ is a radial radio labeling of $< N_G[v_i] >$ and $H_i + K_2$. Thus $rr(< N_{G\diamond H}[v_i] >) = \max\{rr(< N_G[v_i] >), rr(H_i + K_2)\} = \max\{\lambda_i', \chi(H) + 2\}$,(by Fact 4). This completes the proof. □

The following two theorems deal with the radial radio numbers of $N_{G\star H}[v]$, for all $v \in V(G \star H)$.

Theorem 3.5. *If* $rr(< N_G[v] >) = \lambda$*, then* $rr(< N_{G\star H}[v] >) = \lambda + \chi(H) - 1$.

Proof. Let $N_G(v) = \{w_1, w_2, ..., w_{deg_G(v)}\}$. In $G \star H$, the induced subgraph induced by $N_{G\star H}[v]$ contains $deg_G(v)$ copies of H, say $H_1, H_2, ..., H_{deg_G(v)}$. Let $V(H_i) = \{u_j^i : 1 \leq j \leq n_2\}$, where $1 \leq i \leq deg_G(v)$. Then $u_j^i v \notin E(G \star H)$, for all i, j, where $1 \leq i \leq deg_G(v)$ and $1 \leq j \leq n_2$. Here, $N_{G\star H}[v] = N_G[v] \bigcup \{ \bigcup_{i=1}^{deg_G(v)} V(H_i)\}$ and $rad(< N_{G\star H}[v] >) = 1$. Let f be a $rr(< N_G[v] >)-$ radial radio labeling of $< N_G[v] >$ such that
$rr(< N_G[v] >) = \max_{1\leq i\leq deg_G(v)} \{f(w_i)\}$ and let g be the proper vertex coloring of H.

Define $h : N_{G\star H}[v] \to \{1, 2, 3, ...\}$ such that $h(N_G[v]) = f(N_G[v])$; $h(u_i^j) = f(w_i) + g(u_j) - 1$, where $1 \leq i \leq deg_G(v)$ and $1 \leq j \leq n_2$. It is obvious that, h is a radial radio labeling of $< N_{G\star H}[v] >$. Also, $span\ h = \max_{1\leq i\leq deg_G(v)} f(w_i) + \max_{1\leq j\leq n_2} g(u_j^i) - 1 = span\ f + \chi(H) - 1 = \lambda + \chi(H) - 1$. Thus $span\ h = \lambda + \chi(H) - 1$ and hence $rr(< N_{G\star H}[v] >) \leq \lambda + \chi(H) - 1$. We now claim that, $rr(< N_{G\star H} >) \geq \lambda + \chi(H) - 1$. Let c be any radial radio labeling of $< N_{G\star H}[v] >$. Since $rr(< N_G[v] >) = \lambda$, $\max_{u\in N_G(v)} c(u) \geq \lambda$. We note that, $H + K_1$ is an induced subgraph of $< N_{G\star H}[v] >$ and $rad(< N_{G\star H}[v] >) = 1$ and so we need $\chi(H)$ positive integers to label the vertices of each copy of H. Since $d_{<N_{G\star H}[v]>}(w_i, u_j^i) = 2$, we can use the label of w_i to some of the vertices in the set $\{u_j^i : 1 \leq j \leq n_2\}$. This gives that, to label the vertices of H_i, we need $c(v_i)$ to $c(v_i) + \chi(H) - 1$ positive integers. Also, since $d_{<N_{G\star H}[v]>}(u_j^i, u_k^i) = 2$, for all j, k, where $1 \leq j, k \leq n_2$, the labels of $V(H_j)$ can be reused to the vertices of $V(H_k)$. This forces that, $\max_{1\leq j\leq n_2} c(u_j^i) \geq \max_{u\in N_G(v)} c(u) + \chi(H) - 1 = \lambda + \chi(H) - 1$ and hence $span\ c \geq \lambda + \chi(H) - 1$. Thus $rr(< N_{G\star H}[v] >) \geq \lambda + \chi(H) - 1$. From the above discussion, we conclude that, $rr(< N_{G\star H}[v] >) = \lambda + \chi(H) - 1$. □

Theorem 3.6. *If* $rr(< N_H[u] >) = \lambda''$, *then* $rr(< N_{G\star H}[u] >) = \lambda' + \lambda'' - 1$, *where* $\lambda' = rr(< N_G[v] >)$ *and* $uw \in E(G \star H)$, *for all* $w \in N_G(v)$.

Proof. Let $N_G(v) = \{w_1, w_2, ..., w_{deg_G(v)}\}$.
Then $N_{G\star H}[u] = N_H[u] \bigcup \{w_1, w_2, ..., w_{deg_G(v)}\}$ and $uw_i \in E(G \star H)$, for all i, where $1 \leq i \leq deg_G(v)$. Here $rad(< N_{G\star H}[u] >) = 1$. Let f be a $\lambda'-$ radial radio labeling of $rr(< N_G[v] >)$ such that $\lambda' = \max_{1\leq i\leq deg_G(v)} f(w_i)$ and let g be the $\lambda''-$radial radio labeling of $< N_H[u] >$.

Define $h : N_{G\star H}[u] \to \{1, 2, 3, ...\}$ such that $h(N_H[u]) = g(N_H[u])$; $h(w_i) = \lambda'' + f(w_i) - 1$, where $1 \leq i \leq deg_G(v)$. We can easily verify that, h is a radial radio labeling of $< N_{G\star H}[u] >$. Also, $span\ h = \max_{1\leq i\leq deg_G(v)} \{\lambda'' + f(w_i) - 1\} = \lambda'' + \lambda' - 1$. This forces that, $rr(< N_{G\star H}[u] >) \leq \lambda'' + \lambda' - 1$. It is enough to show that, $rr(< N_{G\star H}[u] >) \geq \lambda'' + \lambda' - 1$. Since $v \notin N_{G\star H}[u]$ and $d_{<N_{G\star H}[u]>}(w_i, w_j) = d_{<N_G[v]>}(w_i, w_j)$, for all i, j, where $1 \leq i \neq j \leq deg_G(v)$, $\lambda' - 1$ positive integers are enough to label the vertices of $< N_G[v] >$. Also $rad(< N_{G\star H}[u] >) = rad(< N_H[u] >)$, we need λ'' positive integers to label the vertices of $< N_H[u] >$. We note that, $d_{<N_{G\star H}[u]>}(x, w_j) = 1$, for all $x \in N_H[u]$ and w_j, where $1 \leq j \leq deg_G(v)$ and so we cannot reuse the labels form the set $\{1, 2, 3, ..., \lambda''\}$ to the vertices of $N_G(v)$. This gives that, for any radial radio labeling of c of $< N_{G\star H}[u] >$, $\max_{x\in N_{G\star H}[u]} c(x) \geq \lambda'' + \lambda' - 1$ and hence $rr(< N_{G\star H}[u] >) \geq \lambda' + \lambda'' - 1$. This completes the proof. □

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