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SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATION USING A NEW MODIFIED ADOMIAN METHOD

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ABSTRACT. A new modification on Adomian Decomposition Method (ADM) is presented in this paper to solve second order ordinary differential equation. The efficiency of this method is demonstrated by many examples.

1. INTRODUCTION

Consider the following problem:

$$y'' + p(x)y' + p'(x)y = h(x, y), \tag{1.1}$$

with

$$y(0) = A, y'(0) = B\,,$$

where A and B are constants, $p(x)$ is given functions and $h(x, y)$ is a real function. In 1980s the Adomian method [1,2] was introduced by Adomian George. The quality and easyness of the method, as well as the good results provided by this method, attracted the attention of many researchers, and many adjustments were made, for example in [5–8]. In this paper a new contribution on solving second order differential equation is done. We present a differential operator with ability to solve some type of equation.

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2. Preliminaries

The Adomian decomposition method usually defines the equations with an operator form by considering the highest order derivative in the problem. We can write equation (1.1) as following:

$$Ly = h(x, y), \tag{2.1}$$

where

$$L(.) = \frac{d}{dx} e^{-\int P(x)dx} \frac{d}{dx} e^{\int P(x)dx}(.),$$

and

$$L^{-1}(.) = e^{-\int P(x)dx} \int_0^x e^{\int P(x)dx} \int_0^x (.)dxdx.$$

By applying L^{-1} on (2.1), we have

$$y(x) = \delta(x) + L^{-1}(h(x, y), \tag{2.2}$$

such that

$$L(\delta(x)) = 0.$$

The method by Adomian gives the solution $y(x)$ and the function $h(x, y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{2.3}$$

and

$$h(x, y) = \sum_{n=0}^{\infty} A_n, \tag{2.4}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen [3, 4] to formulate Adomian polynomials. The following algorithm:

$$A_0 = H(y_0),$$

$$A_1 = y_1 H'(y_0),$$

$$A_2 = y_2 H'(y_0) + \frac{1}{2!} y_1^2 H'' y_0),$$

$$A_3 = y_3 H'(y_0) + y_1 y_2 H''(y_0) + \frac{1}{3!} y_1^3 H''' y_0), \ldots \tag{2.5}$$

can be used to build Adomian polynomials, when $H(y)$ is given function. From (2.3), (2.4) and (2.2) we have

$$\sum_{n=0}^{\infty} y_n(x) = \delta(x) + L^{-1}\sum_{n=0}^{\infty} A_n.$$

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$y_0 = \delta(x),$$

$$y_{(n+1)} = L^{-1}A_n, \; n \geq 0, \tag{2.6}$$

thus

$$y_0 = \delta(x),$$
$$y_1 = L^{-1}A_0,$$
$$y_2 = L^{-1}A_1,$$
$$y_3 = L^{-1}A_2, \ldots \tag{2.7}$$

Using the equations (2.5) and (2.7) we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (2.3) can be immediately obtained. For numerical purposes, the n-term approximate

$$\zeta_n = \sum_{k=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

3. Applications

In this section, we applied a new modified on three examples when $p(x)$ had three different functions.

Problem 1. Consider the following problem:

$$y'' + xy' + y = (2+x)e^x + e^{2x} - y^2, \tag{3.1}$$

$$y(0) = 1, y'(0) = 1,$$

where $p(x) = x$, in equation (1.1) and $y(x) = e^x$, the solution of equation (3.1). We put

$$L(.) = \frac{d}{dx} e^{-\frac{x^2}{2}} \frac{d}{dx} e^{\frac{x^2}{2}} (.),$$

and

$$L^{-1}(.) = e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}} \int_0^x (.). \tag{3.2}$$

Rewrite equation (3.1) in an operator form

$$Ly = (2+x)e^x + e^{2x} - y^2, \tag{3.3}$$

applying equation (3.2) on both side of equation (3.3) we get:

$$y(x) = y(0)e^{-\frac{x^2}{2}} + y'(0)e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}} dx + L^{-1}((2+x)e^x + e^{2x}) - L^{-1}(y^2),$$

$$y_0 = e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}} dx + L^{-1}((2+x)e^x + e^{2x}),$$

when we use Taylor series of $e^{-\frac{x^2}{2}}$ and $e^{\frac{x^2}{2}}$ with order 10 and eq.(2.6), we obtain

$$y_0 = 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^6}{60} + \frac{11\,x^7}{1680} - \frac{x^8}{3360} - \frac{127\,x^9}{362880} + \frac{187\,x^{10}}{1814400} + ...,$$

$$y_1 = \frac{-x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{12} - \frac{37\,x^6}{720} - \frac{41\,x^7}{2520} - \frac{11\,x^8}{5760} - \frac{41\,x^9}{181440} - \frac{253\,x^{10}}{518400} + ...,$$

$$y_2 = \frac{x^4}{12} + \frac{x^5}{12} + \frac{x^6}{20} + \frac{13\,x^7}{504} + \frac{211\,x^8}{20160} + \frac{239\,x^9}{60480} + \frac{1727\,x^{10}}{907200} + ...$$

The series solution by (ADM) is given by

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{11\,x^6}{720} + \frac{9\,x^7}{560} + \\ &\quad + \frac{37\,x^8}{4480} + \frac{35\,x^9}{10368} + \frac{1837\,x^{10}}{1209600} + \dots \end{aligned}$$

TABLE 1. Comparison of exact solution and ADM solution for example 1

x	Exact	ADM	Absolute error
0.0	0.00000	0.00000	0.00000
0.2	1.22140	1.22140	0.00000
0.4	1.49182	1.49191	0.00009
0.6	1.82212	1.82339	0.00127
0.8	2.22554	2.23451	0.00897
1.0	2.71828	2.76117	0.04289

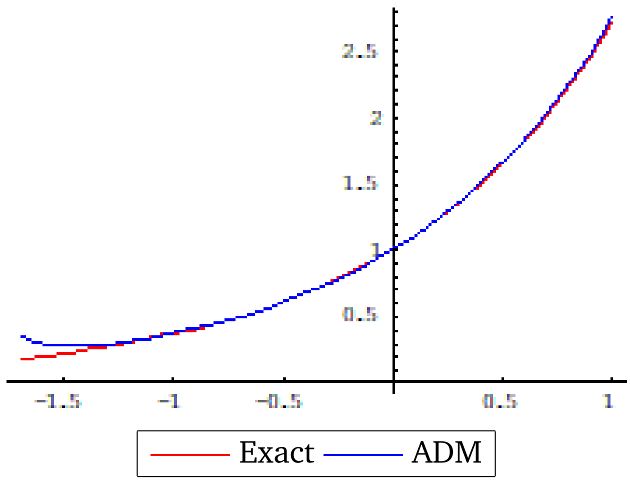

FIGURE 1. The exact solution $y = e^x$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

Problem 2. Consider the following problem:

$$y'' + \frac{1}{1+x}y' - \frac{1}{(1+x)^2}y = \frac{2+6x+3x^2}{(1+x)^2}, \tag{3.4}$$

$$y(0) = 0, y'(0) = 0,$$

where $p(x) = \dfrac{1}{1+x}$, in equation (1.1) and $y(x) = x^2$, is the solution of equation (3.4). We put

$$L(.) = \frac{d}{dx}\frac{1}{(1+x)}\frac{d}{dx}(1+x)(.),$$

so

$$L^{-1}(.)=\frac{1}{(1+x)}\int_0^x(1+x)\int_0^x(.). \tag{3.5}$$

Rewrite equation (3.4) in an operator form

$$Ly=\frac{2+6\,x+3\,x^2}{(1+x)^2}, \tag{3.6}$$

applying equation (3.5) on both side of equation (3.6) we get:

$$y(x)=L^{-1}(\frac{2+6\,x+3\,x^2}{(1+x)^2}),$$

$$y(x)=x^2.$$

In above example, we got the exact solution.

Problem 3. Consider the following problem:

$$y''+e^x y'+e^x y=e^{x^3}-e^y+6\,x+e^x\,x^2\,\left(3+x\right), \tag{3.7}$$

$$y(0)=0, y'(0)=0,$$

where $p(x)=e^x$, in equation (1.1) and $y(x)=x^3$, the solution of equation (3.7). We put

$$L(.)=\frac{d}{dx}e^{-e^x}\frac{d}{dx}e^{e^x}(.),$$

so

$$L^{-1}(.)=e^{-e^x}\int_0^x e^{e^x}\int_0^x(.). \tag{3.8}$$

Rewrite equation (3.7) in an operator form

$$Ly=e^{x^3}-e^y+6\,x+e^x\,x^2\,\left(3+x\right). \tag{3.9}$$

Applying equation (3.8) on both side of equation (3.9) we get:

$$y(x)=L^{-1}(e^{x^3}+6\,x+e^x\,x^2\,\left(3+x\right))-L^{-1}(e^y),$$

$$y_0=L^{-1}(e^{x^3}+6\,x+e^x\,x^2\,\left(3+x\right)),$$

by using Taylor series of e^{-e^x} and e^{e^x} with order 10 and Adomain polynomials we obtain

$$y_0=\frac{x^2}{2}+\frac{5\,x^3}{6}-\frac{x^4}{12}+\frac{x^5}{20}+\frac{x^6}{180}-\frac{x^7}{1008}+\frac{59\,x^8}{8064}-\frac{107\,x^9}{72576}-\frac{683\,x^{10}}{907200}+...,$$

$$y_1=\frac{-x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{30}-\frac{x^6}{360}-\frac{5\,x^7}{504}-\frac{x^8}{288}+\frac{71\,x^9}{60480}-\frac{289\,x^{10}}{181440}+...,$$

$$y_2 = \frac{x^4}{24} - \frac{x^5}{60} + \frac{x^6}{360} + \frac{13\,x^7}{1680} - \frac{x^8}{280} + \frac{17\,x^9}{6720} + \frac{1867\,x^{10}}{1814400} + ...,$$

The series solution by (ADM) is given by

$$y(x) = y_0 + y_1 + y_2 = x^3 + \frac{x^6}{180} - \frac{x^7}{315} + \frac{11\,x^8}{40320} + \frac{809\,x^9}{362880} - \frac{2389\,x^{10}}{1814400} + ...$$

As we note some terms disappear if we continue finding y, we will get the exact solution $y(x) = x^3$.

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