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ADOMIAN DECOMPOSITION METHOD FOR SOLVING OSCILLATORY SYSTEMS OF HIGHER ORDER

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ABSTRACT. Adomian Decomposition techniques is used in this paper to solve non-linear oscillatory systems of higher order for initial value problem. We have given some examples that illustrate the reliability of this method.

1. INTRODUCTION

Many physical phenomena are described in terms of differential equations including linear and non-linear equations. In this paper, we want to convergent the exact solution of the non-linear oscillatory systems by using the Adomian decomposition Method (ADM). ADM is clear and accurate. In addition, it is easy with respect to the numerical applications. This method is discovered by George Adomian in 1980s [1]. In recent years, lots and lots of researchers have applied this method to solve non-linear systems [5–7] . In [4] Yahya Qaid Hasan and Liu Ming Zhu studied non-linear oscillatory systems of second order, and they obtained results of a good efficiency by utilizing this method. The main aim of this paper is to find approximate solutions by this method. We propose a modern differential operator which would enable us to have accurate solutions for solving non-linear oscillatory systems.

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2. Buliding Oscillatory Systems Types

We present a new differential operator as follows:

$$L(.) = e^{-bx}\frac{d}{dx}e^{(b-a)x}\frac{d^m}{dx^m}e^{ax}(.), \tag{2.1}$$

where $m \geq 1.$

When we put $m = 1,$ in equation (2.1), we get a first type of oscillatory systems

$$y'' + (a+b)y' + aby = f(x,y),$$

when we put $m = 2,$ in equation (2.1), we get a second type of oscillatory systems

$$y''' + (2a+b)y'' + (2ab+a^2)y' + a^2by = f(x,y).$$

We continue with this procedure and finally we have:

$$y^{(m+1)} + \sum_{n=0}^{m} a^n b\binom{m}{n} y^{(m-n)} + \sum_{n=1}^{m} a^n\binom{m}{n} y^{(m-n+1)} = f(x,y).$$

3. Adomian decomposition technique

We consider the differential equation as below:

$$y^{(m+1)} + \sum_{n=0}^{m} a^n b\binom{m}{n} y^{(m-n)} + \sum_{n=1}^{m} a^n\binom{m}{n} y^{(m-n+1)} = f(x,y). \tag{3.1}$$

with initial conditions

$$y(0) = a_0,\ y'(0) = a_1,\ y''(0) = a_2, ...\ y^{(m)}(0) = a_i\,,$$

where $i = 0, 1, 2, 3, ...$ The differential operator L for equation (3.1) as below:

$$Ly = f(x,y) \tag{3.2}$$

where

$$L(.) = e^{-bx}\frac{d}{dx}e^{(b-a)x}\frac{d^m}{dx^m}e^{ax}(.),$$

and the invers operator L^{-1} is:

$$L^{-1}(.) = e^{-ax}\underbrace{\int_0^x\int_0^x\int_0^x ... \int_0^x}_{m} e^{(a-b)x}\int_0^x e^{bx}(.)\underbrace{dxdxdx...dxdx}_{(m+1)}.$$

By taken L^{-1} on equation (3.2), we have:

$$y(x) = \delta(x) + L^{-1}(f(x, y)), \tag{3.3}$$

where $\delta(x)$ the terms arising from using the auxiliary conditions.

The solution we obtain by Adomian method gives to the form $y(x)$ and the function $f(x, y)$ is infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{3.4}$$

and

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{3.5}$$

where the ingredient $y_n(x)$ of the solution $y(x)$ shall be determined recursively. Specific algorithms were seen in [2, 3] to formulate Adomian polynomials.

The following algorithm:

$$\begin{aligned} A_0 &= g(y_0), \\ A_1 &= y_1 g'(y_0), \\ A_2 &= y_2 g'(y_0) + \frac{1}{2!} y_1^2 g''(y_0), \\ A_3 &= y_3 g'(y_0) + y_1 y_2 g''(y_0) + \frac{1}{3!} y_1^3 g'''(y_0), \ \ldots \end{aligned} \tag{3.6}$$

can be used to structure Adomian polynomials, when $g(y)$ is a non-linear function. From (3.3),(3.4) and (3.5) we get

$$\sum_{n=0}^{\infty} y_n(x) = \delta(x) + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{3.7}$$

Through using Adomian Decomposition Method, the components $y(x)$ can be determined as

$$\begin{aligned} y_0 &= \delta(x), \\ y_{n+1} &= L^{-1} A_n, n \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} y_1 &= L^{-1} A_0, \\ y_2 &= L^{-1} A_1, \\ y_3 &= L^{-1} A_2, \ldots \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we can define the components $y_n(x)$, and therefore the series solution of $y(x)$ in (3.7) we can get it directly.

$$\phi_n(x) = \sum_{i=0}^{n-1} yi,$$

it can be used to sacrificial the exact solution.

4. Applications of the Method

In this part, we will provide some numerical examples that explain this method. where we give various examples of different order.

Example 1. *When we put $m = 3, a = \frac{1}{3}, b = 3,$ in equation (3.1), we get*

$$y'''' + 4y''' + \frac{10}{3}y'' + \frac{28}{27}y' + \frac{1}{9}y = 24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8 - y^2, \tag{4.1}$$

$$y(0) = y'(0) = y''(0) = y'''(0) = 0$$

with the exact solution $y = x^4$. We can rewrite equation (4.1) as below:

$$Ly = 24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8 - y^2, \tag{4.2}$$

where the differential operator L is:

$$L(.) = e^{-3x}\frac{d}{dx}e^{\frac{8}{3}x}\frac{d^3}{dx^3}e^{\frac{1}{3}x}(.),$$

and the invers differential operator L^{-1} is:

$$L^{-1}(.) = e^{\frac{-1}{3}x}\int_0^x\int_0^x\int_0^x e^{\frac{-8}{3}x}\int_0^x e^{3x}(.)dxdxdxdx\,.$$

By taken L^{-1} on (4.2) we find

$$y = L^{-1}(24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8) - L^{-1}y^2,$$

therefore

$$y(x) = L^{-1}(24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8) - L^{-1}y^2\,. \tag{4.3}$$

Substitute $y_n(x)$ for $y(x)$ in (4.3), we get:

$$\sum_{n=0}^{\infty} y_n(x) = (24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8) - L^{-1}A_n,$$

hence

$$y_0 = (24 + 96x + 40x^2 + \frac{112}{27}x^3 + \frac{1}{9}x^4 + x^8),$$
$$y_{n+1} = -L^{-1}A_n,$$

where A_n polynomials of Adomian given by

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

so

$$y_0 = x^4 + \frac{x^{12}}{11880} - \frac{x^{13}}{38610} + \frac{19\,x^{14}}{3243240} - \frac{37\,x^{15}}{31274100},$$
$$y_1 = \frac{-x^{12}}{11880} + \frac{x^{13}}{38610} - \frac{19\,x^{14}}{3243240} + \frac{37\,x^{15}}{31274100},$$

therefore

$$y(x) = x^4.$$

Then, from y_0 and y_1, we can get the exact solution $y(x) = x^4$, by this method.

Example 2. *When we put $m = 4$, $a = 1$, $b = 1$, in eq. (3.1), we get*

(4.4)

$$y^{(5)}+5y^{(4)}+10y^{(3)}+10y^{(2)}+5y'+y = 121+600x+600x^2+200x^3+25x^4+x^5+(1+x^5)^4-y^4,$$

$$y(0) = 1, y'(0) = y''(0) = y'''(0) = y''''(0) = 0\,,$$

with the exact solution $y = 1 + x^5$. We can rewrite equation (4.4) as below

$$Ly = 121 + 600x + 600x^2 + 200x^3 + 25x^4 + x^5 + (1 + x^5)^4 - y^4, \tag{4.5}$$

where

$$L(.) = e^{-x}\frac{d^5}{dx^5}e^x(.),$$

also

$$L^{-1}(.) = e^{-x}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x e^x(.)dxdxdxdxdx\,.$$

By taken L^{-1} on (4.5) we find

$$y = L^{-1}(121 + 600x + 600x^2 + 200x^3 + 25x^4 + x^5 + (1 + x^5)^4) - L^{-1}y^4,$$

therefore

(4.6)

$$y(x) = e^{-x}[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{42}]+L^{-1}(121+600x+600x^2+200x^3+25x^4+x^5+(1+x^5)^4)\,.$$

Substitute $y_n(x)$ for $y(x)$ in (4.6), we get

$$\sum_{n=0}^{\infty} y_n(x) = e^{-x}[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{42}]+L^{-1}(121+600x+600x^2+200x^3+25x^4+x^5+(1+x^5)^4)$$

$$-L^{-1}A_n,$$

hence

$$y_0 = e^{-x}[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{42}]+L^{-1}(121+600x+600x^2+200x^3+25x^4+x^5+(1+x^5)^4),$$

$$y_{n+1} = -L^{-1}A_n\,.$$

Here A_n polynomials of Adomian are given by

$$A_0 = y_0^4,$$

$$A_1 = 4y_0^3 y_1,$$

so

$$y_0 = 1+\frac{121\,x^5}{120}-\frac{x^6}{144}+\frac{x^7}{336}-\frac{x^8}{1152}+\frac{x^9}{5184}+\frac{59\,x^{10}}{604800},$$

$$y_1 = \frac{-x^5}{120}+\frac{x^6}{144}-\frac{x^7}{336}+\frac{x^8}{1152}-\frac{x^9}{5184}-\frac{179\,x^{10}}{1814400},$$

$$y_2 = \frac{x^{10}}{907200},$$

therefore

$$y(x) = 1+x^5.$$

From this example, we observe that by Adomian Method we get the exact solution.

Example 3. *When we put $m=6, a=1, b=1$ in eq. (3.1), we obtain*

$$y^{(7)}+7y^{(6)}+21y^{(5)}+35y^{(4)}+35y^{(3)}+21y^{(2)}+7y'+y=$$

(4.7) $$5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^14-y^2$$

$$y(0)=y'(0)=y''(0)=y'''(0)=y''''(0)=y'''''(0)=y''''''=0$$

with the exact solution $y=x^7$. We can rewrite equation (4.7) as follows:

(4.8) $$Ly = 5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^{14}-y^2.$$

The differential operator L for equation is

$$L(.) = e^{-x}\frac{d^7}{dx^7}e^{x}(.),$$

also

$$L^{-1}(.) = e^{-x}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x e^x(.)dxdxdxdxdxdxdx\,.$$

By taken L^{-1} on (4.8) we find

$$y = L^{-1}(5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^{14})-L^{-1}y^2\,,$$

therefore

(4.9)

$$y(x) = L^{-1}(5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^{14})-L^{-1}y^2,$$

substitute $y_n(x)$ for $y(x)$ in (4.9), we get

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}(5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^{14})$$

$$-L^{-1}A_n,$$

hence

$$y_0 = 0 + L^{-1}(5040+35280x+52920x^2+29400x^3+7351x^4+882x^5+49x^6+x^{14}),$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0$$

where A_n polynomials of Adomian given by

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

so

$$y_0 = x^7 + 6.01251\,10^{-7}\,x^{11} - 3.5073\,10^{-7}\,x^{12} + 1.07917\,10^{-7}\,x^{13} - 8.09376\,10^{-8}\,x^{14}$$
$$+3.08334\,10^{-8}\,x^{15} - 7.27475\,10^{-9}\,x^{16} + 1.25261\,10^{-9}\,x^{17} - 1.71746\,10^{-10}\,x^{18},$$

$$y_1 = -1.70634\,10^{-9}\,x^{21} + 5.42925\,10^{-10}\,x^{22} - 9.44217\,10^{-11}\,x^{23} + 1.18027\,10^{-11}\,x^{24}$$
$$-1.18077\,10^{-12}\,x^{25},$$

$$y_2 = 3.41267\,10^{-9}\,x^{28} + 2.32682\,10^{-9}\,x^{29} + 8.09329\,10^{-10}\,x^{30},$$

$$y(x) = x^7 + 6.01251\,10^{-7}\,x^{11} - 3.5073\,10^{-7}\,x^{12} + 1.07917\,10^{-7}\,x^{13} - 8.09376\,10^{-8}\,x^{14}$$
$$+3.08334\,10^{-8}\,x^{15} - 7.27475\,10^{-9}\,x^{16} + 1.25261\,10^{-9}\,x^{17} - 1.71746\,10^{-10}\,x^{18}$$
$$+1.97341\,10^{-11}\,x^{19} - 1.96353\,10^{-12}\,x^{20} - 1.70634\,10^{-9}\,x^{21} + 5.42925\,10^{-10}\,x^{22}$$
$$-9.44217\,10^{-11}\,x^{23} + 1.18027\,10^{-11}\,x^{24} - 1.18077\,10^{-12}\,x^{25} + 3.41267\,10^{-9}\,x^{28}$$
$$+2.32682\,10^{-9}\,x^{29} + 8.09329\,10^{-10}\,x^{30}.$$

Table 1. Numerical comparison between ADM and exact solution

x	*Exact*	*ADM*	*Absolute Error*
0.1	1×10^{-7}	1×10^{-7}	00000
0.2	0.0000128	0.0000128	00000
0.3	0.0002187	0.0002187	00000
0.4	0.0016384	0.0016384	00000
0.5	0.0078125	0.0078125	00000
0.6	0.0279936	0.0279936	00000
0.7	0.0823543	0.0823543	00000
0.8	0.209715	0.209715	00000
0.9	0.478297	0.478297	00000
1.0	1.0000	1.0000	00000

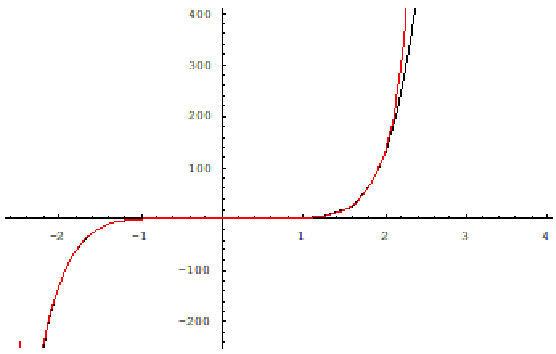

FIGURE 1. The Approximation solution for ADM and Exact solution

From table 1 and figure 1 we see the difference in the solution between the ADM (red line) and Exact (black line) solution.

5. CONCLUSION

In this paper, we found a new differential operator to solve non-linear oscillatory systems of higher order by Adomian Decomposition Method. We gave some non-linear examples to clarity the oscillatory systems of order forth, fifth and seventh. From tables and figures, we have shown that the solutions converge to the exact solvation. Adomian Decomposition Method is effective and powerful method.

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