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MAPPINGS AND DECOMPOSITIONS OF PAIRWISE CONTINUITY ON PAIRWISE ALMOST LINDELÖF SPACES

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ABSTRACT. In this paper we study the effect of some mappings and decompositions of pairwise continuity and pairwise open mappings on pairwise almost Lindelöf spaces. Also we introduce some pairwise decompositions of continuity. We show that some kinds of mappings preserve the pairwise almost Lindelöf property such as pairwise almost continuous and pairwise θ-continuous mappings, and other kinds of mappings implies another properties such as pairwise α-continuous, pairwise precontinuous, pairwise almost precontinuous and pairwise almost α-continuous mappings.

1. INTRODUCTION

Many papers studied the generalizations of Lindelöf spaces and their properties. In 1984, Willard and Dissanayake [15] introduce the notion of almost Lindelöf spaces and, in 1996, Cammoroto and Santoro [1] introduced a new results on these spaces which are the main generalization of Lindelöf spaces.In 1963, Kelly [5] introduce the notion of bitopological spaces and several papers studied this notion, Kim [12], Cooke and Reilly [2], Fora and Hdeib [4] and Kilicman and Salleh [7–11, 13].

In [10] Kilicman and Salleh introduced and studied the pairwise Lindelöf spaces and then introduced and studied the generalizations of pairwise Lindelöf spaces, pairwise nearly Lindelöf spaces [13], pairwise almost Lindelöf spaces

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citekis1, pairwise weakly Lindelöf spaces [11] and extended some previous results about the generalized Lindelöf spaces.

The concepts of decompositions of continuity and their effect on topological spaces, specially covering properties, has been of major interests in many works and a lot of papers studied that effect. In [8] Kilicman and Salleh introduce some pairwise mappings and studied some properties of these mappings. Also, they studied the effect of mappings and decompositions of pairwise continuity on pairwise nearly Lindelöf spaces.

In this work we will introduce some pairwise mappings and give some properties of these mappings, then we will study the effect of some mappings and decompositions of pairwise continuity and pairwise open mappings on pairwise almost Lindelöf spaces. We will show that some mappings will preserve the pairwise almost Lindelöf property while other mappings will imply another properties on the image of a pairwise almost Lindelöf space.

2. Preliminaries

Throughout this paper, all spaces (X, τ_1, τ_2) (or simply X) are always mean a bitopological space. If P is a topological property then $(\tau_i, \tau_j) - P$ means that τ_i has property P with respect to τ_j and $\tau_i - P$ denotes that (X, τ_1, τ_2) has property P with respect to τ_i. By $\tau_i - int(A)$ and $\tau_i - cl(A)$ we mean the interior and the closure of a subset A of a bitopological space X with respect to the topology τ_i, respectively. Also, by τ_i-open cover of X we mean a cover of X by τ_i-open subsets of X and by (τ_i, τ_j)-regular open cover of$X$ we mean a cover of X by (τ_i, τ_j)-regular open subsets of X. The prefixes (τ_i, τ_j)- or τ_i- are replaced by (i, j)- or i- respectively if there is no confusion.

Definition 2.1. *A subset S of a bitopological space (X, τ_1, τ_2) is called*

(1) *i-open [8] if S is open with respect to τ_i.*
(2) *(i, j)-regular open [6] if $S = i - int(j - cl(S))$, S is called pairwise regular open if it is both $(1, 2)$-regular open and $(2, 1)$-regular open.*
(3) *(i, j)-regular closed [6] if $S = i - cl(j - int(S))$, S is called pairwise regular closed if it is both $(1, 2)$-regular closed and $(2, 1)$-regular closed.*
(4) *(i, j)-preopen [8] if $S \subseteq i - int(j - cl(S))$, S is called pairwise preopen if it is both $(1, 2)$-preopen and $(2, 1)$-preopen.*

(5) (i,j)-β-*open [8] if* $S \subseteq j-cl(i-int(j-cl(S)))$, S *is called pairwise* β-*open if it is both* $(1,2)$-β-*open and* $(2,1)$-β-*open.*
(6) (i,j)-*clopen [8] if* S *is* i-*closed and* j-*open in* X, S *is called pairwise clopen if it is both* $(1,2)$-*clopen and* $(2,1)$-*clopen.*
(7) (i,j)-α-*open if* $S \subseteq i-int(j-cl(i-int(S)))$, S *is called pairwise* α-*open if it is both* $(1,2)$-α-*open and* $(2,1)$-α-*open.*
(8) i-*dense if* $i-cl(S) = X$, S *is called pairwise dense if it is both* 1-*dense and* 2-*dense.*

Definition 2.2. *[4, 10] A bitopological space* (X,τ_1,τ_2) *is called* i-*Lindelöf if the topological space* (X,τ_i) *is Lindelöf.* X *is called Lindelöf if it is both* 1-*Lindelöf and* 2-*Lindelöf.*

Definition 2.3. *[5, 6] A bitopological space* (X,τ_1,τ_2) *is called* (i,j)- *regular if for every* $x \in X$ *and every* i-*open set* V *containing* x, *there exists an* i-*regular open set* U *such that* $x \in U \subseteq j-cl(U) \subseteq V$, X *is called pairwise regular if it is both* $(1,2)$-*regular and* $(2,1)$-*regular.*

Definition 2.4. *[14] A bitopological space* (X,τ_1,τ_2) *is called* (i,j)- *almost regular if for every* $x \in X$ *and every* (i,j)-*regular open set* V *containing* x, *there exists an* (i,j)-*regular open set* U *such that* $x \in U \subseteq j-cl(U) \subseteq V$, X *is called pairwise almost regular if it is both* $(1,2)$-*almost regular and* $(2,1)$-*almost regular.*

Definition 2.5. *[6, 14] A bitopological space* (X,τ_1,τ_2) *is called* (i,j)- *semiregular if for every* $x \in X$ *and every* i-*open set* V *containing* x, *there exists an* i-*open set* U *such that* $x \in U \subseteq i-int(j-cl(U)) \subseteq V$, X *is called pairwise semiregular if it is both* $(1,2)$-*semiregular and* $(2,1)$-*semiregular.*

Definition 2.6. *[3] A bitopological space* (X,τ_1,τ_2) *is called* (i,j)- *extremally disconnected if the* i-*closure of every* j-*open set is* j-*open in* X. X *is called pairwise extremally disconnected if it is both* $(1,2)$-*extremally disconnected and* $(2,1)$-*extremally disconnected.*

Definition 2.7. *[13] A bitopological space* (X,τ_1,τ_2) *is called* (i,j)- *nearly Lindelöf if every* i-*open cover* $\{U_\alpha \mid \alpha \in \Delta\}$ *of* X *has a countable subfamily* $\{U_{\alpha_n} \mid n \in \mathbb{N}\}$ *such that* $X = \cup_{n\in\mathbb{N}} i-int(j-cl(U_{\alpha_n}))$, X *is called pairwise nearly Lindelöf if it is both* $(1,2)$-*nearly Lindelöf and* $(2,1)$-*nearly Lindelöf.*

Note that X is (i,j)- nearly Lindelöf if and only if every (i,j)-regular open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of X has a countable subfamily that covers X.

Definition 2.8. *[11] A bitopological space (X,τ_1,τ_2) is called (i,j)- nearly paracompact if every cover of X by (i,j)-regular open sets has a locally finite refinement. X is called pairwise nearly paracompact if it is both $(1,2)$-nearly paracompact and $(2,1)$-nearly paracompact.*

Definition 2.9. *A bitopological space (X,τ_1,τ_2) is called (i,j)- submaximal if every j-dense subset in X is i-Open in X. X is called pairwise submaximal if it is both $(1,2)$-submaximal and $(2,1)$-submaximal.*

Theorem 2.1.
a) *[13] A pairwise semiregular bitopological space (X,τ_1,τ_2) is pairwise nearly Lindelöf if and only if it is pairwise Lindelöf.*
b) *[13] If (X,τ_1,τ_2) is an (i,j)-almost regular and (i,j)-nearly Lindelöf then X is (i,j)- nearly paracompact.*

Definition 2.10. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is called:*

(1) *(i,j)-R-map [8] if $f^{-1}(V)$ is (τ_i,τ_j)-regular open in X for every (σ_i,σ_j)-regular open subset V of Y, f is called pairwise R-map if it is both $(1,2)$-R-map and $(2,1)$-R-map.*
(2) *(i,j)-almost continuous [8] if $f^{-1}(V)$ is τ_i-open in X for every (σ_i,σ_j)-regular open subset V of Y, f is called pairwise almost continuous if it is both $(1,2)$-almost continuous and $(2,1)$-almost continuous.*
(3) *(i,j)-precontinuous [8](resp. (i,j)-β-continuous) if $f^{-1}(V)$ is (τ_i,τ_j)-preopen (resp. (τ_i,τ_j)-β-open) in X for every (σ_i- open subset V of Y, f is called pairwise precontinuous (resp. pairwise β-continuous) if it is both $(1,2)$-precontinuous (resp. $(1,2)$-β-continuous) and $(2,1)$-precontinuous (resp. $(2,1)$-β-continuous).*
(4) *(i,j)-almost precontinuous [8](resp. (i,j)-almost β-continuous) if for every $x\in X$ and every (σ_i,σ_j)-regular open set V in Y containing $f(x)$, there exists a (τ_i,τ_j)-preopen (resp. (τ_i,τ_j)-β-open) set U in X containing x such that $f(U)\subseteq V$, f is called pairwise almost precontinuous (resp. pairwise almost β-continuous) if it is both $(1,2)$-almost precontinuous (resp. $(1,2)$-almost β-continuous) and $(2,1)$-almost precontinuous (resp. $(2,1)$-almost β-continuous).*

(5) *(i,j)-δ-continuous [8](resp. (i,j)-almost δ-continuous) if for every $x \in X$ and every (σ_i, σ_j)-regular open set V in Y containing $f(x)$, there exists a (τ_i, τ_j)-regular open set set U in X containing x such that $f(U) \subseteq V$ (resp. $f(U) \subseteq \sigma_j - cl(V)$), f is called pairwise δ-continuous (resp. pairwise almost δ-continuous) if it is both $(1,2)$-δ-continuous (resp. $(1,2)$-almost δ-continuous) and $(2,1)$-δ-continuous (resp. $(2,1)$-almost δ-continuous).*

Theorem 2.2. *[8] Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then*

a) *If f is a pairwise R-map then f is pairwise δ-continuous.*
b) *If f is a pairwise δ-continuous function then f is pairwise almost continuous.*

3. Some decompositions of pairwise continuity

In this sections we introduce an extension of some mappings and decompositions of continuity to bitopological spaces.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:*

(1) *(i,j)-θ-continuous if for every $x \in X$ and every σ_i-open set V in Y containing $f(x)$, there exists a τ_i-open set U in X containing x such that $f(\tau_j - cl(U)) \subseteq \sigma j - cl(V)$, f is called pairwise θ-continuous if it is both $(1,2)$-θ-continuous and $(2,1)$-θ-continuous.*

(2) *(i,j)-strong θ-continuous if for every $x \in X$ and every σ_i-open set V in Y containing $f(x)$, there exists a τ_i-open set U in X containing x such that $f(\tau_j - cl(U)) \subseteq V$, f is called pairwise strong θ-continuous if it is both $(1,2)$-strong θ-continuous and $(2,1)$-strong θ-continuous.*

(3) *(i,j)-α-continuous[ ] if $f^{-1}(V)$ is (τ_i, τ_j)-α-open in X for every σ_i- open subset V of Y, f is called pairwise α-continuous if it is both $(1,2)$-α-continuous and $(2,1)$-α-continuous.*

(4) *(i,j)-almost α-continuous if for every $x \in X$ and every (σ_i, σ_j)-regular open set V in Y containing $f(x)$, there exists a (τ_i, τ_j)-α-open set set U in X containing x such that $f(U) \subseteq V$, f is called pairwise almost α-continuous if it is both $(1,2)$-almost α-continuous and $(2,1)$-almost α-continuous.*

(5) *(i,j)-weakly quasicontinuous if for every $x \in X$ and every τ_i-open set G containing x and each σ_i-open set V in Y containing $f(x)$, there exists*

a τ_i-open set U in X containing x such that $U \subseteq G$ and $f(U) \subseteq \sigma_j - cl(V)$, f is called pairwise weakly quasicontinuous if it is both $(1,2)$-weakly quasicontinuous and $(2,1)$-weakly quasicontinuous.

(6) *(i,j)-contra-continuous if $f^{-1}(V)$ is τ_j-closed in X for every σ_i-open subset V of Y, f is called pairwise contra-continuous if it is both $(1,2)$-contra-continuous and $(2,1)$-contra-continuous.*

(7) *(i,j)-subcontra-continuous if there exists a σ_i-open base $\mathfrak{B}$ for the topology σ_i such that $f^{-1}(V)$ is τ_j-closed in X for every $V \in \mathfrak{B}$, f is called pairwise subcontra-continuous if it is both $(1,2)$-subcontra-continuous and $(2,1)$-subcontra-continuous.*

Lemma 3.1. *Let $\{A_\alpha \mid \alpha \in \Delta\}$ be a collection of (i,j)-α-open sets in a bitopological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is (i,j)-α-open in X.*

Proof. Let $\{A_\alpha \mid \alpha \in \Delta\}$ be a collection of (i,j)-α-open sets in X. Then $A_\alpha \subseteq i - int(j - cl(i - int(A_\alpha)))$ for each $\alpha \in \Delta$, so $\cup_{\alpha\in\Delta} A_\alpha \subseteq \cup_{\alpha\in\Delta} i - int(j - cl(i - int(A_\alpha))) \subseteq i - int(\cup_{\alpha\in\Delta} j - cl(i - int(A_\alpha))) \subseteq i - int(j - cl(\cup_{\alpha\in\Delta} i - int(A_\alpha))) \subseteq i - int(j - cl(i - int(\cup_{\alpha\in\Delta} A_\alpha)))$, hence $\cup_{\alpha\in\Delta} A_\alpha$ is (i,j)-α-open in X. □

Theorem 3.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then the following are equivalent:*

(1) *f is (i,j)-almost α-continuous.*

(2) *$f^{-1}(V)$ is (τ_i, τ_j)-α-open in X for every (σ_i, σ_j)-regular open subset V of Y.*

Proof. (1) ⇒ (2) Let V be any (σ_i, σ_j)-regular open subset of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is (i,j)-almost α-continuous then there exists an (i,j)-α-open set U_x in X containing x such that $f(U_x) \subseteq V$, so $U_x \subseteq f^{-1}(V)$, hence $f^{-1}(V) = \cup\{U_x \mid x \in f^{-1}(V)\}$ and so, by lemma[ ], $f^{-1}(V)$ is (τ_i, τ_j)-α-open in X.

(2) ⇒ (1) Let $x \in X$ and V be a (σ_i, σ_j)-regular open subset of Y containing $f(x)$, so, by (2), $f^{-1}(V)$ is (τ_i, τ_j)-α-open in X and $x \in f^{-1}(V)$ with $f(f^{-1}(V)) \subseteq V$. Therefore, f is (i,j)-almost α-continuous. □

Lemma 3.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i,j)-almost continuous function. Then f is an (i,j)-θ-continuous function.*

Proof. Let $x \in X$ and V be a σ_i-open set in Y containing $f(x)$. Then $\sigma_i - int(\sigma_j - cl(V))$ is a (σ_i, σ_j)-regular open subset of Y containing $f(x)$. Since f is

(i,j)-almost continuous then $f^{-1}(\sigma_i - int(\sigma_j - cl(V)))$ is a τ_i-open subset of X containing x and since $f^{-1}(\sigma_j - cl(V))$ is τ_j-closed in X then $f(\tau_j - cl(f^{-1}(\sigma_i - int(\sigma_j - cl(V))))) \subseteq f(\tau_j - cl(f^{-1}(\sigma_j - cl(V)))) = f(f^{-1}(\sigma_j - cl(V))) = \sigma_j - cl(V)$, so if $U = f^{-1}(\sigma_i - int(\sigma_j - cl(V)))$ then U is τ_i-open in X, $x \in U$ and $f(\tau_j - cl(U) \subseteq \sigma_j - cl(V)$. Therefore, f is an (i,j)-θ-continuous function. □

Corollary 3.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise almost continuous function, then f is a pairwise θ-continuous function.*

The converse of lemma 3.2 is not true as we can see in the next example.

Example 1. *Let $X = \{a,b,c\}$ with the topologies $\tau_1 = \{\phi, X, \{b\}, \{b,c\}\}$, $\tau_2 = \{\phi, X\}$, $\sigma_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma_2 = \{\phi, X, \{a\}\}$ and define $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ as $f(a) = f(b) = f(c) = c$. Then f is an (i,j)-θ-continuous function that is not (i,j)-almost continuous because $\{b\}$ is a (σ_1, σ_2)-regular open in X but $f^{-1}(\{b\}) = \{c\}$ is not τ_1-open in X.*

Lemma 3.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i,j)-θ-continuous function. Then f is an (i,j)-almost δ-continuous function.*

Proof. Let $x \in X$ and V be a σ_i-open set in Y containing $f(x)$. Since f is (i,j)-θ-continuous then there exists a τ_i-open set U in X containing x such that $f(\tau_j - cl(U)) \subseteq \sigma_j - cl(V)$. Since $U \subseteq \tau_i - int(\tau_j - cl(U))$ then $\tau_i - int(\tau_j - cl(U))$ is a (τ_i, τ_j)-regular open set in X containing x such that $f(\tau_i - int(\tau_j - cl(U))) \subseteq f(\tau_j - cl(U)) \subseteq \sigma_j - cl(V)$. Therefore, f is an (i,j)-almost δ-continuous function. □

Corollary 3.2. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise θ-continuous function, then f is a pairwise almost δ-continuous function.*

The converse of lemma 3.3 is not true as we can see in the next example.

Example 2. *Let $X = \{a,b,c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, $\tau_2 = \{\phi, X, \{c\}\}$, $\sigma_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma_2 = \{\phi, X, \{a\}\}$ and define $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ as $f(a) = b$ and $f(b) = f(c) = c$. Then f is a pairwise almost δ-continuous function[ ], but f is not $(1,2)$-θ-continuous function.Indeed, we have $c \in X$, $f(c) = c$, $c \in \{c\} = V$, V is σ_i-open in X, and $c \in \{b,c\} = U$ (the only τ_1-open set in X containing c). But $\tau_2 - cl(U) = X$ and $\sigma_2 - cl(V) = \{b,c\}$, so $f(\tau_2 - cl(U)) \not\subseteq \sigma_2 - cl(V)$. Therefore, f is not $(1,2)$-θ-continuous function.*

4. Mappings on pairwise almost Lindelöf spaces

In this section we study the effect of some kinds of pairwise mappings on pairwise almost Lindelöf spaces.

Definition 4.1. *[7] A bitopological space (X, τ_1, τ_2) is said to be (i, j)-almost Lindelöf if every i-open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of X has a countable subfamily $\{U_{\alpha_n} \mid n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} j - cl(U_{\alpha_n})$. X is called pairwise almost Lindelöf if it is both $(1, 2)$-almost Lindelöf and $(2, 1)$-almost Lindelöf.*

Note that every pairwise Lindelöf space is pairwise nearly Lindelöf and every pairwise nearly Lindelöf space is pairwise almost Lindelöf.

The following theorems are needed in our work.

Theorem 4.1. *[7] Let (X, τ_1, τ_2) be a pairwise semiregular and (j, i)-extremally disconnected bitopological space. Then X is (i, j)-almost Lindelöf if and only if X is i-Lindelöf.*

Theorem 4.2.
a) *[7] A pairwise almost regular bitopological space (X, τ_1, τ_2) is pairwise almost Lindelöf if and only if it is pairwise nearly Lindelöf.*
b) *[7] An (i, j)-regular space (X, τ_1, τ_2) is (i, j)- almost Lindelöf if and only if it is i-Lindelöf.*
c) *[7] A bitopological space (X, τ_1, τ_2) is pairwise almost Lindelöf if and only if every (i, j)-regular open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of X has a countable subfamily $\{U_{\alpha_n} \mid n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} j - cl(U_{\alpha_n})$.*

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-θ-continuous surjection. Then if X is (τ_i, τ_j)-almost Lindelöf then Y is (σ_i, σ_j)-almost Lindelöf.*

Proof. Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a σ_i-open cover of Y. For each $x \in X$ let $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j)-θ-continuous then there exists a τ_i-open set U_x in X containing x such that $f(\tau_j - cl(U_x)) \subseteq \sigma_j - cl(V_{\alpha_x})$, so $\{U_x \mid x \in X\}$ is a τ_i-open cover of X, hence there exists a countable subfamily $\{U_{x_n} \mid n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} \tau_j - cl(U_{x_n})$, so $Y = f(X) = f(\cup_{n\in\mathbb{N}} \tau_j - cl(U_{x_n}) = \cup_{n\in\mathbb{N}} f(\tau_j - cl(U_{x_n})) \subseteq \cup_{n\in\mathbb{N}} \sigma_j - cl(V_{\alpha_{x_n}})$. Therefore, Y is (σ_i, σ_j)-almost Lindelöf. □

Corollary 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise θ-continuous surjection. Then if X is pairwise almost Lindelöf then Y is pairwise almost Lindelöf.*

From Lemma 3.2 and Theorem 4.3 we have:

Theorem 4.4. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an* (i, j)*-almost continuous surjection. Then if* X *is* (τ_i, τ_j)*-almost Lindelöf then* Y *is* (σ_i, σ_j)*-almost Lindelöf.*

Corollary 4.2. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a pairwise almost continuous surjection. Then if* X *is pairwise almost Lindelöf then* Y *is pairwise almost Lindelöf.*

Note that every (i, j)-almost Lindelöf and (i, j)-extremally disconnected is (i, j)-nearly Lindelöf, so we have

Theorem 4.5. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an* (i, j)*-*θ*-continuous (or* (i, j)*-almost continuous) surjection from* X *onto an* (σ_j, σ_i)*-extremally disconnected space* Y*. Then if* X *is* (τ_i, τ_j)*-almost Lindelöf then* Y *is* (σ_i, σ_j)*-nearly Lindelöf.*

Corollary 4.3. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an pairwise* θ*-continuous (or pairwise almost continuous) surjection from* X *onto a pairwise extremally disconnected space* Y*. Then if* X *is pairwise almost Lindelöf then* Y *is pairwise nearly Lindelöf.*

From Theorem 4.1 Theorem 4.3 and Theorem 4.4 we have:

Theorem 4.6. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an* (i, j)*-*θ*-continuous (or* (i, j)*-almost continuous) surjection from* X *onto a pairwise semiregular and* (σ_j, σ_i)*-extremally disconnected space* Y*. Then if* X *is* (τ_i, τ_j)*-almost Lindelöf then* Y *is* σ_i*-Lindelöf.*

Corollary 4.4. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an pairwise* θ*-continuous (or pairwise almost continuous) surjection from* X *onto a pairwise semiregular and pairwise extremally disconnected space* Y*. Then if* X *is pairwise almost Lindelöf then* Y *is pairwise Lindelöf.*

Also, from Theorem 2.1, Theorem 4.2, Theorem 4.3 and Theorem 4.4 we have:

Theorem 4.7. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an* (i, j)*-*θ*-continuous (or* (i, j)*-almost continuous) surjection. Then*

(1) *If* X *is* (τ_i, τ_j)*-almost Lindelöf and* Y *is* (σ_i, σ_j)*-almost regular then* Y *is* (σ_i, σ_j)*-nearly Lindelöf and* (σ_i, σ_j)*-nearly paracompact.*

(2) *If* X *is* (τ_i, τ_j)*-almost Lindelöf and* Y *is* (σ_i, σ_j)*-regular then* Y *is* σ_i*-Lindelöf.*

Corollary 4.5. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a pairwise* θ*-continuous (or pairwise almost continuous) surjection. Then*

(1) *If X is pairwise almost Lindelöf and Y is pairwise almost regular then Y is pairwise nearly Lindelöf and pairwise nearly paracompact.*
(2) *If X is pairwise almost Lindelöf and Y is pairwise regular then Y is pairwise Lindelöf.*

Moreover, from Theorem 2.2 and Theorem 4.4 we have:

Theorem 4.8. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-δ-continuous (or (i, j)-R-map)surjection. Then if X is (τ_i, τ_j)-almost Lindelöf then Y is (σ_i, σ_j)-almost Lindelöf.*

Corollary 4.6. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an pairwise δ-continuous (or pairwise R-map) surjection. Then if X is pairwise almost Lindelöf then Y is pairwise Lindelöf.*

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-strong θ-continuous surjection. Then if X is (τ_i, τ_j)-almost Lindelöf then Y is σ_i-Lindelöf.*

Proof. Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a σ_i-open cover of Y. For each $x \in X$ let $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j)-strong θ-continuous then there exists a τ_i-open set U_x in X containing x such that $f(\tau_j - cl(U_x)) \subseteq V_{\alpha_x}$, so $\{U_x \mid x \in X\}$ is a τ_i-open cover of X, hence there exists a countable subfamily $\{U_{x_n} \mid n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}}\tau_j - cl(U_{x_n})$, so $Y = f(X) = f(\cup_{n\in\mathbb{N}}\tau_j - cl(U_{x_n}) = \bigcup_{n\in\mathbb{N}} f(\tau_j - cl(U_{x_n})) \subseteq \cup_{n\in\mathbb{N}}V_{\alpha_{x_n}}$. Therefore, Y is σ_i-Lindelöf. □

Corollary 4.7. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise strongly θ-continuous surjection. Then if X is pairwise almost Lindelöf then Y is pairwise Lindelöf.*

Theorem 4.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-weakly quasicontiuous and (i, j)-precontinuous surjection. Then if X is (τ_i, τ_j)-nearly Lindelöf then Y is (σ_i, σ_j)-almost Lindelöf.*

Proof. Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a σ_i-open cover of Y. Then $\{f^{-1}(V_\alpha) \mid \alpha \in \Delta\}$ is a cover of X. Since f is (i, j)-precontinuous then $f^{-1}(V_\alpha)$ is (τ_i, τ_j)-preopen in X, so $f^{-1}(V_\alpha) \subseteq \tau_i - int(\tau_j - cl(f^{-1}(V_\alpha)))$ for every $\alpha \in \Delta$, hence $\{\tau_i - int(\tau_j - cl(f^{-1}(V_\alpha))) \mid \alpha \in \Delta\}$ is a (τ_i, τ_j)-regular open cover of X, hence there exists a countable subset $\{\alpha_n \mid n \in \mathbb{N}\}$ of Δ such that $X = \cup_{n\in\mathbb{N}}\tau_i - int(\tau_j - cl(f^{-1}(V_{\alpha_n}))) \subseteq \cup_{n\in\mathbb{N}}f^{-1}(\sigma_j - cl(V_{\alpha_n}))$ because f is (i, j)-weakly quasicontiuous. Thus $Y = f(X) = f(\cup_{n\in\mathbb{N}}f^{-1}(\sigma_j - cl(V_{\alpha_n}))) = f(f^{-1}(\cup_{n\in\mathbb{N}}\sigma_j - cl(V_{\alpha_n}))) = \cup_{n\in\mathbb{N}}\sigma_j - cl(V_{\alpha_n})$. Therefore, Y is (σ_i, σ_j)-almost Lindelöf. □

Corollary 4.8. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise weakly quasicontiuous and pairwise precontinuous surjection. Then if X is pairwise nearly Lindelöf then Y is pairwise almost Lindelöf.*

Theorem 4.11. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-almost δ-continuous surjection. Then if X is (τ_i, τ_j)-nearly Lindelöf then Y is (σ_i, σ_j)-almost Lindelöf.*

Proof. Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a σ_i-open cover of Y. For each $x \in X$ let $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x} \subseteq \sigma_i - int(\sigma_j - cl(V_{\alpha_x}$. Since f is (i, j)-almost δ-continuous and $\sigma_i - int(\sigma_j - cl(V_{\alpha_x}))$ is a (σ_i, σ_j)-regular open subset of Y then there exists a (τ_i, τ_j)-regular open subset U_x of X such that $f(U_x) \subseteq \sigma_j - cl(\sigma_i - int(\sigma_j - cl(V_{\alpha_x}))) \subseteq \sigma_j - cl(V_{\alpha_x})$, so $\{U_x \mid x \in X\}$ is a τ_i-regular open cover of X, hence there exists a countable subfamily $\{U_{x_n} \mid n \in \mathbb{N}\}$ such that $X = \cup_{n\in\mathbb{N}} U_{x_n}$. Thus $Y = f(X) = f(\cup_{n\in\mathbb{N}} U_{x_n}) \subseteq \cup_{n\in\mathbb{N}} f(U_{x_n}) \subseteq \cup_{n\in\mathbb{N}} \sigma_j - cl(V_{\alpha_{x_n}})$. Therefore, Y is (σ_i, σ_j)-almost Lindelöf. □

Corollary 4.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise almost δ-continuous surjection. Then if X is pairwise nearly Lindelöf then Y is pairwise almost Lindelöf.*

From Theorem 4.2 and Theorem 4.11 we have:

Theorem 4.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-almost δ-continuous surjection from a bitopological space X onto a pairwise almost regular space Y. Then if X is (τ_i, τ_j)-nearly Lindelöf then Y is (σ_i, σ_j)-nearly Lindelöf.*

Theorem 4.13. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-almost α-continuous(or (i, j)-almost precontinuous) and (i, j)-contra-continuous surjection. Then if X is (τ_i, τ_j)-almost Lindelöf then Y is (σ_i, σ_j)-nearly Lindelöf.*

Proof. we prove the case of (i, j)-almost α-continuous and the case of (i, j)-almost precontinuous is similar.

Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a (σ_i, σ_j)-regular open cover of Y. Then, $f^{-1}(V_\alpha)$ is an (τ_i, τ_j)-α-open subset of X, so $f^{-1}(V_\alpha) \subseteq \tau_i - int(\tau_j - cl(\tau_i - int(f^{-1}(V_\alpha))))$ for each $\alpha \in \Delta$, hence $\{\tau_i - int(\tau_j - cl(\tau_i - int(f^{-1}(V_\alpha)))) \mid \alpha \in \Delta\}$ is a τ_i-open cover of X, so there exists a countable subset $\{\alpha_n \mid n \in \mathbb{N}\}$ of Δ such that $X = \cup_{n\in\mathbb{N}} \tau_j - cl(\tau_i - int(\tau_j - cl(\tau i - int(f^{-1}(V_{\alpha_n}))))) \subseteq \cup_{n\in\mathbb{N}} \tau_j - cl(\tau i - int(f^{-1}(V_{\alpha_n}))) \subseteq \cup_{n\in\mathbb{N}} \tau_j - cl(f^{-1}(V_{\alpha_n}))$. Since f is (i, j)-contra-continuous then $f^{-1}(V_{\alpha_n})$ is τ_j-closed in X for every $n \in \mathbb{N}$, so $X = \cup_{n\in\mathbb{N}} f^{-1}(V_{\alpha_n})$, hence $Y =$

$f(X) = f(\cup_{n\in\mathbb{N}} f^{-1}(V_{\alpha_n})) = f(f^{-1}(\cup_{n\in\mathbb{N}} V_{\alpha_n})) = \cup_{n\in\mathbb{N}} V_{\alpha_n}))$. Therefore, Y is (σ_i, σ_j)-nearly Lindelöf. □

Corollary 4.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise almost α-continuous (or pairwise almost precontinuous) surjection. Then if X is pairwise almost Lindelöf then Y is pairwise nearly Lindelöf.*

Theorem 4.14. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i, j)-α-continuous(or (i, j)-precontinuous) and (i, j)-subcontra-continuous surjection. Then if X is (τ_i, τ_j)-almost Lindelöf then Y is σ_i-Lindelöf.*

Proof. we prove the case of (i, j)-precontinuous and the case of (i, j)-almost α-is similar.

Let $\mathfrak{B}$ be a σ_i-open base for σ_i such that $f^{-1}(V)$ is τ_j-closed in X for every $V \in \mathfrak{B}$. Let $\{U_\alpha \mid \alpha \in \Delta\}$ be a σ_i-open cover of Y. For each $x \in X$ let $\alpha_x \in \Delta$ such that $f(x) \in U_{\alpha_x}$, so there exists $V_x \in \mathfrak{B}$ such that $f(x) \in V_x \subseteq U_{\alpha_x}$, Since f is (i, j)-precontinuous then $f^{-1}(V_x)$ is (τ_i, τ_j)-peropen in X for all $x \in X$, so $f^{-1}(V_x) \subseteq \tau_i - int(\tau_j - cl(f^{-1}(V_x)))$. Since $f^{-1}(V_x$ is τ_j-closed in X then $f^{-1}(V_x) \subseteq \tau_i - int(f^{-1}(V_x))$, so $f^{-1}(V_x) = \tau_i - int(f^{-1}(V_x))$, hence $f^{-1}(V_x)$ is τ_i-open in X, so $\{f^{-1}(V_x) \mid x \in X\}$ is a τ_i-open cover of X, so there exists a countable subset $\{\alpha_n \mid n \in \mathbb{N}\}$ of Δ such that $X = \cup_{n\in\mathbb{N}} \tau_j - cl(f^{-1}(V_{x_n})) = \cup_{n\in\mathbb{N}} f^{-1}(V_{x_n}) = f^{-1}(\cup_{n\in\mathbb{N}} V_{x_n})$. Thus $Y = f(X) = f(f^{-1}(\cup_{n\in\mathbb{N}} V_{x_n})) = \cup_{n\in\mathbb{N}} V_{x_n}$. Therefore, Y is σ_i-Lindelöf. □

Corollary 4.11. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise α-continuous (or pairwise precontinuous) surjection. Then if X is pairwise almost Lindelöf then Y is pairwise Lindelöf.*

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