ADV MATH
SCI JOURNAL

SOLUTIONS OF EMDEN-FOWLER TYPE EQUATIONS OF DIFFERENT ORDER BY ADOMIAN DECOMPOSITION METHOD

SOMAIA ALI ALAQEL AND YAHYA QAID HASAN[1]

ABSTRACT. This study efficiently applies Adomian decomposition method to approximate solutions of linear and nonlinear Emden-Fowler type equations of higher order. A few illustrative problems are discussed to confirm the accuracy and applicability of the presented approach .

1. INTRODUCTION

The Adomian decomposition method (ADM) [1–3] was first introduced by Adomian in the years 1980s. After that many researchers offered various modifications on this method. ADM is powerful for solving linear and nonlinear problems and has been used as a highly effective technique for solving different types of differential equations with initial and boundary conditions such as Emden-Fowler Equation [5, 8–10] which is considered as one of the most equations that has been great interest by numerous mathematicians and physicists in many studies. Several approaches were investigated to find solutions for this type of equations and to overcome the difficulty of the singular point as [4, 6, 7]. This article is attempt to handle Emden-fowler type equations of higher order with boundary conditions by using ADM. We therefore introduce

a novel differential operator to solve different orders of Emden-Fowler equations.

2. STRUCTURE OF THE EMDEN EQUATIONS OF HIGHER ORDER

To establish different order of Emden-Fowler type equations we will propose the generalized formula as follows

$$(2.1) \qquad x^{-1}\frac{d^{n-1}}{dx^{n-1}}x^{1-m}\frac{d^2}{dx^2}x^m(y) = f(x,y),$$

where $m \geq 0$ and $n \geq 1$. To obtain the first type, we put $n = 1$ in equation (2.1) gives

$$y'' + \frac{2m}{x}y' + \frac{m(m-1)}{x^2}y = f(x,y).$$

To obtain the second type , we set $n = 2$ in equation (2.1) to find

$$y''' + \frac{1+2m}{x}y'' + \frac{m(m-1)}{x^2}y' - \frac{m(m-1)}{x^3}y = f(x,y).$$

To obtain the third type, we put $n = 3$ in equation (2.1) yields:

$$y^{(4)} + \frac{2+2m}{x}y''' + \frac{m(m-1)}{x^2}y'' - \frac{2m(m-1)}{x^3}y' + \frac{2m(m-1)}{x^4}y = f(x,y).$$

To obtain the fourth type, we put $n = 4$ in equation (2.1) gives:

$$y^{(5)} \quad + \quad \frac{3+2m}{x}y^{(4)} + \frac{m(m-1)}{x^2}y''' - \frac{3m(m-1)}{x^3}y'' + \frac{6m(m-1)}{x^4}y'$$

$$-\frac{6m(m-1)}{x^5}y = f(x,y)$$

$$\cdots$$

$$y^{(n+1)} \quad + \quad \frac{(n-1)+2m}{x}y^{(n)} + \frac{m(m-1)}{x^2}y^{(n-1)}$$

$$(2.2) \qquad + \sum_{r=2}^{n}(-1)^{r+1}\frac{(n-1)!}{(n-r)!}\frac{m(m-1)}{x^{1+r}}y^{(n-r)} = f(x,y).$$

3. Essentials of the Method

Consider a general Emden-Fowler equation in the form (2.2) with the following boundary conditions: $y(b) = a_1, y'(b) = a_2, ..., y^{(n)}(b) = a_n$, where the function $f(x, y)$ is known, $a_1, a_2, ..., a_n$ are constants, if $m \neq 1$, $b \neq 0$, and if $m = 1$, b takes any value. Equation (2.2) can be rewritten as

$$(3.1) \qquad Ly = f(x, y),$$

where

$$L(.) = x^{-1} \frac{d^{n-1}}{dx^{n-1}} x^{1-m} \frac{d^2}{dx^2} x^m (.),$$

then the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-m} \int_0^x \int_0^x x^{1-m} \underbrace{\int_b^x \int_b^x \cdots \int_b^x}_{n-1} x(.) dx dx dx dx ... dx .$$

By using L^{-1} to both sides of (3.1) to obtain

$$(3.2) \qquad y = \gamma(x) + L^{-1} f(x, y),$$

such that

$$L\gamma(x) = 0.$$

The Adomian decomposition method assumes that solution $y(x)$ by an infinite series

$$(3.3) \qquad y(x) = \sum_{n=0}^{\infty} y_n(x),$$

and the nonlinear term $f(x, y)$ by an infinite series of polynomials

$$(3.4) \qquad f(x, y) = \sum_{n=0}^{\infty} A_n,$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and the A_n are the Adomian polynomials, specific algorithms were seen

in [3] to formulate Adomian polynomials . The following algorithm :

$$
\begin{aligned}
A_0 &= Z(y_0), \\
A_1 &= Z'(y_0)y_1, \\
A_2 &= Z'(y_0)y_{(2)} + \frac{1}{2}Z''(y_0)y_1^2, \\
A_3 &= Z'(y_o)y_{(3)} + Z''(y_0)y_1y_2 + \frac{1}{3!}Z'''(y_0)y_1^3, \\
&\quad \cdots
\end{aligned}
$$

(3.5)

From (3.2) , (3.3) and (3.4) we get

(3.6)
$$
\sum_{n=0}^{\infty} y_{(n)} = \gamma(x) + L^{-1}\sum_{n=0}^{\infty} A_n.
$$

To determine the components $y_n(x)$,we use Adomian decomposition method by using the relation

$$
y_0 = \gamma(x) + L^{-1}f(x),
$$

$$
y_{n+1} = -L^{-1}A_n, n \geq 0,
$$

therefore

$$
\begin{aligned}
y_0 &= \gamma(x) + L^{-1}f(x), \\
y_1 &= -L^{-1}A_0, \\
y_2 &= -L^{-1}A_1, \\
y_3 &= -L^{-1}A_3, \\
&\quad \cdots
\end{aligned}
$$

(3.7)

Using the equation (3.5) and (3.7) we can determine the components $y_n(x)$, and therefore, we can directly obtain series solution of $y(x)$ in (3.6). In addition, and for numerical reasons, we can be the n-term approximate

$$
\Psi_n = \sum_{n=0}^{n-1} y_n(x),
$$

in order to approximate the exact solution.

4. Applications of ADM

In this part, we present some examples and apply ADM to obtain approximate and exact solutions of the Emden-Fowler type equations of different order.

Example 1. *First, we consider the linear Emden-Fowler type equation*

$$(4.1) \qquad y''' + \frac{5}{x}y'' + \frac{2}{x^2}y' - \frac{2}{x^3}y = \frac{e^{x^3}\left(-2 + 42\,x^3 + 99\,x^6 + 27\,x^9\right)}{x^3},$$

$$y(\tfrac{1}{2}) = 1.13315, \ y'(\tfrac{1}{2}) = 0.849861, \ y''(\tfrac{1}{2}) = 4.03684,$$

where

$$L(.) = x^{-1}\frac{d}{dx}x^{-1}\frac{d^2}{dx^2}x^2(.)\,.$$

Then the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-2}\int_0^x \int_0^x x \int_{\frac{1}{2}}^x x(.)dxdxdx.$$

We write equation (4.1) by ADM operator form as

$$(4.2) \qquad Ly = \frac{e^{x^3}\left(-2 + 42\,x^3 + 99\,x^6 + 27\,x^9\right)}{x^3}.$$

Applying L^{-1} to (4.2) we get the exact solution

$$y(x) = e^{x^3}.$$

Example 2. *Next, assume the nonlinear Emden-Fowler type equation*

$$y^{(4)} + \frac{14}{x}y''' + \frac{30}{x^2}y'' - \frac{60}{x^3}y' + \frac{60}{x^4}y =$$

$$(4.3) \qquad = e^x\left(540 + 780\,x + 270\,x^2 + 30\,x^3 + x^4\right) - x^{16}e^{4x} + y^4,$$

$$y(-1) = 0.367879, \ y'(-1) = -1.10364, \ y''(-1) = 1.8394, \ y'''(-1) = 0.367879,$$

the exact solution is $y(x) = x^4 e^x$, where

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^{-5}\frac{d^2}{dx^2}x^6(.).$$

Then the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-6}\int_0^x \int_0^x x^5 \int_{-1}^x \int_{-1}^x x(.)dxdxdxdx,$$

equation (4.3) in ADM operator form becomes

(4.4) $Ly = e^x \left(540 + 780\,x + 270\,x^2 + 30\,x^3 + x^4\right) - x^{16}e^{4x} + y^4.$

Applying L^{-1} to (4.4) we get

$$y = 0.464229\,x + 0.81459\,x^2 + L^{-1}(e^x\left(540 + 780\,x + 270\,x^2 + 30\,x^3 + x^4\right))$$
$$- L^{-1}(x^{16}e^{4x}) + L^{-1}y^4,$$

therefore

$$y = 0.0000286088\,x + 0.0000229384\,x^2 + e^x\,x^4 + L^{-1}y^4.$$

We use the recursive relation

$$y_0 = 0.0000286088\,x + 0.0000229384\,x^2 + e^x\,x^4,$$
$$y_{n+1} = L^{-1}A_n, \ n \geq 0,$$

where the nonlinear term y^4 has Adomian polynomials A_n as the following

$$A_0 = y_0^4,$$
$$A_1 = 4y_0^3 y_1,$$

so the solution components are

$$y_0 = 0.0000286088\,x + 0.0000229384\,x^2 + e^x\,x^4,$$
$$y_1 = -0.0000152908\,x - 0.0000229363\,x^2 - 8.91966\,10^{-6}\,x^3 -$$
$$1.64375\,10^{-18}\,x^4 - \ldots$$
$$y_2 = -3.38524\,10^{-10}\,x - 5.07786\,10^{-10}\,x^2 - 1.97472\,10^{-10}\,x^3$$
$$-8.29769\,10^{-23}\,x^6 + \ldots$$

this yields the solution in a series form

$$y(x) = y_0 + y_1 + y_2 = x^4 e^x + 0.0000133176\,x + 1.63957\,10^{-9}\,x^2 - 8.91985\,10^{-6}\,x^3$$
$$-1.64365\,10^{-18}\,x^4 - \ldots$$

Example 3. *Consider the following equation*

(4.5) $y^{(5)} + \dfrac{23}{x}y^{(4)} + \dfrac{90}{x^2}y''' - \dfrac{270}{x^3}y'' + \dfrac{540}{x^4}y' - \dfrac{270}{x^5}y = 5040 + e^{x^5} - e^y,$

$$y(1) = 1, y'(1) = 5, y''(1) = 20, y'''(1) = 60, y^{(4)}(1) = 120,$$

the exact solution is $y(x) = x^5$. Here:

$$L(.) = x^{-1} \frac{d^2}{dx^2} x^{-9} \frac{d^2}{dx^2} x^{10}(.),$$

and the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-10} \int_0^x \int_0^x x^9 \int_1^x \int_1^x \int_1^x x(.)dxdxdxdxdx.$$

In ADM operator form equation (4.5) becomes

(4.6) $$Ly = 5040 + e^{x^2} - e^y.$$

Applying L^{-1} to (4.6) we get

$$y = \frac{-63\,x}{11} + \frac{140\,x^2}{11} - \frac{105\,x^3}{13} + x^5 + L^{-1}(e^{x^5}) - L^{-1}e^y.$$

We use the recursive relation

$$y_0 = -0.00180375\,x + 0.0037636\,x^2 - 0.00219399\,x^3 + 1.0002\,x^5 + \ldots,$$

$$y_{n+1} = -L^{-1}A_n, \; n \geq 0,$$

where the nonlinear term e^y has Adomian polynomials A_n as the following

$$
\begin{aligned}
A_0 &= e^{y_0}, \\
A_1 &= y_1 e^{y_0}, \\
A_2 &= (y_2 + \frac{1}{2}y_1^2)e^{y_0},
\end{aligned}
$$

and so on. So the solution components are given by

$$
\begin{aligned}
y_0 &= -0.00180375\,x + 0.0037636\,x^2 - 0.00219399\,x^3 + 1.0002\,x^5 + \ldots \\
y_1 &= -0.00180361\,x + 0.00376324\,x^2 - 0.00219373\,x^3 + 0.000198413\,x^5 + \ldots \\
y_2 &= 1.48956 \times 10^{-7}\,x - 3.67325 \times 10^{-7}\,x^2 + 2.63754 \times 10^{-7}\,x^3 - \\
&\quad 1.25251 \times 10^{-7}\,x^6 + \ldots \\
y_3 &= -1.72379\,10^{-11}\,x + 4.73759\,10^{-11}\,x^2 - 3.82888 \times 10^{-11}\,x^3 \\
&\quad +1.03442 \times 10^{-11}\,x^6 + \ldots
\end{aligned}
$$

and so on. Thus, the approximate solution is

$$
\begin{aligned}
y(x) &= y_0 + y_1 + y_2 + y_3 = -2.92299\,10^{-7}\,x + 7.24058\,10^{-7}\,x^2 \\
&\quad -5.22375 \times 10^{-7}\,x^3 + x^5 + 2.50501 \times 10^{-7}\,x^6 + \ldots
\end{aligned}
$$

TABLE 1. Comparison of numerical errors between the exact solution $y = x^5$ and the ADM solution

x	Exact solution	ADM solution	Absolute error
0.0	0.000	0.0000000	0.000000
0.1	0.00001	0.0000097	0.000000
0.2	0.00032	0.0003199	0.000000
0.3	0.00243	0.0024299	0.000000
0.4	0.01024	0.0102399	0.000000
0.5	0.03125	0.0312499	0.000000
0.6	0.07776	0.0777599	0.000000
0.7	0.16807	0.1680699	0.000000
0.8	0.32768	0.3276999	0.000000
0.9	0.59049	0.5904899	0.000000
1.0	1.000	0.9999999	0.000000

Example 4. *We finally study the nonlinear Emden-Fowler type equation when we set $m = 1$ and $n = 5$ in (2.2), we obtain*

$$y^{(6)} + \frac{6}{x}y^{(5)} =$$

$$(4.7) \qquad 144\left(35 - 4080\,x^6 + 20005\,x^{12} - 13705\,x^{18} + 1056\,x^{24} + x^{30}\right)e^{-6y},$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0, y^{(4)}(0) = 0, y^{(5)}(0) = 0\,,$$

the exact solution is $y(x) = \log(1 + x^6)$. Here

$$L(.) = x^{-1}\frac{d^6}{dx^6}x(.),$$

and the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-1}\int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x x(.)dxdxdxdxdxdx.$$

equation (4.7) in ADM operator form becomes:

$$(4.8) \quad Ly = 144\left(35 - 4080\,x^6 + 20005\,x^{12} - 13705\,x^{18} + 1056\,x^{24} + x^{30}\right)e^{-6y}.$$

Applying L^{-1} to (4.8) we have

$$y = L^{-1}(144\left(35 - 4080\,x^6 + 20005\,x^{12} - 13705\,x^{18} + 1056\,x^{24} + x^{30}\right)e^{-6y}).$$

We use the recursive relation

$$y_0 = 0,$$
$$y_{n+1} = L^{-1}(144\left(35 - 4080\,x^6 + 20005\,x^{12} - 13705\,x^{18} + 1056\,x^{24} + x^{30}\right))A_n$$
$$n \geq 0\,,$$

where the nonlinear term e^{-6y} has Adomian polynomials A_n as the following

$$A_0 = e^{-6y_0},$$
$$A_1 = -6y_1 e^{-6y_0},$$
$$A_2 = (-6y_2 + 18y_1^2)e^{-6y_0}\,,$$

and so on. So the solution components are

$$y_0 = 0,$$
$$y_1 = x^6 - 0.475524\,x^{12} + 0.147464\,x^{18} - 0.0154771\,x^{24} + 0.000286847\,x^{30} + \dots$$
$$y_2 = -0.0244755\,x^{12} + 0.181187\,x^{18} - 0.148732\,x^{24} + 0.0388222\,x^{30} - \dots$$
$$y_3 = 0.00468185\,x^{18} - 0.0843324\,x^{24} + 0.118886\,x^{30} - \dots$$

and so on. This yields the solution in a series form

$$y(x) = y_0 + y_1 + y_2 + y_3 = x^6 - 0.5\,x^{12} + 0.333333\,x^{18} - 0.248541\,x^{24} + 0.157995\,x^{30}\,.$$

Note that, the solution by ADM converges to the exact solution

$$y(x) = \log(1 + x^6) = x^6 - 0.5\,x^{12} + 0.333333\,x^{18} - 0.25\,x^{24} + 0.2\,x^{30} + \dots$$

5. CONCLUSION

The Emden-fowler type equations of higher order are investigated by suggested modification of ADM. Numerical examples showed that the method is successful for finding solutions of these equations with some iterations approximate the exact solutions very well. The results demonstrated that the proposed strategy is powerful and valid.

REFERENCES

[1] G. ADOMIAN : *A review of the decomposition in applied mathematics,* Mathematical analysis and applications, **1** (1988), 501-544.

[2] G. ADOMIAN, R. RACH : *Inversion of nonlinear stochastic operators*, J. Math. Anal. Appl., **91** (1983), 39–46.

[3] G. ADOMIAN: *Solving frontier problem of physics: the decomposition method,* Kluwer academic publishers London, 1994.

[4] M. S. H. CHOWDHURY, I. HASHIM : *Solutions of Emden-Fowler equations by homotopy perturbation method,* Nonlinear Anal Thear : TMA, **70** (2009), 2480–2484.

[5] Y. Q. HASAN, L. Z. MING: *Solving singular boundary value problems of higher-ordar ordinary differential equation by modified Adomain decomposition method,* Commun Nonlinear Sci Numer. Simulat., **14** (2009), 2592–2596.

[6] M. SINGH , A. K. VERMA: *An effective computational technique for a class of Lane-Emden equations,* J. Math. Chem., **54** (2016), 231-251.

[7] R. SINGH, H. GARG, V. GULERIA: *Haar wavelet collocation method for LaneâĂŞEmden equations with Dirichlet, Neumann and Neumann-Robin boundary conditions,* Comput. Appl. Math., **346** (2019), 150–161.

[8] A. M. WAZWAZ: *Solving two Emden-Folwer type equations of third order by The variational iteration method,* Appl. Math. Inf. Sci., **9** (2015), 2429–2436.

[9] A. M. WAZWAZ, R. RACH, J. DUAN: *Solving new fourth-order Emden-Fowler type equations by Adomian decomposition method,* Int. J. Comput. Math. Eng. Sci. Mech., **5** (2015), 1–30.

[10] A. M. WAZWAZ: *Analytical solution for the time-dependent Emden-Fowler type of equations by Adomian decomposition method,* Appl. Math. Comput., **166** (2005), 638–651.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SHEBA REGION
YEMEN
E-mail address: somaiaaalaqel@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SHEBA REGION
YEMEN
E-mail address: yahya217@yahoo.com