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APEX OF FUZZY VERTICES AND BOTTOM OF FUZZY VERTICES IN A FUZZY GRAPH

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ABSTRACT. In this paper, apex of fuzzy vertices and bottom of fuzzy vertices are introduced, and also theorems related to these concepts are stated and proved.

1. INTRODUCTION

In 1965, Zadeh [8] introduced the notion of fuzzy set as a method of presenting uncertainty. Since complete information in science and technology is not always available. Thus we need mathematical models to handle various types of systems containing elements of uncertainty. After that Rosenfeld [6] introduced fuzzy graphs. Yeh and Bang [7] also introduced fuzzy graphs independently. Fuzzy graphs are useful to represent relationships which deal with uncertainty and it differs greatly from classical graph. Nagoor Gani and Ratha [3] introduced fuzzy regular graphs, total degree and totally regular fuzzy graphs. Ramakrishnan and Lakshmi [4,5] introduced depth of μ, height of μ. In this paper, some results on apex of P and bottom of P are given.

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Key words and phrases. Fuzzy subset, Fuzzy relation, Strong fuzzy relation, Fuzzy graph, Fuzzy loop, Fuzzy pseudo graph, Degree of fuzzy vertex, Total Degree of fuzzy vertex, order of the fuzzy graph, size of the fuzzy graph, Fuzzy regular graph, Fuzzy totally regular graph, Fuzzy complete graph, apex of P, bottom of P.

2. PRELIMINARIES

Definition 2.1. *[7] Let S be any nonempty set. A mapping $X : S \to [0,1]$ is called a fuzzy subset of X.*

Example 1. *A fuzzy subset $B = \{(p,0.3),(q,0.4),(r,0.6)\}$ of a set $S = \{p,q,r\}$.*

Definition 2.2. *[9] Let S be a fuzzy subset in a set M, the strongest fuzzy relation on M, that is a fuzzy relation T with respect to S given by $T(x,y) = min\,\{S(x),S(y)\}$ for all x and y in M.*

Definition 2.3. *[1] Let M be any nonempty set, N be any set and $f : N \to M \times M$ be any function. Then P is a fuzzy subset of M, T is a fuzzy relation on M with respect to P and Q is a fuzzy subset of N such that*

$$Q(n) \ \le T(x,y)_{n \in f^{-1}(x,y)}.$$

Then the ordered triple $F_g = (P,Q,f)$ is called a fuzzy graph $(f_g,)$ where the elements of P are called fuzzy points $(f_p s)$ or fuzzy vertices and the elements of Q are called fuzzy lines or fuzzy edges of the fuzzy graph$F$. If $f(n) = (x,y)$, then the fuzzy points $(x,P(x)),(y,P(y))$ are called fuzzy adjacent points and fuzzy point $(x,P(x))$, fuzzy line $(n,Q(n))$ are called incident with each other. If two district fuzzy lines $(n_1,Q(n_1))$ and $(n_2,Q(n_2))$ are incident with a common fuzzy point, then they are called fuzzy adjacent lines.

Definition 2.4. *[1] A fuzzy line joining a fuzzy point to itself is called a fuzzy loop.*

Definition 2.5. *[1] Let $F_g = (P,Q,f)$ be a fuzzy graph. If more than one fuzzy line joining two fuzzy vertices is allowed, then the fuzzy graph F_g is called a fuzzy pseudo graph.*

Definition 2.6. *[1] $F_g = (P,Q,f)$ is called a fuzzy simple graph (f_{sig}) if it has neither fuzzy multiple lines nor fuzzy loops.*

Example 2. *$F_g = (P,Q,f)$, where $M = (x_1,x_2,x_3,x_4,x_5\}$ $N = (a,b,c,d,e,h,g)$ and $f : N \to M \times M$ is defined by $f(a) = (x_1,x_2)$, $f(b) = (x_2,x_2)$, $f(c) = (x_2,x_3)$, $f(d) = (x_3,x_4)$, $f(e) = (x_3,x_4)$, $f(h) = (x_4,x_5)$, $f(g) = (x_1,x_5)$. A fuzzy subset $A = \{(x_1,0.3),(x_2,0.5),(x_3,0.6),(x_4,0.7),(x_5,0.9)\}$ of X.*

Definition 2.7. *[1] The fuzzy graph $F_{g_1} = (P_1,Q_1,f)$ is called a fuzzy subgraph (f_{sg}) of $F_g = (P,Q,f)$ if $P_1 \subseteq P$ and $Q_1 \subseteq Q$.*

Definition 2.8. *[1] Let $F_g = (P, Q, f)$ be a fuzzy graph. Then the degree of a fuzzy vertex is defined by*

$$deg(x) = \sum_{n \in f^{-1}(x,y)} Q(n) + 2 \sum_{n \in f^{-1}(x,x)} Q(n).$$

Definition 2.9. *[1] Let $F_g = (P, Q, f)$ be a fuzzy graph. The total degree of fuzzy vertex x is defined by $deg_T(x) = deg(x) + P(x)$ for all x in M.*

Definition 2.10. *[1] The minimum degree of the fuzzy graph $F_g = (P, Q, f)$ is $\delta(F_g) = \cap \{deg(x) : x \in M\}$ and the maximum degree of F_g is $\Delta(F_g) = \cup \{deg(x) : x \in M\}$.*

Definition 2.11. *[2] Let $F_g = (P, Q, f)$ be a fuzzy graph. Then the order of fuzzy graph F_g is defined to be $O(F_g) = \sum_{x \in M} P(x)$.*

Definition 2.12. *[2] Let $F_g = (P, Q, f)$ be a fuzzy graph. Then the size of the fuzzy graph F_g is defined to be $S(F_g) = \sum_{x \in N} Q(n)$.*

Definition 2.13. *[3] A fuzzy graph $F_g = (P, Q, f)$ is called fuzzy k_1- regular graph if $deg(v) = k_1$ for all $v \in V$.*

Definition 2.14. *[3] A fuzzy graph F_g is fuzzy k_1- totally regular graph if each vertex of F_g has the same total degree k_1.*

Theorem 2.1. *[1] The sum of the degree of all fuzzy vertices in a fuzzy graph is equal to twice the sum of the membership value of all fuzzy edges. That is $\sum_{x \in M} deg(x) = 2S(F_g)$.*

Definition 2.15. *[1] A fuzzy graph $F_g = (P, Q, f)$ is called a fuzzy complete graph (f_{cg}) if every pair of distinct fuzzy vertices are fuzzy adjacent and*

$$Q(n) = T(x, y)_{n \in f^{-1}(x,y)}$$

for all x and y in M.

Definition 2.16. *[1] A fuzzy graph $F_g = (P, Q, f)$ is a fuzzy strong graph if*

$$Q(n) = T(x, y)_{n \in f^{-1}(x,y)}$$

for all $n \in N$.

3. APEX OF P AND BOTOM OF P

Definition 3.1. *Let $F_g = (P, Q, f)$ be a f_g. Then the bottom of P is defined by $d(P) = min\{P(x) : x \in M\}$.*

Definition 3.2. *Let $F_g = (P, Q, f)$ be a f_g. Then the apex of P is defined by $h(P) = max\{P(x) : x \in M\}$.*

Example 3. *In Figure 3.1. we have:*

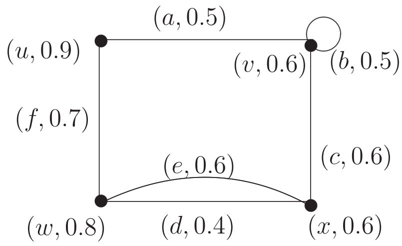

Fig 3.1. *Fuzzy graph* F_g

Here $d(P) = 0.6$, $h(P) = 0.9$.

Remark 3.1. *Clearly $d(P) \leq h(P)$ and $Q(n) \leq h(P)$.*

Theorem 3.1. *Let $F_g = (P, Q, f)$ be any f_g with respect to set M and N where $|M| = v$ and $|N| = e$. Then $S(F_g) \leq eh(P)$.*

Proof. Suppose $F_g = (P, Q, f)$ is any f_g with $v-$ fuzzy vertices. Obviously, $Q(n) \leq h(P) \Rightarrow \sum_{n \in N} Q(n) \leq \sum_{n \in N} h(P) \Rightarrow S(F_g) \leq eh(P)$. □

Corollary 3.1. *Let $F_g = (P, Q, f)$ be any f_g with respect to set M and N where $|M| = v$ and $|N| = e$. Then $\sum_{x \in M} deg(x) \leq 2eh(P)$.*

Theorem 3.2. *Let $F_g = (P, Q, f)$ be any f_{sig} $v-$fuzzy vertices. Then*

$$\frac{2S(F_g)}{v(v-1)} \leq h(P).$$

Proof. By Theorem 3.1 we have:

$$S(F_g) \geq eh(P) \Rightarrow \frac{S(F_g)}{e} \geq h(P) \Rightarrow \frac{2S(F_g)}{v(v-1)} \leq h(P).$$

□

Corollary 3.2. *Let $F_g = (P, Q, f)$ be any f_{sig} $v-$fuzzy vertices. Then*

$$\sum_{x \in M} deg(x) \leq v(v-1)h(P).$$

Theorem 3.3. *Let $F_g = (P, Q, f)$ be any f_{cg} with $v-$fuzzy vertices. Then*

$$\frac{2S(F_g)}{v(v-1)} \leq h(P).$$

Proof. By Theorem 3.1 , $\frac{S(F_g)}{e} \leq h(P) \Rightarrow S(F_g) \leq eh(P)$. Since F_g is f_{cg}, $S(F_g) \leq \frac{v(v-1)h(P)}{2}$ which implies that $\frac{2S(F_g)}{v(v-1)} \leq h(P)$. □

Theorem 3.4. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$fuzzy vertices and P be constant function if and only if $d(P) = \frac{2S(F_g)}{v(v-1)} = h(P)$.*

Proof. Assume that F_g is a f_{cg} with $p-$ fuzzy vertices and let $P(x) = t$ for all x in M. That is $Q(n) = T(x, y)_{n \in f^{-1}(x,y)}$ for all x and y in M. Then: $Q(n) = P(x) \cap P(y) = t$ for all x and y in M, so

$$\begin{aligned} d(P) &= Q(n) = h(P) \\ &\Rightarrow \sum_{n \in N} d(P) = \sum_{n \in N} Q(n) = \sum_{n \in N} h(P) \\ &\Rightarrow ed(P) = S(F_g) = eh(P) \\ &\Rightarrow \frac{v(v-1)}{2} d(P) = S(F_g) = \frac{v(v-1)}{2} h(P). \end{aligned}$$

Hence $d(P) = \frac{2S(F_g)}{v(v-1)} = h(P)$.

Conversely, assume that $d(P) = \frac{2S(F_g)}{v(v-1)} = h(P)$. Suppose F_g is not f_{cg}. By Theorem 3.2, $\frac{2S(F_g)}{v(v-1)} \leq h(P)$, which is a contradiction. □

Corollary 3.3. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices and P be $t-$constant function. Then $\frac{2S(F_g)}{v(v-1)} = t$.*

Corollary 3.4. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices and P be a constant function. Then $\sum_{x \in M} deg(x) = v(v-1)h(P) = v(v-1)d(P)$.*

Theorem 3.5. *Let $F_g = (P, Q, f)$ be any f_g with respect to set M and N where $|M| = v$ and $|N| = e$. Then*

(i) $\dfrac{O(F_g) + S(F_g)}{v + e} \leq h(P)$;

(ii) $\dfrac{O(F_g) - S(F_g)}{v - e} \leq h(P)$.

Proof. By Theorem 3.1, $S(F_g) \leq eh(P)$. Obviously,

$$P(x) \leq h(P) \Rightarrow \sum_{x \in M} P(x) \leq \sum_{x \in M} h(P) \Rightarrow O(F_g) \leq vh(P).$$

Hence $\dfrac{O(F_g) + S(F_g)}{v + e} \leq h(P)$. Similarly, we can prove another part. □

Theorem 3.6. *Let $F_g = (P, Q, f)$ be any f_{sig} with $v-$ fuzzy vertices. Then*

(i) $O(F_g) + S(F_g) \leq \dfrac{v(v+1)h(P)}{2}$;

(ii) $O(F_g) - S(F_g) \leq \dfrac{v(3-v)h(P)}{2}$.

Remark 3.2. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices. Then*

(i) $O(F_g) + S(F_g) \leq \dfrac{v(v+1)h(P)}{2}$;

(ii) $O(F_g) - S(F_g) \leq \dfrac{v(3-v)h(P)}{2}$.

Theorem 3.7. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices and P be constant function. Then*

(i) $\dfrac{v(v+1)d(P)}{2} = O(F_g) + S(F_g) = \dfrac{v(v+1)h(P)}{2}$;

(ii) $\dfrac{v(v-3)d(P)}{2} = O(F_g) - S(F_g) = \dfrac{v(3-v)h(P)}{2}$.

Proof. By Theorem 3.4, we have

$$d(P) = \frac{2S(F_g)}{v(v-1)} = h(P) \Rightarrow \frac{v(v-1)d(P)}{2} = S(F_g) = \frac{v(v-1)h(P)}{2}.$$

Obviously,

$$\begin{aligned} d(P) &= P(x) = h(P) \\ &\Rightarrow \sum_{x \in M} d(P) = \sum_{x \in M} P(x) = \sum_{x \in M} h(P) \\ &\Rightarrow vd(P) = O(F_g) = vh(P). \end{aligned}$$

Hence $\dfrac{v(v+1)d(P)}{2} = O(F_g) + S(F_g) = \dfrac{v(v+1)h(P)}{2}$.

Similarly, we can prove another part. □

Corollary 3.5. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices and P be $t-$constant function. Then*

(i) $O(F_g) + S(F_g) = \dfrac{v(v+1)t}{2}$;

(ii) $O(F_g) - S(F_g) = \dfrac{v(3-v)t}{2}$.

Theorem 3.8. *Let $F_g = (P, Q, f)$ be any f_g with respect to set M and N where $|M| = v$ and $|N| = e$. Then*

$$\frac{\sum_{x\in M} deg_T(x) + O(F_g)}{v+e} \leq 2h(P).$$

Theorem 3.9. *Let $F_g = (P, Q, f)$ be a f_{sig} with $v-$ fuzzy vertices. Then*

$$\sum_{x\in M} deg_T(x) + O(F_g) \leq v(v+1)h(P).$$

Theorem 3.10. *Let $F_g = (P, Q, f)$ be f_{cg} with $v-$ fuzzy vertices and P be constant function. Then*

$$\sum_{x\in M} deg_T(x) = v^2 h(P) = v^2 d(P).$$

Proof. By Theorem 3.7,

$$\frac{v(v+1)d(P)}{2} = O(F_g) + S(F_g) = \frac{v(v+1)h(P)}{2}$$

which implies that

$$\sum_{x\in M} deg_T(x) + O(F_g) = v(v+1)h(P)$$

which implies that

$$\sum_{x\in M} deg_T(x) = v^2 h(P) = v^2 d(P).$$

□

Theorem 3.11. *If F_g is a fuzzy k_1- regular graph with $v-$ fuzzy vertices with $e \leq \dfrac{v(v-1)}{2}$, then $h(P) \geq \dfrac{k_1}{v-1}$.*

Proof. Suppose F_g is a fuzzy k_1- regular graph with $v-$ fuzzy vertices. Here $d(x) = k_1$ for all $x \in M$, $\sum_{x\in M} d(x) = \sum_{x\in M} k_1 = vk_1$. We get $2S(F_g) = vk_1$ implies that $S(F_g) = \frac{vk_1}{2}$. By Theorem 3.2, $\frac{vk_1}{2} \leq \frac{v(v-1)h(P)}{2}$ which implies that $\frac{k_1}{v-1} \leq h(P)$ implies that $h(P) \geq \frac{k_1}{v-1}$.

□

Remark 3.3. *But the converse of the above theorem need not be true. For example, consider the fuzzy graph F_g given in Fig 3.2.*

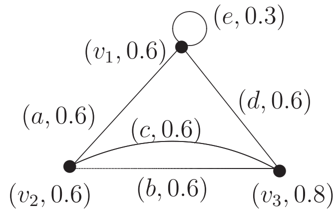

Fig 3.2 *Fuzzy graph* F_g

h(P)=0.8, $\frac{k_1}{v-1} = 0.6$, F_g is a $1.8-$ regular graph and $h(P) \geq \frac{k_1}{v-1}$ but $e \succ \frac{v(v-1)}{2}$.

Theorem 3.12. *For any fuzzy graph F_g, $\delta(F_g) \leq \frac{2eh(P)}{v}$.*

Proof. For any fuzzy graph, $\delta(F_g) \leq \frac{2S(F_g)}{v}$. By Theorem 3.1, $S(F_g) \leq eh(P)$, which implies that $\delta(F_g) \leq \frac{2eh(P)}{v}$.

□

Theorem 3.13. *Let $F_g = (P, Q, f)$ be a f_{sig} with v–fuzzy vertices. Then*

$$\delta(F_g) \leq (v-1)h(P).$$

Proof. For any f_g, we have $\delta(F_g) \leq \frac{2S(F_g)}{v}$. By Theorem 3.2 ,

$$\frac{2S(F_g)}{v} \leq (v-1)h(P),$$

which implies that $\delta(F_g) \leq (v-1)h(P)$. □

Theorem 3.14. *Let $F_g = (P, Q, f)$ be a f_{cg} with $v-$ fuzzy vertices and P be $t-$constant function. Then $\Delta(F_g) = \delta(F_g) = (v-1)h(P) = (v-1)d(P)$.*

Proof. By corollary 3.4, $deg(x) = (v-1)t$ for all $x \in M$ and $h(P) = d(P) = k_1$ also $\Delta(F_g) = \delta(F_g) = (v-1)k_1$, implies that:

$$\begin{aligned} \frac{\delta(F_g)}{(v-1)} &= \frac{\Delta(F_g)}{(v-1)} = k_1 \\ &\Rightarrow h(P) = d(P) = \frac{\delta(F_g)}{v-1} = \frac{\Delta(F_g)}{v-1} \end{aligned}$$

implies that $\delta(F_g) = \Delta(F_g) = (v-1)h(P) = (v-1)d(P)$. □

Theorem 3.15. *If $F_g = (P, Q, f)$ is a fuzzy c_1- totally regular graph with $v-$ fuzzy vertices. Then $O(F_g) \geq v[c_1 - (v-1)h(P)]$.*

Proof. For any f_g, we have $S(F_g) = \dfrac{vc_1 - O(F_g)}{2}$. By Theorem 3.2,

$$\begin{aligned} S(F_g) &\leq \frac{v(v-1)h(P)}{2} \\ &\Rightarrow \frac{vc_1 - O(F_g)}{2} \leq \frac{v(v-1)h(P)}{2} \\ &\Rightarrow vc_1 - v(v-1)h(P) \leq O(F_g). \end{aligned}$$

Hence $O(F_g) \geq v[c_1 - (v-1)h(P)]$. □

Theorem 3.16. *If $F_g = (P, Q, f)$ is a fuzzy c_1- totally regular graph with $v-$ fuzzy vertices. Then $S(F_g) \geq \dfrac{v(c_1 - h(P))}{2}$.*

Proof. For any f_g,

$$\begin{aligned} S(F_g) &= \frac{(vc_1 - O(F_g))}{2} \\ &\Rightarrow -2S(F_g) + vc_1 = O(F_g) \\ &\Rightarrow -2S(F_g) + vc_1 \leq c_1 h(P) \\ &\Rightarrow S(F_g) \geq \frac{v(c_1 - h(P))}{2}. \end{aligned}$$

□

Theorem 3.17. *If $F_g = (P, Q, f)$ is a fuzzy c_1- totally regular graph with $v-$ fuzzy vertices. Then $S(F_g) \leq \dfrac{v(c_1 - d(P))}{2}$.*

Proof. For any fuzzy graph ,

$$\begin{aligned} S(F_g) &= \frac{(vc_1 - O(F_g))}{2} \\ &\Rightarrow -2S(F_g) + vc_1 = O(F_g) \\ &\Rightarrow -2S(F_g) + vc_1 \geq vd(P) \\ &\Rightarrow S(F_g) \leq \frac{v(c_1 - d(P))}{2}. \end{aligned}$$

□

Theorem 3.18. *If $F_g = (P, Q, f)$ is both fuzzy k_1 regular graph and fuzzy c_1- totally regular graph with $v-$ fuzzy vertices. Then $h(P) \geq \frac{k_1}{v-1}$.*

Proof. By Theorem 3.15, $O(F_g) \geq v[c_1 - (v-1)h(P)]$. For any f_g,

$$\begin{aligned} O(F_g) &= v(c_1 - k_1) \\ &\Rightarrow v[c_1 - (v-1)h(P)] \leq v(c_1 - k_1) \\ &\Rightarrow c_1 - (v-1)h(P) \leq (c_1 - k_1) \\ &\Rightarrow k_1 \leq (v-1)h(P) \\ &\Rightarrow h(P) \geq \frac{k_1}{v-1}. \end{aligned}$$

□

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