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THE SOLUTION OF HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY BY A NEW STRATEGY OF ADOMAIN METHOD

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ABSTRACT. We investigates the efficiency of using a proposed modification to Adomain Decomposition Method (ADM) in order to solve third order boundary value problems into higher-order ordinary differential equation. Firstly, a new modification (MADM) is introduced. Secondly, the modified ADM form is applied in order to construct numerical solutions of non-linear problems. From the illustrated examples as well as the obtained results, it can be concluded that the MADM is proved to be an effective algorithm.

1. INTRODUCTION

There are several forms of physical phenomena in physic, engineering and other areas of science are mathematically shown as boundary value problems linked with non-linear third order ordinary differential equations of the kind:

$$y''' = f(x, y, y', y'') .$$

In the literature, there are few studies that investigate numerical solutions of higher-order boundary value problems [1–5]. These kinds of problems are proved to be unique by [6]. Using the ADM method, which involves Dirichlet, Neumann or Robin conditions, different types of boundary value problems

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were solved. Adomain Method was currently used by Deeba et al., [3] in order to obtain analytical and numerical solutions of Breatu equation. Additionally, Wazwaz [7, 8] proposed a reliable algorithm to solve non-linear boundary value problems. The achievement of the flow up solutions by mixed boundary circumstances has been Wazwaz's additional strong confirmation that has demonstrated the reliability and trustworthiness of decomposition method use Wazwaz [9]. The use of decomposition method by Wazwaz [10], was actually proved by giving the numerical outcomes that were of the fifth order boundary value problems and the use of the sixthâĂŞdegree B-spline method showed also its efficiency in making a noticeable attainment in displaying the differences and similarities (contrast) between the errors. According to the numerical outcomes given by Wazwaz for the purpose of demonstrating the use of decomposition method, and the use of sixth-degree B-spline method, it becomes so clear those numerical outcomes indicate that unlike the B-spline method, the decomposition method is easier to use and more accurate.

In this article, we give a strategy to the differential operator and therefore the inverse operator to solve third-order boundary value problems into higher-order ordinary differential equations. This research aims at solving a particular kind of equations in general.

2. Discussion of MADM

To achieve the main aim of this article, we write the general higher order boundary ordinary differential equation taken as:

$$y^{(n+2)} = f(x, y, y', ..., y^{(n+1)}), \qquad n \geq 1, \tag{2.1}$$

with the condition given as:

$$y(a) = b_0, y'(a) = b_1, ..., y^{(n)}(a) = b_n, y^{(n-1)}(0) = c_n,$$

where f is a differential operator of linear or non-linear of order less than $(n + 2)$, and $a, b_0, b_1, ..., b_n, c_n,$ are real finite constant.

We introduce the new differential operator, as:

$$L(.) = x^{-2}\frac{d}{dx}x^{3-m}\frac{d^{2-m}}{dx^{2-m}}x^{3m-1}\frac{d^m}{dx^m}x^{-2m}\frac{d^{n-1}}{dx^{n-1}}(.), \qquad n \geq 1, \tag{2.2}$$

where $m = 0$ or $m = 1$.

The problem (2.1) is then written in an operator form as:

$$L(.) = f(x, y, y', ..., y^{(n+1)}), \tag{2.3}$$

we assumed that the operator L is inverter and has an inverse L^{-1}, which is an integral operator

$$L^{-1}(.) = \underbrace{\int_a^x \int_a^x ... \int_a^x}_{(n-1)} x^{2m} \underbrace{\int_a^x}_{(m)} x^{1-3m} \underbrace{\int_a^x ... \int_a^x}_{(2-m)} x^{m-3} \int_0^x x^2(.), \ \underbrace{dxdx...dx}_{(n+2)-times}.$$

Applying L^{-1} to both sides of (2.3) and using the condition, we get

$$y(x) = \beta(x) + L^{-1} f(x, y, y', ..., y^{(n+1)}), \tag{2.4}$$

where the function $\beta(x)$ which represent the term comes out from conditions. The method of Adomian decomposition introduce the solution $y(x)$

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{2.5}$$

and the non-linear $f(x, y, y', y'', ..., y^{(n+1)})$ by an infinite series of polynomials

$$f(x, y, y', y'', ..., y^{(n+1)}) = \sum_{n=0}^{\infty} A_n, \tag{2.6}$$

where $y_n(x)$ of the solution $y(x)$ and A_n are Adomain polynomials. By

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{n} \lambda^i y_i \right) \right]_{\lambda=0}, n = 0, 1, 2, ..., \tag{2.7}$$

which gives

$$\begin{aligned}
A_0 &= N(y_0), \\
A_1 &= y_1 N'(y_0), \\
A_2 &= y_2 N'(y_0) + \frac{1}{2} y_1^2 N''(y_0), \\
A_3 &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{1}{3!} y_1^3 N'''(y_0), \\
&\cdots
\end{aligned}$$

Substituting from equation (2.5) and equation (2.6) into equation (2.4), we have

$$\sum_{n=0}^{\infty} y_n(x) = \beta(x) + L^{-1} \sum_{n=0}^{\infty} A_n \,. \tag{2.8}$$

The components y_n can be specified as:

$$\begin{aligned} y_0 &= \beta(x), \\ y_{n+1} &= L^{-1} A_n, \;\; n \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} y_0 &= \beta(x), \\ y_1 &= L^{-1} A_0, \\ y_2 &= L^{-1} A_1, \\ y_3 &= L^{-1} A_2 \\ & \dots \end{aligned}$$

As can be noticed in (2.7) and (2.8), we can determine the components $y_n(x)$. Consequently, we can immediately obtain the series solution of $y(x)$ in (2.5). To put it numerically, the $n-$ term approximate

$$\phi_n = \sum_{k=0}^{n-1} y_k,$$

is to be used to approximate the accurate solution. In order to test the validity of the above presented approach, we can test it on a variety of several liner and non-linear boundary value problem.

3. Implementation of MADM

In this section, when n=1,3, in the differential operator (2.2). We will give an example of non-linear boundary value problems of third order when m=0, and an example of fifth order when m=1, and in every case one boundary condition.

3.1. **Example.** We will give an example on the equation (2.1), when n=1. We have:

$$y''' = 2e^{-3y}, \tag{3.1}$$

$$y(0) = 0, y(1) = 0.693, y'(1) = 0.5\,.$$

It can be rewritten equation (3.1), as

$$Ly = 2e^{-3y}, \tag{3.2}$$

where the differential operator (2.2) at m=0, we get

$$L(.) = x^{-2}\frac{d}{dx}x^{3}\frac{d^2}{dx^2}x^{-1}(.),$$

and

$$L^{-1}(.) = x\int_1^x\int_1^x x^{-3}\int_0^x x^2(.), dxdxdx\,.$$

Take the L^{-1} on both side of (3.2) and using condition yields

$$L^{-1}(Ly) = L^{-1}(2e^{-3y}),$$

$$y(x) = 0.886x - 0.193x^2 + L^{-1}(2e^{-3y})\,, \tag{3.3}$$

we can determine the components $y_n(x)$ for $y(x)$ into (3.3)

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) &= 0.886x - 0.193x^2 + L^{-1}(2e^{-3y}),\\ y_0 &= 0.886x - 0.193x^2,\\ y_{n+1} &= 2L^{-1}A_n, n \geq 0,\end{aligned}$$

we get the series of $2e^{-3y}$, by Adomian polynomials

$$\begin{aligned}A_0 &= 2e^{-3y_0},\\ A_1 &= 2e^{-3y_0}y_1\,.\end{aligned}$$

Then

$$\begin{aligned}y_0 &= 0.886x - 0.193x^2,\\ y_1 &= 0.117997\,x - 0.316379\,x^2 + 0.333333\,x^3 + ... + 0.000241362\,x^{13},\\ y_2 &= 0.00123928\,x - 0.00305195\,x^2 + 0.0098331\,x^4 + ... + 6.40812\,10^{-9}\,x^{26}\,.\end{aligned}$$

The solution in a series from is given by

$$y(x) = y_0 + y_1 + y_2 = 1.00524\,x - 0.512431\,x^2 + 0.333333\,x^3 + ... + 6.40812\,10^{-9}\,x^{26},$$

The next Table 1. and Figure 1. offer comparison between the exact solution and MADM solution.

x	Exact	MADN	Error
0.0	0.000000	0.000000	0.000000
0.2	0.182322	0.182912	0.000590
0.4	0.336472	0.337025	0.000553
0.6	0.470004	0.470288	0.000284
0.8	0.587787	0.587794	0.000007
1.0	0.693147	0.693000	0.000147

TABLE 1. A numerical comparative analysis between the exact solution $y(x) = Log[x+1]$, and approximate solution$\sum_{n=0}^{1} y_n$.

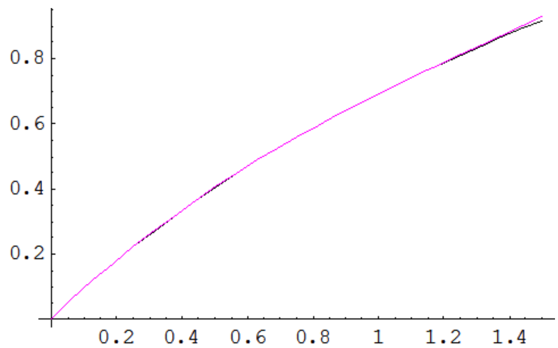

FIGURE 1. Showing between the approximate solution with the exact solution $y(x) = Log[x+1]$.

3.2. **Example.** Take into account the nonlinear boundary value problem at n=3 in equation (2.1):

$$y^{(5)} = e^x - e^y, \tag{3.4}$$

with the conditions:

$$y''(0) = 0, y(0.5) = 0.5, y'(0.5) = 1, y''(0.5) = y'''(0.5) = 0,$$

it can be rewritten equation (3.4) as

$$L(y) = e^x - e^y, \tag{3.5}$$

when m = 1, in an operator(2.2), we get:

$$L(.) = x^{-2}\frac{d}{dx}x^2\frac{d}{dx}x^2\frac{d}{dx}x^{-2}\frac{d^2}{dx^2}(.),$$

and L^{-1} as:

$$L^{-1}(.) = \int_{0.5}^{x}\int_{0.5}^{x} x^2 \int_{0.5}^{x} x^{-2} \int_{0.5}^{x} x^{-2} \int_{0}^{x} x^2(.), dxdxdxdxdx.$$

Take the L^{-1} to both side of (3.5), we get

$$y(x) = x + L^{-1}(e^x - e^y)\,. \tag{3.6}$$

The components $y_n(x)$ for $y(x)$ into(3.6) using the series of the decomposition for $x + e^x$ and the series of polynomial for $-e^y$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(x) &= 0.000238412 + 0.998822\,x + 4.27319\,10^{-20}\,x^2 + \cdots + \\ &\quad 7.64716\,10^{-13}\,x^{15} + L^{-1}\sum_{n=0}^{1} A_n\,.\end{aligned}$$

The nonlinear term e^y, we get it as

$$\begin{aligned} A_0 &= e^{y_0}, \\ A_1 &= e^{y_0} y_1, \end{aligned}$$

we have

$$\begin{aligned} y_1 &= -0.000238417 + 0.00117759\,x + 3.25261\,10^{-19}\,x^2 + ... + 0.\,x^4\,\log(x), \\ y_2 &= 4.6299\,10^{-9} - 2.70623\,10^{-8}\,x + 3.78992\,10^{-7}\,x^3 + ... + 0.\,x^{19}\,\log(x), \end{aligned}$$

then, the solution of general given by

$$y(x) = y_0 + y_1 + y_2 = 3.82414\,10^{-13} + 1.\,x + 3.67993\,10^{-19}\,x^2 + ... + 0.\,x^{19}\,\log(x)\,.$$

We can get from table 2 that the (MADM) is effective to the exact solution.

x	Exact	MADN	Error
0.1	0.10000	0.10000	0.00000
0.3	0.30000	0.30000	0.00000
0.5	0.50000	0.50000	0.00000
0.7	0.70000	0.70000	0.00000
0.9	0.90000	0.90000	0.00000
1.0	1.00000	1.00000	0.00000

TABLE 2. Comparison between the exact solution $y(x) = x$, and approximate solution$\sum_{n=0}^{2} y_n$.

4. CONCLUSION

We tested the validity of ADM for solving many application problems and it has been proved that the ADM is successful. Al though ADM can be used with simple calculation, it has some difficulties with respect to solving boundary problems. Many approaches have been presented to deal with such difficulties, however they require additional computational work due to the exact that all boundary conditions are not included in the canonical form.

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