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# DEGREE EXPONENT ADJACENCY POLYNOMIAL OF SOME GRAPHS

PUSHPALATHA MAHALANK[1], H. S. RAMANE, AND A. R. DESAI

ABSTRACT. The degree exponent adjacency polynomial of a graph $G$ is the characteristic polynomial of the degree exponent adjacency matrix $DEA(G)$, whose $(i,j)$-th entry is $d_i^{d_j}$, whenever the vertex $v_i$ is adjacent to vertex $v_j$, otherwise it is zero, where $d_i$ is the degree of a vertex $v_i$. In this paper, we obtain the degree exponent adjacency polynomial of graphs obtained from regular graphs.

## 1. INTRODUCTION

The most studied graph polynomial is the characteristic polynomial of its adjacency matrix [2]. The other polynomials of graphs are the characteristic polynomial of Laplacian matrix [6], signless Laplacian matrix [3] and distance matrix [1]. Recently many other graph polynomials have been studied such as degree sum polynomial [5, 10], degree subtraction polynomial [9], degree sum adjacency polynomial [13], degree subtraction adjacency polynomial [8], Zagreb polynomial [7] etc. In [11] the degree exponent polynomial is obtained for variuos graphs. In this paper, we introduce the degree exponent adjacency polynomial of a graph and obtain it for some graph operations of regular graphs.

[1]*corresponding author*

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**Definition 1.1.** *Let $G$ be a graph with $n$ vertices and $m$ edges. The degree of a vertex $v_i$ is the number of edges incident to it and is denoted by $d_i = deg_G(v_i)$. If the degrees of all vertices are equlal to $r$, then $G$ is called an $r$-regular graph.*

**Definition 1.2.** *Let the vertex set of $G$ be $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set be $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of $G$ is a square matrix $A(G) = [a_{ij}]$ of order $n$, where*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.3.** *The characteristic polynomial of $A(G)$ is called adjacency polynomial and is defined as*

$$\phi(G : \lambda) = det(\lambda I - A(G)),$$

*where $I$ is an identity matrix.*

**Definition 1.4.** *The incident matrix of $G$ is a $n \times m$ matrix $B(G) = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident to } e_j \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 1.1.** *[2] If $G$ is a regular graph of degree $r$ and $L(G)$ is the line graph of $G$, then*

(i) $B(G)B(G)^T = A(G) + rI$ , *and*
(ii) $B(G)^T B(G) = 2I + A(L(G))$.

**Definition 1.5.** *The degree exponent matrix of a graph $G$ is an $n \times n$ matrix $DE(G) = [de_{ij}]$, where*

$$de_{ij} = \begin{cases} d_i^{d_j} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

**Definition 1.6.** *We define here degree exponent adjacency matrix as $n \times n$ matrix $DEA(G) = [dea_{ij}]$, where*

$$dea_{ij} = \begin{cases} d_i^{d_j} & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 1.7.** *The characteristic polynomial of $DEA(G)$ is called the degree exponent adjacency polynomial of $G$ and is defined as*

$$\psi(G : \lambda) = det(\lambda I - DEA(G)),$$

*where $I$ is an identity matrix.*

**Lemma 1.2.** *[2] If $M$ and/or $Q$ is a nonsingular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N| = |Q||M - NQ^{-1}P|.$$

## 2. Degree exponent adjacency polynomial of some graphs

**Theorem 2.1.** *Let G be an $r$-regular graph on $n$ vertices. Then*

$$\psi(G:\lambda) = r^{nr}\,\phi\left(G:\frac{\lambda}{r^r}\right).$$

*Proof.* $DEA(G) = r^r A(G)$. Hence

$$\begin{aligned} \psi(G:\lambda) &= |\ \lambda I - DEA(G)\ | \\ &= |\ \lambda I - r^r A(G)\ | \\ &= (r^r)^n \left|\frac{\lambda}{r^r} I - A(G)\right| \\ &= r^{nr}\phi\left(G:\frac{\lambda}{r^r}\right). \end{aligned}$$

□

**Definition 2.1.** *A subdivision graph of $G$ is $S(G)$ obtained by inserting a new vertex on each edge of $G$ [4]. If $u \in V(G)$ then $deg_{S(G)}(u) = deg_G(u)$ and if $v$ is subdivided vertex then $d_{S(G)}(v) = 2$.*

**Theorem 2.2.** *Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then*

$$\psi(S(G):\lambda) = r^{2n}\,2^{nr}\lambda^{m-n}\,\phi\left(G:\frac{\lambda^2\ - r^3\,2^r}{r^2\,2^r}\right).$$

*Proof.* $DEA(S(G)) = \begin{bmatrix} O & 2^r B(G)^T \\ r^2 B(G) & O \end{bmatrix}$, where $O$ is the zero matrix.

Therefore,

$$\psi(S(G):\lambda) = \begin{vmatrix} \lambda I_m & -2^r B(G)^T \\ -r^2 B(G) & \lambda I_n \end{vmatrix}.$$

Using Lemma 1.2

$$\begin{aligned}
\psi(S(G):\lambda) &= \lambda^m \left|\lambda\, I_n - r^2\, 2^r\, \frac{B(G)B(G)^T}{\lambda}\right| \\
&= \lambda^{m-n}\, |\, \lambda^2\, I_n - r^2\, 2^r\, (A(G)+rI)\, | \\
&= \lambda^{m-n}\, |\, (\lambda^2 - r^3\, 2^r)\, I_n - r^2\, 2^r\, A(G)\, | \\
&= \lambda^{m-n}\, (r^2\, 2^r)^n \left|\frac{\lambda^2\ -\ r^3\, 2^r}{r^2\, 2^r}\, I - A(G)\right| \\
&= r^{2n}\, 2^{nr}\, \lambda^{m-n}\, \phi\left(G\colon \frac{\lambda^2\ - r^3\, 2^r}{r^2\, 2^r}\right).
\end{aligned}$$

□

**Definition 2.2.** *The semitotal point graph of $G$ is $T_1(G)$ with vertex set $V(G) \cup E(G)$ and two vertices in $T_1(G)$ are adjacent if they are both adjacent vertices or one is a vertex and other is an incident edge to it [12]. If $u \in V(G)$, then $deg_{T_1(G)}(u) = 2\ deg_G(u)$ and if $e \in E(G)$, then $deg\ _{T_1(G)}(e) = 2$.*

**Theorem 2.3.** *Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then*

$$\psi(T_1(G)\colon \lambda) = \lambda^{m-n}[(2r)^{2r}\lambda + 2^{2r}(2r)^2]^n\ \phi\left(G\colon \frac{\lambda^2\ - 2^{2r}(2r)^2 r}{(2r)^{2r}\lambda +\ 2^{2r}(2r)^2}\right).$$

*Proof.* $DEA(T_1(G)) = \begin{bmatrix} O & 2^{2r}B(G)^T \\ (2r)^2 B(G) & (2r)^{2r}A(G) \end{bmatrix}$, where $O$ is the zero matrix. Therefore

$$\psi(T_1(G)\colon \lambda) = \begin{vmatrix} \lambda I_m & -2^{2r}B(G)^T \\ -(2r)^2 B(G) & \lambda I - (2r)^{2r}A(G) \end{vmatrix}.$$

Using Lemma 1.2

$$\begin{aligned}
\psi(T_1(G)\colon \lambda) &= \lambda^m \left|\lambda\, I - (2r)^{2r}A(G) - 2^{2r}\, (2r)^2\, \frac{B(G)B(G)^T}{\lambda}\right| \\
&= \lambda^{m-n}\, |\, \lambda^2\, I - (2r)^{2r}\lambda\, A(G) - 2^{2r}\, (2r)^2\, (A(G)+rI)\, | \\
&= \lambda^{m-n}\, |\, (\lambda^2 - 2^{2r}(2r)^2 r)I - ((2r)^{2r}\, \lambda + 2^{2r}(2r)^2)A(G)\, | \\
&= \lambda^{m-n}[(2r)^{2r}\lambda + 2^{2r}(2r)^2]^n \left|\left(\frac{\lambda^2 - 2^{2r}(2r)^2 r}{(2r)^{2r}\lambda + 2^{2r}(2r)^2}\right) I - A(G)\right| \\
&= \lambda^{m-n}[(2r)^{2r}\lambda + 2^{2r}(2r)^2]^n\ \phi\left(G\colon \frac{\lambda^2 - 2^{2r}(2r)^2 r}{(2r)^{2r}\lambda + 2^{2r}(2r)^2}\right).
\end{aligned}$$

□

**Definition 2.3.** *The semitotal line graph of $G$ is $T_2(G)$ with vertex set $V(G) \cup E(G)$ and two vertices in $T_2(G)$ are adjacent if they are adjacent edges in $G$ or one is a vertex and other is an edge incident to it in $G$. If $e = uv \in E(G)$, then $deg_{T_2(G)}(e) = deg_G(u) + deg_G(v)$ and if $u \in V(G)$, then $deg_{T_2(G)}(u) = deg_G(u)$.*

**Theorem 2.4.** *Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then*

$$\psi(T_2(G)\colon \lambda) = \lambda^{n-m}\ [(2r)^{2r}\lambda + (2r)^r\ r^{2r}]^m\ \phi\left(L(G)\colon \frac{\lambda^2\ - 2(2r)^r\ r^{2r}}{(2r)^{2r}\lambda +\ (2r)^r\ (r)^{2r}}\right),$$

*where $L(G)$ is the line graph of $G$.*

*Proof.* $DEA(T_2(G)) = \begin{bmatrix} (2r)^{2r}A(L(G)) & (2r)^r B(G)^T \\ r^{2r}B(G) & O \end{bmatrix}$, where $A(L(G))$ is the adjacency matrix of the line graph of $G$ and $O$ is the zero matrix. Therefore

$$\psi(T_2(G)\colon \lambda) = \begin{vmatrix} \lambda I_m - (2r)^{2r}A(L(G)) & -(2r)^r B(G)^T \\ -r^{2r}B(G) & \lambda I_n \end{vmatrix}.$$

Using Lemma 1.2, we have

$$\begin{aligned}
\psi(T_2(G)\colon \lambda) &= \lambda^n \left|\lambda\ I_m - (2r)^{2r}A(L(G)) - \frac{(2r)^r B(G)^T\ r^{2r}B(G)}{\lambda}\right| \\
&= \lambda^n \left|\lambda\ I_m - (2r)^{2r}A(L(G)) - \frac{(2r)^r\ r^{2r}}{\lambda}(2I + A(L(G)))\right| \\
&= \lambda^{n-m}\ |\ (\lambda^2 - 2(2r)^r\ r^{2r})\ I - ((2r)^{2r}\ \lambda + (2r)^r\ r^{2r})A(L(G))\ | \\
&= \lambda^{n-m}\ [(2r)^{2r}\ \lambda + (2r)^r\ r^{2r}]^m\ \left|\left(\frac{\lambda^2\ - 2(2r)^r\ r^{2r}}{(2r)^{2r}\ \lambda +\ (2r)^r\ r^{2r}}\right)\ I\ - A(L(G))\right| \\
&= \lambda^{n-m}\ [(2r)^{2r}\lambda + (2r)^r\ r^{2r}]^m\ \phi\left(L(G)\colon \frac{\lambda^2\ - 2(2r)^r\ r^{2r}}{(2r)^{2r}\lambda +\ (2r)^r\ (r)^{2r}}\right).
\end{aligned}$$

□

**Definition 2.4.** *The vertices and edges a graph are referred as its elements. The total graph of $G$ is $T(G)$ with vertex set $V(G) \cup E(G)$ and two vertices in $T(G)$ are adjacent if the corresponding elements are adjacent or incident in $G$ [4]. If $u \in V(G)$, then $deg_{T(G)}(u) = 2\ deg_G(u)$ and if $e = uv \in E(G)$, then $deg_{T(G)}(e) = deg_G(u) + deg_G(v)$.*

**Theorem 2.5.** *Let $G$ be an $r$-regular graph on $n$ vertices and $m$ edges. Then*

$$\psi(T(G)\colon \lambda) = (2r)^{2r(m+n)} (x+2)^{m-n} \prod_{i=1}^{n} \left[x^2 - (2\lambda_i + r - 2)x + \lambda_i^2 + (r-3)\lambda_i - r\right],$$

*where $\lambda_i$, $i = 1, 2, \dots, n$ are the eigenvalues of $A(G)$ and $x = \dfrac{\lambda}{(2r)^{2r}}$.*

*Proof.* $DEA(T(G)) = \begin{bmatrix} (2r)^{2r}A(G) & (2r)^{2r}B(G) \\ (2r)^{2r}B(G)^T & (2r)^{2r}A(L(G)) \end{bmatrix}$. Therefore

$$\begin{aligned}
\psi(T(G)\colon \lambda) &= \begin{vmatrix} \lambda I - (2r)^{2r}A(G) & -(2r)^{2r}B(G) \\ -(2r)^{2r}B(G)^T & \lambda I - (2r)^{2r}A(L(G)) \end{vmatrix} \\
&= \left((2r)^{2r}\right)^{m+n} \begin{vmatrix} \frac{\lambda}{(2r)^{2r}} I - A(G) & -B(G) \\ -B(G)^T & \frac{\lambda}{(2r)^{2r}} I - A(L(G)) \end{vmatrix} \\
&= (2r)^{2r(m+n)} \begin{vmatrix} \frac{\lambda}{(2r)^{2r}} I + rI - B(G)B(G)^T & -B(G) \\ -B(G)^T & \frac{\lambda}{(2r)^{2r}} I + 2I - B(G)^T B(G) \end{vmatrix}.
\end{aligned}$$

Multiply first row by $B(G)^T$ and subtract from second row, we get

$$\psi(T(G)\colon \lambda) = (2r)^{2r(m+n)} \begin{vmatrix} \frac{\lambda}{(2r)^{2r}} I + rI - B(G)B(G)^T & -B(G) \\ -\left(\frac{\lambda}{(2r)^{2r}} + r + 1\right) B(G)^T + B(G)^T B(G) B(G)^T & \left(\frac{\lambda}{(2r)^{2r}} + 2\right) I \end{vmatrix}.$$

Multiplying second row by $\dfrac{B(G)}{\left(\frac{\lambda}{(2r)^{2r}} + 2\right)}$ and adding to first row and taking $x = \dfrac{\lambda}{(2r)^{2r}}$ we get

$$\begin{aligned}
&\psi(T(G)\colon\lambda) = (2r)^{2r(m+n)} \\
&\left|\begin{matrix} (x+r)\,I - BB^T + \dfrac{B}{(x+2)}\left(-\,(x+r+1)\,B^T + B^TBB^T\right) & O \\ -\,(x+r+1)\,B^T + B^TBB^T & (x+2)\,I \end{matrix}\right| \\
= \;& (2r)^{2r(m+n)}(x+2)^m \left|(x+r)I - BB^T + \frac{BB^T}{(x+2)}\left[-(x+r+1)I + BB^T\right]\right| \\
= \;& (2r)^{2r(m+n)}(x\;+\;2)^m \left|xI - A(G) + \frac{(A(G)+rI)}{(x+2)}\left[A(G) - (x+1)I\right]\right| \\
= \;& (2r)^{2r(m+n)}(x\;+\;2)^{m-n} \left|A(G)^2 - (x-r+3)A(G) + (x^2-(r-2)x-r)I\right| \\
= \;& (2r)^{2r(m+n)}(x+2)^{m-n} \prod_{i=1}^{n}\left[\lambda_i^2 - (x-r+3)\lambda_i + x^2 - (r-2)x - r\right] \\
= \;& (2r)^{2r(m+n)}(x+2)^{m-n} \prod_{i=1}^{n}\left[x^2 - (2\lambda_i+r-2)x + \lambda_i^2 + (r-3)\lambda_i - r\right],
\end{aligned}$$

where $\lambda_i$, $i = 1, 2, \ldots, n$ are the eigenvalues of $A(G)$. □

## 3. CONCLUSION

A degree exponent adjacency matrix of a graph is introduced and obtained its characteristic polynomial for subdivison graph, semitotal point graph, semitotal line graph and total graph of regular graphs. The study can be extended to find properties of eigenvalues of the degree exponent adjacncy polynomial.

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DEPARTMENT OF MATHEMATICS
SDM COLLEGE OF ENGINEERING AND TECHNOLOGY
DHARWAD - 580002, INDIA
*E-mail address*: mahalankpushpalatha@gmail.com

DEPARTMENT OF MATHEMATICS
KARNATAK UNIVERSITY
DHARWAD - 580003, INDIA
*E-mail address*: hsramane@yahoo.com

DEPARTMENT OF MATHEMATICS
SDM COLLEGE OF ENGINEERING AND TECHNOLOGY
DHARWAD - 580002, INDIA
*E-mail address*: ardesai69@rediffmail.com