

EQUITABLE COLORING OF JUMP GRAPH OF GRAPHS

DAFIK¹, K. PRAVEENA, M. VENKATACHALAM, AND IKA HESTI AGUSTIN

ABSTRACT. We discuss the equitable chromatic number of the jump graph of Path, Cycle, Wheel graph, Helm graph, Sunlet graph and Tadpole graph in this paper. A proper vertex coloring of a graph is equitable if the sizes of color classes differ in size by at most one. Let G be the nonempty graph. The jump graph $J(G)$ of G is the graph whose vertices are the edges of G and where two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G . The jump graph $J(G)$ is the complement of the line graph.

1. INTRODUCTION

Graph coloring is a special case of graph labeling in graph theory. Vertex coloring is the most common graph coloring problem which states that it is the way of coloring the vertices of a graph such that no two adjacent vertices share the same color. The other graph coloring problems like Edge Coloring (assigning a color to each edge so that no two adjacent edges share the same color) and Face Coloring (assigning a color to each face or region so that no two faces that share a boundary have the same color) can be transformed into vertex coloring [6]. A vertex coloring of a graph G is a mapping $C : V(G) \rightarrow S$. The elements of S are called colors; the vertices of same color form a color class. If $|S| = k$, we say that C is k -coloring. A coloring is proper if

¹corresponding author

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adjacent vertices have different colors. A graph is k -colorable if it has a proper k -coloring. The chromatic number $\chi(G)$ is the least k such that G is k -colorable [3]. Obviously, $\chi(G)$ exists as assigning distinct colors to vertices that yields a proper $|V(G)|$ -coloring. A proper coloring C of a graph G , which assigns colors to the vertices of G such that the numbers of vertices in any two color classes differ by at most one, is called an equitable coloring of G . In 1973, Meyer [8] introduced first the notion of equitable colorability. In 1998, Lih [5] surveyed the progress on the equitable coloring of graphs.

2. PRELIMINARIES

A graph G is said to be equitably k -colorable if its vertices can be partitioned into k classes V_1, V_2, \dots, V_k such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every i, j . The smallest integer k for which G is equitably k -colorable is known as the equitable chromatic number of G and denoted by $\chi_=(G)$ [2]. The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G . We call the complement of line graph $L(G)$ as the jump graph $J(G)$ of G , found in [1]. Let G be a non-empty graph. The jump graph $J(G)$ of a graph G is the graph whose vertices are the edges of G and where two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G . Since both $L(G)$ and $J(G)$ are defined on the edge set of a graph G , it follows that isolated vertices of G (if G has) play no role in line graph and jump graph transformation. We assume that the graph under consideration is nonempty and has no isolated vertices [1, 6, 7]. The jump graph $J(G)$ is totally disconnected if and only if G is a star [9]. For any integer $m \geq 4$, the wheel graph W_m is the m -vertex graph obtained by joining a vertex v to each of the $m - 1$ vertices u_1, u_2, \dots, u_{m-1} of the cycle graph C_{m-1} . The m -sunlet graph on $2m$ vertices is obtained by attaching m pendant edges to the cycle C_m and is denoted by S_m [4]. The *Helm graph* H_m is the graph obtained from an m -wheel graph by adjoining a pendant edge at each node of the cycle [4]. The (m, n) -tadpole graph denoted by $T_{m,n}$ also called dragon graph is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

In this paper we determine the equitable chromatic number of Jump graph of path $\chi_=(J(P_m))$, cycle $\chi_=(J(C_m))$, wheel graph $\chi_=(J(W_m))$, helm graph $\chi_=(J(H_m))$, sunlet graph $\chi_=(J(S_m))$, tadpole graph $\chi_=(J(T_{m,n}))$.

3. EQUITABLE COLORING OF JUMP GRAPH OF PATH

Theorem 3.1. *The Equitable chromatic number of jump graph of path $J(P_m)$, where m is any positive integer and $m > 3$ is*

$$\chi_=(J(P_m)) = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even;} \\ \frac{m-1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let $V(P_m) = \{v_1, v_2, v_3, \dots, v_m\}$ and $E(P_m) = \{v'_1, v'_2, v'_3, \dots, v'_{m-1}\}$ be the vertices and edges of P_m , respectively. By the definition of Jump graph,

$$V(J(P_m)) = E(P_m) = \{v'_1, v'_2, v'_3, \dots, v'_{m-1}\}$$

The neighbouring vertices of $J(P_m)$ are non-adjacent to each other and so same color can be assigned to these neighbouring vertices. Since, the total number of vertices is $m - 1$ and same color can be assigned to two vertices each, the total number of colors required will be either $\frac{m}{2}$ or $\frac{m-1}{2}$. The assigning of colors is done by the following ways:

(i) If m is even.

The vertices $\{v'_1, v'_2, v'_3, v'_4, \dots, v'_{m-1}\}$ are assigned the colors $\{1, 1, 2, 2, \dots, \frac{m}{2}\}$, respectively. In this case color $\frac{m}{2}$ is assigned once and all other remaining colors are assigned twice satisfying equitable coloring. Therefore, $\chi_=(J(P_m)) = \frac{m}{2}$.

(ii) If m is odd.

The vertices $\{v'_1, v'_2, v'_3, v'_4, \dots, v'_{m-1}\}$ are assigned the colors $\{1, 1, 2, 2, \dots, \frac{m-1}{2}\}$, respectively. Here each color is assigned two times which satisfies equitability. Therefore, $\chi_=(J(P_m)) = \frac{m-1}{2}$.

□

4. EQUITABLE COLORING OF JUMP GRAPH OF CYCLE

Theorem 4.1. *The Equitable chromatic number of jump graph of cycle $J(C_m)$, where m is any positive integer and $m > 3$ is*

$$\chi_=(J(C_m)) = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even;} \\ \frac{m+1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let $V(C_m) = \{v_1, v_2, v_3, \dots, v_m\}$ and $E(C_m) = \{v'_1, v'_2, v'_3 \dots, v'_m\}$ be the vertices and edges of C_m , respectively taken in cyclic order. By the definition of jump graph

$$V(J(C_m)) = E(C_m) = \{v'_1, v'_2, v'_3 \dots, v'_m\}.$$

Similar to the previous theorem, the neighbouring vertices of $J(C_m)$ are non-adjacent to each other and so same color can be assigned to these neighbouring vertices. Since, the total number of vertices is m and same color can be assigned to two vertices each, the total number of colors required will be either $\frac{m}{2}$ or $\frac{m+1}{2}$. The assigning of colors is done by the following ways:

(i) If m is even.

We assign the colors $\{1, 1, 2, 2, \dots, \frac{m}{2}\}$ to the vertices $\{v'_1, v'_2, v'_3, v'_4, \dots, v'_m\}$, respectively. In this case all the colors are assigned twice satisfying equitability. Therefore, $\chi_=(J(C_m)) = \frac{m}{2}$.

(ii) If m is odd

We assign the colors $\{1, 1, 2, 2, \dots, \frac{m+1}{2}\}$ to the vertices $\{v'_1, v'_2, v'_3, v'_4, \dots, v'_m\}$, respectively. In this case the color $\frac{m+1}{2}$ is assigned once and all other remaining colors are assigned twice which satisfies the condition of equitable coloring.

$$\text{Therefore, } \chi_=(J(C_m)) = \frac{m+1}{2}.$$

□

Corollary 4.1. *Since, $J(P_2)$, $J(P_3)$ and $J(C_3)$ are null graphs*

$$\chi_=(J(P_2)) = \chi_=(J(P_3)) = \chi_=(J(C_3)) = 1.$$

5. EQUITABLE COLORING OF JUMP GRAPH OF WHEEL GRAPH

Theorem 5.1. *The Equitable chromatic number of jump graph of wheel graph $J(W_m)$, $m > 3$ and m being any positive integer is*

$$\chi(J(W_m)) = \left\lceil \frac{2(m-1)}{3} \right\rceil.$$

Proof. Let $V(W_m) = \{v\} \cup \{u_i : 1 \leq i \leq m-1\}$ and $E(W_m) = \{v'_i : 1 \leq i \leq m-1\} \cup \{u'_i : 1 \leq i \leq m-1\}$ where v'_i is the edge $\{vu_i(1 \leq i \leq m-1)\}$ and u'_i is the edge $\{u_i u_{i+1}(1 \leq i \leq m-2)\}$ and u'_{m-1} is the edge $u_{m-1}u_1$. By definition,

$$V(J(W_m)) = E(W_m) = \{v'_i : 1 \leq i \leq m-1\} \cup \{u'_i : 1 \leq i \leq m-1\}.$$

The total number of vertices of $(J(W_m))$ is $2(m-1)$. Since, minimum degree of each vertex of $(J(W_m))$ is 3, atleast 3 colors must be used. Hence, the total number of colors used can be $\left\lceil \frac{2(m-1)}{3} \right\rceil$. The assigning of colors can be done by the following three cases:

Case 1: If $m \pmod{3} \equiv 0$.

The partition of $V(J(W_m))$ is done as follows:

$$\begin{aligned} V_1 &= \{v'_1, v'_2\} \cup \{u'_1\} \\ V_2 &= \{v'_3\} \cup \{u'_2, u'_3\} \\ V_3 &= \{v'_4, v'_5\} \cup \{u'_4\} \\ &\vdots \\ V_{\lceil \frac{2(m-3)}{3} \rceil} &= \{v'_{m-3}\} \cup \{u'_{m-4}, u'_{m-3}\} \\ V_{\lceil \frac{2(m-2)}{3} \rceil} &= \{v'_{m-2}\} \cup \{u'_{m-2}\} \\ V_{\lceil \frac{2(m-1)}{3} \rceil} &= \{v'_{m-1}\} \cup \{u'_{m-1}\} \end{aligned}$$

The number of vertices in each of the color classes is either 2 or 3 satisfying equitability.

Case 2: If $m \pmod{3} \equiv 1$.

The partition of $V(J(W_m))$ is done as follows:

$$\begin{aligned}
 V_1 &= \{v'_1, v'_2\} \cup \{u'_1\} \\
 V_2 &= \{v'_3\} \cup \{u'_2, u'_3\} \\
 V_3 &= \{v'_4, v'_5\} \cup \{u'_4\} \\
 &\vdots \\
 V_{\lceil \frac{2(m-2)}{3} \rceil} &= \{v'_{m-3}, v'_{m-2}\} \cup \{u'_{m-3}\} \\
 V_{\lceil \frac{2(m-1)}{3} \rceil} &= \{v'_{m-1}\} \cup \{u'_{m-2}, u'_{m-1}\}
 \end{aligned}$$

Each of the color classes has 3 vertices satisfying equitability.

Case 3: If $m \pmod{3} \equiv 2$.

The partition of $V(J(W_m))$ is done as follows:

$$\begin{aligned}
 V_1 &= \{v'_1, v'_2\} \cup \{u'_1\} \\
 V_2 &= \{v'_3\} \cup \{u'_2, u'_3\} \\
 V_3 &= \{v'_4, v'_5\} \cup \{u'_4\} \\
 &\vdots \\
 V_{\lceil \frac{2(m-2)}{3} \rceil} &= \{v'_{m-2}\} \cup \{u'_{m-3}, u'_{m-2}\} \\
 V_{\lceil \frac{2(m-1)}{3} \rceil} &= \{v'_{m-1}\} \cup \{u'_{m-1}\}
 \end{aligned}$$

The number of vertices in each of the color classes is either 2 or 3 satisfying equitability.

Thus, in all the three cases equitable coloring is satisfied.

Therefore, $\chi_=(J(W_m)) = \left\lceil \frac{2(m-1)}{3} \right\rceil$. □

6. EQUITABLE COLORING OF JUMP GRAPH OF HELM GRAPH

Theorem 6.1. *The Equitable chromatic number of jump graph of helm graph $J(H_m)$, $m \geq 3$ and m being any positive integer is*

$$\chi_=(J(H_m)) = m.$$

Proof. Let $V(H_m) = \{v\} \cup \{u_i : 1 \leq i \leq m\} \cup \{w_i : 1 \leq i \leq m\}$ and $E(H_m) = \{v'_i : 1 \leq i \leq m\} \cup \{u'_i : 1 \leq i \leq m-1\} \cup \{u'_m\} \cup \{w'_i : 1 \leq i \leq m\}$ where v'_i is

the edge $\{vu_i(1 \leq i \leq m)\}$, u'_i is the edge $\{u_iu_{i+1}(1 \leq i \leq m-1)\}$, u'_m is the edge u_mu_1 and w'_i is the edge $\{v_iw_i(1 \leq i \leq m)\}$. By the definition of jump graph,

$$V(J(H_m)) = E(H_m) = \left\{v'_i : 1 \leq i \leq m\right\} \cup \left\{u'_i : 1 \leq i \leq m-1\right\} \cup \left\{u'_m\right\} \cup \left\{w'_i : 1 \leq i \leq m\right\}.$$

Each vertex $\{v'_i(1 \leq i \leq m)\}$ is non-adjacent with the vertices u'_i and $w'_i(1 \leq i \leq m)$. Hence color i is assigned to the vertices v'_i, u'_i and $w'_i(1 \leq i \leq m)$. Thus, the colors $1, 2, 3, \dots, m$ appears 3 times each satisfying m -equitable coloring. Therefore, $\chi_=(J(H_m)) = m$. \square

7. EQUITABLE COLORING OF JUMP GRAPH OF SUNLET GRAPH

Theorem 7.1. *The Equitable chromatic number of jump graph of sunlet graph $J(S_m)$, $m \geq 3$ and m being any positive integer is*

$$\chi_=(J(S_m)) = m.$$

Proof. Let $V(S_m) = \{v_i : 1 \leq i \leq m\} \cup \{u_i : 1 \leq i \leq m\}$ and $E(S_m) = \{v'_i : 1 \leq i \leq m-1\} \cup \{v'_m\} \cup \{u'_i : 1 \leq i \leq m\}$ where v'_i is the edge $\{v_iv_{i+1}(1 \leq i \leq m-1)\}$, v'_m is the edge v_mv_1 , u'_i is the edge $v_iu_i(1 \leq i \leq m)$. By the definition of jump graph,

$$V(J(S_m)) = E(S_m) = \left\{v'_i : 1 \leq i \leq m-1\right\} \cup \left\{v'_m\right\} \cup \left\{u'_i : 1 \leq i \leq m\right\}.$$

Each vertex $\{v'_i(1 \leq i \leq m)\}$ is adjacent to all other vertices of $\{v'_i(1 \leq i \leq m)\}$ except for the two neighbouring vertices. Each vertex of $\{u'_i(1 \leq i \leq m)\}$ is adjacent to all other vertices of $\{u'_i(1 \leq i \leq m)\}$. Thus, a clique is formed for the vertices $\{u'_i(1 \leq i \leq m)\}$.

Hence, to satisfy equitability colors $\{1, 2, 3, \dots, m\}$ are assigned in the same order to the vertices $\{v'_1, v'_2, v'_3, \dots, v'_m\}$ as well as to the vertices $\{u'_1, u'_2, u'_3, \dots, u'_m\}$, respectively. Thus, each color $\{1, 2, 3, \dots, m\}$ are assigned two times satisfying equitable coloring. Therefore, $\chi_=(J(S_m)) = m$. \square

8. EQUITABLE COLORING OF JUMP GRAPH OF TADPOLE GRAPH

Theorem 8.1. *The Equitable chromatic number of jump graph of Tadpole graph $J(T_{m,n})$, $m > 3$ and m being any positive integer is*

$$\chi(J(T_{m,n})) = \left\lceil \frac{m+n}{2} \right\rceil.$$

Proof. Let $V(T_{m,n}) = \{v_i : 1 \leq i \leq m\} \cup \{p_j : 1 \leq j \leq n\}$ and $E(T_{m,n}) = \{v'_i : 1 \leq i \leq m\} \cup \{p'_j : 1 \leq j \leq n\}$ where v'_i is the edge $\{v_i v_{i+1} (1 \leq i \leq m-1)\}$, v'_m is the edge $v_m v_1$, p'_1 is the edge $v_1 p_1$ and p'_j is the edge $\{p_{j-1} p_j (2 \leq j \leq n)\}$. Then:

$$V(J(T_{m,n})) = E(T_{m,n}) = \{v'_i : 1 \leq i \leq m\} \cup \{p'_j : 1 \leq j \leq n\}.$$

By definition, the neighbouring vertices of jump graph of tadpole graph are nonadjacent to each other and can thus be assigned same color. Since neighbouring two vertices are assigned one color, the total number of colors used are $\frac{m+n}{2}$ where $m+n$ is the total number of vertices.

The assigning of colors is as follows: The colors $\{1, 1, 2, 2, \dots, \lceil \frac{m+n}{2} \rceil\}$ are assigned consecutively to the vertices $\{v'_1, v'_2, \dots, v'_m, p'_1, p'_2, \dots, p'_n\}$ such that each color appears either once or twice satisfying equitable coloring. Therefore, $\chi(J(T_{m,n})) = \left\lceil \frac{m+n}{2} \right\rceil$. \square

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CGANT - UNIVERSITY OF JEMBER AND MATHEMATICS EDUCATION DEPARTMENT

UNIVERSITY OF JEMBER, INDONESIA.

E-mail address: d.dafik@unej.ac.id

DEPARTMENT OF COMPUTER SCIENCE

DR.G.R.DAMODARAN COLLEGE OF SCIENCE

COIMBATORE, TAMIL NADU, INDIA

E-mail address: praveena4ks@gmail.com

DEPARTMENT OF MATHEMATICS

KONGUNADU ARTS AND SCIENCE COLLEGE

COIMBATORE, TAMIL NADU, INDIA

E-mail address: venkatmaths@gmail.com

CGANT - RESEARCH GROUP

UNIVERSITY OF JEMBER, INDONESIA

E-mail address: d.dafik@unej.ac.id