LAX OPERATOR APPROACH FOR GLOBAL EXISTENCE AND UNIQUENESS CONDITION WITH KAC VAN-MOERBEKE HIERARCHY

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ABSTRACT. By using a supersymmetric approach (communication approach), this paper gives the modified application of Toda hierarchy and Kac-Van Moerbeke hierarchy.

1. INTRODUCTION

In this paper we investigate the relation between Kac-Van Moerbeke hierarchy and Toda hierarchy, see [1–4]. Let $\varsigma(\phi) \in l^\infty(z, R)$ where $\varsigma(s, \phi) \neq 0$, $(s, \phi) \in z \times R$. Let the function $\phi \in R$ be differentiable and it divides into even and odd process of $\varsigma(\phi)$;

$$
\begin{aligned}
\varsigma_e(s, \phi) &= \varsigma(2s, \phi) \\
\varsigma_o(s, \phi) &= \varsigma(2s + 1, \phi) \quad (s, \phi) \in z \times R.
\end{aligned}
$$

Let α be the bounded operator applying in $l^2(z)$.

$$
\begin{aligned}
\alpha(\phi) &= \varsigma_o(\phi)S^+ + \varsigma_e(\phi) \\
\alpha(\phi)^* &= \varsigma_o^-(\phi)S^- + \varsigma_e(\phi).
\end{aligned}
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2. MAIN RESULT

We have:

$$F_1(\phi) = \alpha(\phi)^* \alpha(\phi) \quad \text{and} \quad F_1(\phi) = \alpha(\phi)\alpha(\phi)^*$$

and define $F_k(\phi) = a_k(\phi)S^+ + a_k^-(\phi)S^- + b_k(\phi)$.

If we take the values for $k = 1, 2$ we have:

$$a_1(\phi) = \varsigma_e(\phi)\varsigma_o(\phi), \quad b_1(\phi) = \varsigma_e(\phi)^2 + \varsigma_o^-(\phi)^2$$
$$a_2(\phi) = \varsigma_e^+(\phi)\varsigma_o(\phi), \quad b_2(\phi) = \varsigma_e(\phi)^2 + \varsigma_o(\phi)^2 \quad .$$

As we know, $\bar{\varsigma}(s, \phi) = \varsigma(s+1, \phi)$ will be equivalent to:

(2.1) $$\bar{\varsigma}_e(s, \phi) = \varsigma_o(s, \phi), \quad \bar{\varsigma}_o(s, \phi) = \varsigma_e(s+1, \phi),$$

and hence
$$\begin{aligned}\bar{a}_1(\phi) &= a_2(\phi) \\ \bar{b}_1(\phi) &= b_2(\phi) \\ \bar{a}_2(\phi) &= a_1^+(\phi) \\ \bar{b}_2(\phi) &= b_1^+(\phi).\end{aligned}$$
Now we have,
$$\begin{aligned}\bar{p}_{1,\beta}(\phi) &= p_{2,\beta}(\phi), \quad \bar{q}_{1,\beta}(\phi) = q_{2,\beta}(\phi) \\ \bar{p}_{2,\beta}(\phi) &= p_{1,\beta}^+(\phi), \quad \bar{q}_{2,\beta}(\phi) = q_{1,\beta}^+(\phi).\end{aligned}$$

Using Hamilton operator,

$$\left\langle \delta_m, F_2^{j+1}\delta_s \right\rangle = \left\langle \delta_m, \alpha F_1^j \alpha^* \delta_s \right\rangle,$$

we can write $p_{2,\beta}, q_{2,\beta}$ in terms of $p_{1,\beta}, q_{1,\beta}$.

When we solve the equation with $m = n$ it gives:

$$p_{2,\beta+1}(\phi) = \varsigma_o(\phi)^2 p_{1,\beta}^+(\phi) + \varsigma_e(\phi)^2 p_{1,\beta}(\phi) + q_{1,\beta}(\phi),$$

and for $m = n + 1$, we obtain

$$q_{2,\beta+1}(\phi) - q_{1,\beta+1}(\phi) = \varsigma(\phi)^2 \left[2\{\varsigma_e^+(\phi)^2 p_{1,\beta}^+(\phi) - \varsigma_e(\phi)^2 p_{1,\beta}(\phi) + q_{1,\beta}^+(\phi) - q_{1,\beta+1}(\phi)\} \right].$$

Using the equation (2.1), we can get the two more solutions:

$$\begin{aligned}p_{1,\beta+1}(\phi) &= \varsigma_e(\phi)^2 p_{2,\beta}(\phi) + \varsigma_o(\phi)^2 p_{2,\beta}^-(\phi) + q_{2,\beta}^-(\phi) \\ q_{2,\beta+1}^-(\phi) - q_{1,\beta+1}(\phi) &= \varsigma_e(\phi)^2 \left[-2\varsigma_o(\phi)^2 p_{2,\beta}(\phi) - \varsigma_o^-(\phi)^2 p_{2,\beta}^-(\phi) - q_{2,\beta}(\phi) - q_{2,\beta}^-(\phi) \right].\end{aligned}$$
(2.2)

These relations will be very important for next steps.

We now explain matrix-valued operators $D(\phi), B_{2r+2}(\phi)$ *in* $l^2(Z, C^2)$ as follows:

$$D(\phi) = \begin{bmatrix} 0 & \alpha(\phi)^* \\ \alpha(\phi) & 0 \end{bmatrix}$$

and

$$(2.3) \qquad B_{2r+2}(\phi) = \begin{bmatrix} A_{1,2r+2}(\phi) & 0 \\ 0 & A_{2,2r+2}(\phi) \end{bmatrix}$$

where $r \in N_0$. Here, $A_{k,2r+2}(\phi)$ are defined as Lax operator.
i.e.

$$A_{k,2r+2}(\phi) = -F_k(\phi)^{r+1} + \sum_{j=0}^{r} [2a_k(\phi)p_{k,j}(\phi)S^+ - q_{k,\beta}]F_k(\phi)^{\beta} + p_{k,r+1}.$$

The sequences $[p_{k,\beta}(s,\phi)]$, and $[q_{k,\beta}(s,\phi)]$, $0 \leq j \leq r+1$ are defined as Toda hierarchy. We choose the similar summation constants in $A_{1,2r+2}(\phi)$ and $A_{2,2r+2}(\phi)$. From equation (2.3), we have:

(2.4)
$$B_2(\phi) = \begin{bmatrix} \varsigma_e(\phi)\varsigma_o(\phi)S^+ - \varsigma_e^-(\phi)\varsigma_o^-(\phi)S^- & 0 \\ 0 & \varsigma_e^+(\phi)\varsigma_o(\phi)S^+ - \varsigma_e(\phi)\varsigma_o^-(\phi)S^- \end{bmatrix}.$$

Further, for the skew-symmetric operator $A_{2r+2}(t)$ known as Lax operator, it holds:

$$\frac{d}{dt}F(\phi) - [A_{2r+2}(\phi), F(\phi)] = 0 \qquad \phi \in R.$$

Using equation (2.4), we have:

$$\frac{d}{d\phi}D = [B_{2r+2}, D] = \begin{bmatrix} 0 & A_{1,2r+2}\alpha^* - \alpha^* A_{2,2r+2} \\ A_{2,2r+2}\alpha - \alpha A_{1,2r+2} & 0 \end{bmatrix}.$$

Taking $F_1^{\beta}\alpha^* = \alpha^* F_2^{\beta}$ & $S^- = \frac{1}{a_2^-}(F_2 - a_2 S^+ - b_2)$, we have:

$$\alpha * A_{2,2r+2} - A_{1,2r+2}\alpha* = \sum_{j=1}^{r} \frac{\varsigma_o \varsigma_e^+}{\varsigma_e} 2\varsigma_e^2 (p_{2,\beta} - p_{1,j}) + (q_{2,\beta}^- - q_{1,\beta}) S^+ F_2^{r-j}$$

$$+ \sum_{j=1}^{r} \frac{1}{\varsigma_e} (q_{2,\beta+1}^- - q_{1,\beta+1}) + 2[(a_1)^2 p_{1,j} - (a_2)^2 p_{2,\beta}]$$

$$-\varsigma_o^2 (q_{2,\beta}^- - q_{1,\beta}) + \varsigma_e^2 (q_{2,\beta} - q_{2,\beta}^-) F_2^{r-\beta} + (q_{2,r+1}^- - q_{1,r+1})$$

$$(2.5) \qquad - \varsigma_o^- (p_{2,r+1}^- - p_{1,r+1}) S^- + \varsigma_e (p_{2,r+1} - p_{1,r+1}).$$

Simplifying $\langle \delta_n, (F_1^l \alpha^* - \alpha^* F_2^l)\delta_n \rangle = 0$ explicitly gives that

$$q_{2,\beta}^- - q_{1,\beta} + 2\varsigma_e^2 (p_{2,\beta} - p_{1,\beta}) = 0.$$

Next, simplifying the equation (2.2) and (2.5) we get:

$$\left(q_{2,\beta+1}^- - q_{1,\beta+1}\right) + 2\left((a_1)^2 p_{1,\beta} - (a_2)^2 p_{2,\beta}\right) - \varsigma_o^2\left(q_{2,\beta}^- - q_{1,\beta}\right) + \varsigma_e^2\left(q_{2,\beta} - q_{2,\beta}^-\right) = 0.$$

Hence, we get:

$$A_{1,2r+2}\alpha^* - \alpha^* A_{2,2r+2} = -\varsigma_o^-\left(p_{2,r+1}^- - p_{1,r+1}\right)S^- + \varsigma_e\left(p_{2,r+1} - p_{1,r+1}\right) = 0.$$

As we know the equation

(2.6) $$\frac{d}{dt}D(\phi) - [B_{2r+2}(\phi), D(\phi)] = 0,$$

will be equivalent to: $KM_r(\varsigma) = (KM_r(\varsigma)_e, KM_r(\varsigma)_o)$,

(2.7) $$\Rightarrow \left(\begin{array}{c} \frac{d}{d\phi}\varsigma_e - \varsigma_e(p_{2,r+1} - p_{1,r+1}) \\ \frac{d}{d\phi}\varsigma_o - \varsigma_o(p_{2,r+1} - p_{1,r+1}^+) \end{array} \right) = 0.$$

Hence, the transformation shows that the equation for ς_e and ς_o are for one equation ς. More explicitly, combining $p_{k,\beta}$ and $q_{k,\beta}$ into the single sequence.

s.t. $\left. \begin{array}{l} G_\beta(2s) = p_{1,\beta}(s) \\ G_\beta(2s+1) = p_{2,\beta}(s) \end{array} \right\}$ respectively $\left(\begin{array}{l} F_\beta(2s) = q_{1,\beta}(s) \\ F_\beta(2s+1) = q_{2,\beta}(s) \end{array} \right)$.

At the end we can write the equation (2.7) as

$$KM_r(\varsigma) = \frac{d}{d\phi}\varsigma - \frac{d}{d\phi}\left(G_{r+1}^+ - G_{r+1}\right).$$

By the lemma,

(2.8) $$q_{\beta+1} - q_{\beta+1}^- - 2\left(a^2 p_\beta^+ - (a^-)^2 p_\beta^-\right) - b(q_\beta - q_\beta^-) = 0,$$

we can see that G_β and F_β satisfy the relation $G_0 = 1, \quad F_0 = c_1$

$$2G_{\beta+1} - F_\beta - F_\beta^{--} - 2(\varsigma^2 + (\varsigma^-)^2)G_\beta = 0$$

$$F_{\beta+1} - F_{\beta+1}^{--} - 2[(\varsigma\varsigma^+)^2 G_\beta^{++} - (\varsigma^-\varsigma^{--})^2 G_\beta^{--}] - \left(\varsigma^2 + (\varsigma^{--})^2\right)(F_\beta - F_\beta^{--}) = 0,$$

(2.9)

where $0 \le \beta < r$. We finally got that here Toda Concept shifting $r \in N_0$ yield the Kac-van Moerbeke hierarchy (KMH).

This can be written as $KM_r(\varsigma) = 0$ and from equation (2.9), $\begin{array}{l} G_1 = \varsigma^2 + (\varsigma^-)^2 + c_1 \\ F_1 = 2(\varsigma\varsigma^+)^2 + c_2 \end{array}$.

Hence,

$$KM_0(\varsigma) = \frac{d}{dt}\varsigma - \varsigma\left((\varsigma^+)^2 + (\varsigma^-)^2\right) = 0$$

$$KM_1\left(\varsigma\right) = \frac{d}{dt}\varsigma - \varsigma\left(\left(\varsigma^+\right)^4 - \left(\varsigma^-\right)^4 + \left(\varsigma^{++}\right)^2\left(\varsigma^+\right)^2 + \left(\varsigma^+\right)^2\varsigma^2 - \varsigma^2(\varsigma^-)^2 - \left(\varsigma^-\right)^2(\varsigma^{--})^2\right) +$$
$$c_1\left(-\varsigma\right)\left(\left(\varsigma^+\right)^2 - \left(\varsigma^-\right)^2\right) = 0$$

This Lax equation also equal to equation (2.6).

CONCLUSION

In this paper, we have discussed about the Kac-van Moerbeke hierarchy. By using a commutation application, we can say that the Kac-van Moerbeke hierarchy is a changed Toda hierarchy absolutely in the processing that the modified mKdVH is related to the KdVH. The relation between the TL hierarchy and its updated application will be the Commuting at initial stage by difference expressions of the KM hierarchy which would be given as Backlund transformation depended on the factorization of difference expressions.

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