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ON $\mathbb{R}$-LINEAR CODES AND ITS COVERING RADIUS OF SOME CLASSES OF LINEAR CODES IN $\mathbb{R}$

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ABSTRACT. In this paper, we determine the covering radius of codes over $\mathbb{R} = \mathbb{Z}_2 R^*$, where $R^* = \mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$. We have defined the block repetition codes over $\mathbb{R}$ and obtain the covering radius with respect to Euclidean weight and Lee weight for block repetition codes, simplex code of α-type and β-type in $\mathbb{R}$.

1. INTRODUCTION

Recently, there has been substantial interest in the class of additive codes. In [10, 11], Delsarte contributes to the algebraic theory of association scheme. Additive codes over $\mathbb{Z}_2\mathbb{Z}_4$ have been extensively studied in [1, 3–5]. Enormous results were made available on the simplex codes over finite fields and finite rings. A few of them are [6, 12, 13].

In [7], the authors, in particular, gave lower and upper bounds on the covering radius of codes over the ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2$ where $u^2 = 0$ with respect to different distance, and they explained the covering radius of various repetition codes, Simplex Codes (Type α and β over R.) Further, in [8], they examined the covering radius of Simplex and MacDonald codes of types α and β over $\mathbb{Z}_2\mathbb{Z}_4$.

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The above results motivate us to investigate in this area. In this paper, we obtain the covering radius of codes over $\mathbb{R}$ and covering radius for block repetition codes, simplex codes of types α and β were determined over $\mathbb{R}$. The rest of this paper is organized as follows.

In Section 2, some properties related to additive codes over $\mathbb{R} = \mathbb{Z}_2R^*$, where $R^* = \{0, 1, u, 1+u\}, u^2 = 0$ are given. The covering radius of codes over $\mathbb{R}$ is considered in Section 3. In Section 4, we define the block repetition codes over $\mathbb{R}$ and find their covering radius. The construction of simplex codes over $\mathbb{R}$ of types α and β and the covering radius of these codes is also considered in Sections 5.

2. Preliminaries

Throughout this paper, $\mathbb{R} = \mathbb{Z}_2R^*$, where $R^* = \{0, 1, u, 1+u\}, u^2 = 0$. In this section, some preliminary results are given based on [3] and [5]. A non empty set C is a $\mathbb{R}$-additive code if it is a subgroup of $\mathbb{Z}_2^\gamma \times {R^*}^\delta$. In this case, C is also isomorphic to an abelian structure $\mathbb{Z}_2^\lambda \times R^{*\mu}$ for some λ and μ. We say that C is of type $2^\lambda R^{*\mu}$ as a group. It follows that it has $\mid C \mid = 2^{\lambda+2\mu}$ codewords, and the number of order two codewords in C is $\mid C \mid = 2^{\lambda+\mu}$.

We consider the following extension of the Gray map

$$\phi : \mathbb{Z}_2^\gamma \times {R^*}^\delta \to \mathbb{Z}_2^n, \text{ with } n = \gamma + 2\delta,$$

given by

$$\phi(u, v) = (u, \phi(v_1), \cdots \phi(v_\delta)), \forall\ u \in \mathbb{Z}_2^\gamma, \forall\ (v_1, \cdots, v_\delta) \in {R^*}^\delta,$$

where

$$\phi : R^* \to \mathbb{Z}_2^2,$$

is the Gray map given by $\phi(0) = (0,\ 0), \phi(1) = (0,\ 1), \phi(u) = (1,\ 1)$, and $\phi(1+u) = (1,\ 0)$. Then the binary image of a $\mathbb{R}-$additive code under the extended Gray map is called a $\mathbb{R}-$linear code of length $n = \gamma + 2\delta$.

The Hamming weight of u, denoted by $wt_H(u)$ and $wt_L(v)$ and $wt_E(v)$ the Lee and Euclidean weights of v, respectively, where $u \in \mathbb{Z}_2^\gamma$ and $v \in {R^*}^\delta$ are defined as

$$wt_E(v_i) = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i = 1 \text{ or } (1+u) \\ 4 & \text{if } v_i = u \end{cases}$$

and

$$wt_L(v_i) = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i = 1 \text{ or } (1+u) \\ 2 & \text{if } v_i = u \end{cases}.$$

The Lee weight of x is defined as $wt_L(x) = wt_H(u) + wt_L(v)$, with $x = (u,\ v) \epsilon \mathbb{Z}_2^\gamma \times R^{*^\delta}$, and $u = (u_1,\ \cdots,\ u_\gamma)\ \epsilon\ \mathbb{Z}_2^\gamma$ and $v = (v_1,\ \cdots,\ v_\delta)\ \epsilon\ R^{*^\delta}$ and the Euclidean weight of x is defined as $wt_E(x) = wt_H(u) + wt_E(v)$. The Gray map defined above is an isometry which transforms the Lee distance defined over $\mathbb{Z}_2^\gamma \times R^{*^\delta}$ to the Hamming distance defined over $\mathbb{Z}_2^n$, with $n = \gamma + 2\delta$.

3. THE COVERING RADIUS OF CODES OVER ℝ

The covering radius of a code C over $\mathbb{R}$ is introduced in this section.

Let C be a code of length n with minimum distance d over a code alphabet A. Then the spheres of radius $\left\lfloor \dfrac{d-1}{2} \right\rfloor$ around the codewords may not cover the whole space.

The least non-negative integer r such that sphere of radius r around the codewords cover the whole space A^n is called the *covering radius* of the code. That is, the covering radius of C is

$$r(C) = \max_{u \in A^n} \left\{ \min_{c \in C} \{d(u,c)\} \right\}.$$

For a binary code C, its covering radius $r(C)$ is defined as follows

$$r(C) = \max_{u \epsilon \mathbb{F}_2} \{ \min_{c \epsilon C} d_H(u,\ c) \}. \tag{3.1}$$

The extension of this definition to codes over $\mathbb{R}$ is that the covering radius of a code C is the smallest number r such that the spheres of radius r around the codewords cover $\mathbb{Z}_2^\gamma \times R^{*^\delta}$. Hence, the covering radius of a code C over $\mathbb{R}$, with respect to the Lee and Euclidean distances, is given by

$$r_L(C) = \max_{u \epsilon \mathbb{Z}_2^\gamma \times R^{*^\delta}} \{ \min_{c \epsilon C} d_L(u,c) \} \text{ and } r_E(C) = \max_{u \epsilon \mathbb{Z}_2^\gamma \times R^{*^\delta}} \{ \min_{c \epsilon C} d_E(u,c) \},$$

respectively. It is easy to see that $r_L(C)$ and $r_E(C)$ are the minimum values r_L and r_E such that

$$\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}} = \bigcup_{c \epsilon C} S_{r_L}(c),$$

and

$$\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}} = \bigcup_{c \epsilon C} S_{r_E}(c),$$

respectively, where

$$S_{r_L}(u) = \{v \epsilon \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}; d_L(u,v) \leq r_L\},$$

and

$$S_{r_E}(u) = \{v \epsilon \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}; d_E(u,v) \leq r_E\},$$

for $u \,\epsilon\, \mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$.

The following result, for codes over $\mathbb{Z}_4$, given by Aoki et al. in [2] is also valid for codes over $\mathbb{R}$. Its proof follows from the definition of the covering radius and the fact that the map ϕ is a weight preserving map.

Proposition 3.1. *Let C be a code over $\mathbb{Z}_2^{\gamma} \times R^{*^{\delta}}$ and $\phi(C)$ be the Gray map image of C. Then $r_L(C) = r_H(\phi(C))$.*

The following result is useful for determining the covering radius of codes over rings. This is a generalization of the result in [9] for codes over finite fields.

Proposition 3.2. *If C_0 and C_1 are codes over $\mathbb{R}$, of lengths n_0 and n_1, minimum distance d_0 and d_1, and generated by matrices G_0 and G_1, respectively, and if C is the code generated by*

$$G = \left[\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array}\right],$$

then $r_d(C) \leq r_d(C_0) + r_d(C_1)$, and the covering radius of the concatenation of C_0 and C_1, denoted C_c, satisfies

$$r_d(C_c) \geq r_d(C_0) + r_d(C_1),$$

for all distances d over $\mathbb{R}$.

Proof. We define the generator matrix of C_c as $[G_0', G_1]$, so that C_c is a code of length $n_0 + n_1$ and minimum distance d, where $d \geq min\{d_0, d_1\}$. Then the covering radius satisfies

$$r_d(C_c) \geq r_d(C_0) + r_d(C_1).$$

□

4. THE COVERING RADIUS OF THE BLOCK REPETITION CODES OVER $\mathbb{R}$

In order to determine the covering radius of simplex codes of types α and β over $\mathbb{R}$, we define some classes of block codes over $\mathbb{R}$ and the approach in [14] is used to obtain the covering radius.

The block repetition code C^n over $\mathbb{R}$ is a $\mathbb{R}-$additive code of length $n = \sum_{j=1}^{7} n_j$ with generator matrix

$$G = (\overbrace{0101\cdots01}^{n_1}\overbrace{0u0u\cdots0u}^{n_2}\overbrace{01+u01+u\cdots01+u}^{n_3}\overbrace{1010\cdots10}^{n_4}\overbrace{1111\cdots11}^{n_5}$$
$$\overbrace{1u1u\cdots1u}^{n_6}\overbrace{11+u11+u\cdots11+u}^{n_7}).$$

If, for a fixed $1 \le i \le 7$, we have for all $1 \le j \ne i \le 7, n_j = 0$, then the code $C^n = C^{n_i}$ is denoted by C_i. Therefore, the seven basic repetition codes are given below. We have that

$C_1 = \{(00\cdots00), (01\cdots01), (0u\cdots0u), (01+u\cdots01+u)\}$,
is an additive code of length $n = n_1$ generated by

$G_1 = [0101\cdots01]$.

$C_2 = \{(00\cdots00), (0u\cdots0u)$,
is an additive code of length $n = n_2$ generated by

$G_2 = [0u0u\cdots0u]$.

$C_3 = \{(00\cdots00), (01\cdots01), (0u\cdots0u), (01+u\cdots01+u)\}$,
is an additive code of length $n = n_3$ generated by

$G_3 = [01+u01+u\cdots01+u]$,

$C_4 = \{(00\cdots00), (10\cdots10)\}$,
is an additive code of length $n = n_4$ generated by

$G_4 = [1010\cdots10]$,

$C_5 = \{(00\cdots00), (01\cdots01), (0u\cdots0u), (01+u\cdots01+u), (10\cdots10),$
$(11\cdots11), (1u\cdots1u), (11+u\cdots11+u)\}$,
is an additive code of length $n = n_5$ generated by

$G_5 = [1111\cdots11]$,

$C_6 = \{(00\cdots00), (0u\cdots0u), (10\cdots10), (1u\cdots1u)\}$,
is an additive code of length $n = n_6$ generated by

$G_6 = [1u1u\cdots1u]$,

$C_7 = \{(00\cdots00), (01\cdots01), (0u\cdots0u), (01+u\cdots01+u),$

$(10\cdots 10), (11\cdots 11), (1u\cdots 1u), (11+u\cdots 11+u)\}$,

is an additive code of length $n = n_7$ generated by

$G_7 = [11+u11+u\cdots 11+u]$,

The following theorems provide the covering radius of C_j, for $1 \leq j \leq 7$.

Theorem 4.1. *The covering radius of $C_j, 1 \leq j \leq 7$, with respect to the Euclidean weight is given by*

(1) $3n/4 \leq r_E(C_1) = r_E(C_3) \leq 2n$,
(2) $n \leq r_E(C_2) \leq 3n$,
(3) $n/4 \leq r_E(C_4) \leq 4n$,
(4) $n \leq r_E(C_5) = r_E(C_7) \leq 2n$,
(5) $5n/4 \leq r_E(C_6) \leq 5n/2$.

Proof. For $c\epsilon C_j, 1 \leq j \leq 7$, let $t_i(c), 0 \leq i \leq 7$ denote the number of occurrences of symbol i in the codeword c. Considering 1 to 5, we have that

$$r_E(C_j) = \max_{x\in\mathbb{R}^n}\{d_E(x, C_j); 1 \leq j \leq 7\}.$$

Let $x\epsilon\mathbb{R}^n$. If x is given by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum\limits_{j=0}^{7} t_j = n$, then

$$d_E(x, \overline{00}) = n - t_0 + 3t_2 + t_5 + 4t_6 + t_7,$$
$$d_E(x, \overline{01}) = n - t_1 + 3t_3 + t_4 + t_6 + 4t_7,$$
$$d_E(x, \overline{0u}) = n - t_2 + 3t_0 + 4t_4 + t_5 + t_7,$$
$$d_E(x, \overline{01+u}) = n - t_3 + 3t_1 + t_4 + 4t_5 + t_6,$$
$$d_E(x, \overline{10}) = n - t_4 + t_1 + 4t_2 + t_3 + 3t_6,$$
$$d_E(x, \overline{11}) = n - t_5 + t_0 + t_2 + 4t_3 + 3t_7,$$
$$d_E(x, \overline{1u}) = n - t_6 + 4t_0 + t_1 + t_3 + 3t_4,$$
$$d_E(x, \overline{11+u}) = n - t_7 + t_0 + 4t_1 + t_2 + 3t_5.$$

Therefore,

$$d_E(x, C_1) = d_E(x, C_3) =$$
$$\min\{(n - t_0 + 3t_2 + t_5 + 4t_6 + t_7), (n - t_1 + 3t_3 + t_4 + t_6 + 4t_7),$$
$$(n - t_2 + 3t_0 + 4t_4 + t_5 + t_7), (n - t_3 + 3t_1 + t_4 + 4t_5 + t_6)\}$$
$$\leq \frac{4n + 2(t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7) + 2(t_4 + t_5 + t_6 + t_7)}{4} \leq 2n$$

and hence

$$r_E(C_1) = r_E(C_3) \leq 2n.$$

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{01\cdots01}^{\frac{n}{4}}\overbrace{0u\cdots0u}^{\frac{n}{4}}\overbrace{01+u\cdots01+u}^{\frac{n}{4}})\ \epsilon\mathbb{R}^n,$$

then

$$d_E(x,\overline{00}) = d_E(x,\overline{01}) = d_E(x,\overline{0u}) = d_E(x,\overline{01+u}) = \frac{n}{8} + 4(\frac{n}{8}) + \frac{n}{8} = \frac{3n}{4}.$$

Thus $r_E(C_1) = r_E(C_3) \geq \frac{3n}{4}$ and we have, $\frac{3n}{4} \leq r_E(C_1) = r_E(C_3) \leq 2n,$

$$\begin{aligned} d_E(x,C_2) &= \min\{(n-t_0+3t_2+t_5+4t_6+t_7),(n-t_2+3t_0+4t_4+t_5+t_7)\} \\ &\leq \frac{2n+2(t_0+t_2+t_5+t_7)+4(t_4+t_6)}{2} \leq 3n. \end{aligned}$$

Therefore, $r_E(C_2) \leq 3n.$

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{0u\cdots0u}^{\frac{n}{2}})\ \epsilon\ \mathbb{R}^n,$$

then $d_E(x,\overline{00}) = d_E(x,\overline{0u}) = 4(\frac{n}{4}) = n$. Thus $r_E(C_2) \geq n$, and so, $n \leq r_E(C_2) \leq 3n,$

$$\begin{aligned} d_E(x,C_4) &= \min\{(n-t_0+3t_2+t_5+4t_6+t_7),(n-t_4+t_1+4t_2+t_3+3t_6)\} \\ &\leq \frac{2n-t_0+t_1+7t_2+t_3-t_4+t_5+7t_6+t_7}{2} \leq 4n \end{aligned}$$

and hence $r_E(C_4) \leq 4n.$

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{10\cdots10}^{\frac{n}{2}})\ \epsilon\ \mathbb{R}^n,$$

then $d_E(x,\overline{00}) = d_E(x,\overline{10}) = \frac{n}{4}$. Thus $r_E(C_4) \geq \frac{n}{4}$, and so $\frac{n}{4} \leq r_E(C_4) \leq 4n,$

$$\begin{aligned} &d_E(x,C_5) = d_E(x,C_7) \\ &\quad = \min\{(n-t_0+3t_2+t_5+4t_6+t_7),(n-t_1+3t_3+t_4+t_6+4t_7), \\ &\qquad (n-t_2+3t_0+4t_4+t_5+t_7),(n-t_3+3t_1+t_4+4t_5+t_6), \\ &\qquad (n-t_4+t_1+4t_2+t_3+3t_6),(n-t_5+t_0+t_2+4t_3+3t_7), \\ &\qquad (n-t_6+4t_0+t_1+t_3+3t_4),(n-t_7+t_0+4t_1+t_2+3t_5)\} \leq 2n, \end{aligned}$$

then $r_E(C_5) = r_E(C_7) \leq 2n.$

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{8}}\overbrace{01\cdots01}^{\frac{n}{8}}\overbrace{0u\cdots0u}^{\frac{n}{8}}\overbrace{01+u\cdots01+u}^{\frac{n}{8}}\overbrace{10\cdots10}^{\frac{n}{8}}$$
$$\overbrace{11\cdots11}^{\frac{n}{8}}\overbrace{1u\cdots1u}^{\frac{n}{8}}\overbrace{11+u\cdots11+u}^{\frac{n}{8}})\ \epsilon\ \mathbb{R}^n,$$

then

$$\begin{aligned} d_E(x,\overline{00}) &= d_E(x,\overline{01}) = d_E(x,\overline{0u}) = d_E(x,\overline{01+u}) \\ &= d_E(x,\overline{10}) = d_E(x,\overline{11}) = d_E(x,\overline{1u}) = d_E(x,\overline{11+u}) \\ &= \frac{n}{16} + 4(\frac{n}{16}) + \frac{n}{16} + \frac{n}{16} + \frac{n}{8} + \frac{n}{16} + 4(\frac{n}{16}) + \frac{n}{8} = n. \end{aligned}$$

Thus $r_E(C_5) = r_E(C_7) \geq n$, and so $n \leq r_E(C_5) = r_E(C_7) \leq 2n$,

$$\begin{aligned} d_E(x,C_6) = \min\{&(n - t_0 + 3t_2 + t_5 + 4t_6 + t_7), (n - t_2 + 3t_0 + 4t_4 + t_5 + t_7), \\ &(n - t_4 + t_1 + 4t_2 + t_3 + 3t_6), (n - t_6 + 4t_0 + t_1 + t_3 + 3t_4)\} \\ \leq\ & \frac{4n + 2n + 4n}{4}, \end{aligned}$$

so then $r_E(C_6) \leq \dfrac{5n}{2}$.

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{0u\cdots0u}^{\frac{n}{4}}\overbrace{10\cdots10}^{\frac{n}{4}}\overbrace{1u\cdots1u}^{\frac{n}{4}})\ \epsilon\ \mathbb{R}^n,$$

then

$$d_E(x,\overline{00}) = d_E(x,\overline{0u}) = d_E(x,\overline{10}) = d_E(x,\overline{1u}) = 4(\frac{n}{8}) + \frac{n}{8} + \frac{n}{8} + 4(\frac{n}{8}) = \frac{5n}{4}.$$

Thus $r_E(C_6) \geq \dfrac{5n}{4}$, and so $\dfrac{5n}{4} \leq r_E(C_6) \leq \dfrac{5n}{2}$. □

Example 1. *Consider the repetition codes of length* $n = 4$ *over* $\mathbb{R} = \mathbb{Z}_2 R^*$, *where* $R^* = \mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$. *Then, we have the seven basic repetition codes of length* $n = 4$ *respectively.*

$C_1 = \{k(01010101) \mid k\ \epsilon\ \mathbb{R}\}$, $C_2 = \{k(0u0u0u0u) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_3 = \{k(01+u01+u01+u01+u) \mid k\ \epsilon\ \mathbb{R}\}$, $C_4 = \{k(10101010) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_5 = \{k(11111111) \mid k\ \epsilon\ \mathbb{R}\}$, $C_6 = \{k(1u1u1u1u) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_7 = \{k(11+u11+u11+u11+u) \mid k\ \epsilon\ \mathbb{R}\}$.

Further we have, $r_E(C_1) = r_E(C_3) = 8$, $r_E(C_2) = 12$, $r_E(C_4) = 16$, $r_E(C_5) = r_E(C_7) = 8$ *and* $r_E(C_6) = 10$. *The Covering radius of these repetition codes with* *respect to the Euclidean weight have achieved the upper bound that appeared in Theorem 4.1.*

Theorem 4.2. *The Covering radius of* $C_j, 1 \leq j \leq 7,$ *with respect to the Lee weight is given by*

(1) $n/2 \leq r_L(C_1) = r_E(C_3) \leq 2n,$
(2) $n/2 \leq r_L(C_2) \leq 2n,$
(3) $n/4 \leq r_L(C_4) \leq 2n,$
(4) $3n/4 \leq r_L(C_5) = r_L(C_7) \leq 3n/2,$
(5) $3n/4 \leq r_L(C_6) \leq 3n/2.$

Proof. For $c \epsilon C_j, 1 \leq j \leq 7$, let $t_i(c), 0 \leq i \leq 7$ denote the number of occurrences of symbol i in the codeword c. Considering 1 to 5, we have that

$$r_L(C_j) = \max_{x \epsilon \mathbb{R}^n} \{d_L(x, C_j); 1 \leq j \leq 7\}.$$

Let $x \epsilon \mathbb{R}^n$. If x is given by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^{7} t_j = n$, then

$$\begin{aligned}
d_L(x, \overline{00}) &= n - t_0 + t_2 + t_5 + 2t_6 + t_7,\\
d_L(x, \overline{01}) &= n - t_1 + t_3 + t_4 + t_6 + 2t_7,\\
d_L(x, \overline{0u}) &= n - t_2 + t_0 + 2t_4 + t_5 + t_7,\\
d_L(x, \overline{01+u}) &= n - t_3 + t_1 + t_4 + 2t_5 + t_6,\\
d_L(x, \overline{10}) &= n - t_4 + t_1 + 2t_2 + t_3 + t_6,\\
d_L(x, \overline{11}) &= n - t_5 + t_0 + t_2 + 2t_3 + t_7,\\
d_L(x, \overline{1u}) &= n - t_6 + 2t_0 + t_1 + t_3 + t_4,\\
d_L(x, \overline{11+u}) &= n - t_7 + t_0 + 2t_1 + t_2 + t_5.
\end{aligned}$$

Therefore, $d_L(x, C_1) = d_L(x, C_3) = \min\{(n - t_0 + t_2 + t_5 + 2t_6 + t_7), (n - t_1 + t_3 + t_4 + t_6 + 2t_7), (n - t_2 + t_0 + 2t_4 + t_5 + t_7), (n - t_3 + t_1 + t_4 + 2t_5 + t_6)\} \leq \frac{4n + 4(t_4 + t_5 + t_6 + t_7)}{4} \leq 2n$, then

$$r_E(C_1) = r_E(C_3) \leq 2n.$$

If

$x = (\overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{01 \cdots 01}^{\frac{n}{4}} \overbrace{0u \cdots 0u}^{\frac{n}{4}} \overbrace{01+u \cdots 01+u}^{\frac{n}{4}}) \, \epsilon \, \mathbb{R}^n$, then $d_L(x, \overline{00}) = d_L(x, \overline{01}) =$

$d_L(x,\overline{0u}) = d_L(x,\overline{01+u}) = \frac{n}{8} + 2(\frac{n}{8}) + \frac{n}{8} = \frac{n}{2}$. Thus $r_L(C_1) = r_L(C_3) \geq \frac{n}{2}$, and so $\frac{n}{2} \leq r_L(C_1) = r_L(C_3) \leq 2n$,

$$\begin{aligned} d_L(x,C_2) &= \min\{(n-t_0+t_2+t_5+2t_6+t_7),(n-t_2+t_0+2t_4+t_5+t_7)\} \\ &\leq \frac{2n+2(t_4+t_5+t_6+t_7)}{2} \leq 2n. \end{aligned}$$

Then $r_L(C_2) \leq 2n$.

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{2}}\overbrace{0u\cdots0u}^{\frac{n}{2}})\ \epsilon\ \mathbb{R}^n,$$

then $d_L(x,\overline{00}) = d_L(x,\overline{0u}) = 2(\frac{n}{4}) = \frac{n}{2}$. Thus $r_L(C_2) \geq \frac{n}{2}$ and so $\frac{n}{2} \leq r_L(C_2) \leq 2n$,

$$d_L(x,C_4) = \min\{(n-t_0+t_2+t_5+2t_6+t_7),(n-t_4+t_1+2t_2+t_3+t_6)\} \leq \frac{2n+2n}{2} \leq 2n,$$

so then $r_L(C_4) \leq 2n$.

If $x = (\overbrace{00\cdots00}^{n/1}\overbrace{10\cdots10}^{n/2})\ \epsilon\ \mathbb{R}^n$, then $d_L(x,\overline{00}) = d_L(x,\overline{10}) = (\frac{n}{4}) = \frac{n}{4}$. Thus $r_L(C_4) \geq \frac{n}{4}$, and hence $\frac{n}{4} \leq r_L(C_4) \leq 2n$,

$$\begin{aligned} d_L(x,C_5) =& d_L(x,C_7) \\ =& \min\{(n-t_0+t_2+t_5+2t_6+t_7),(n-t_1+t_3+t_4+t_6+2t_7), \\ & (n-t_2+t_0+2t_4+t_5+t_7),(n-t_3+t_1+t_4+2t_5+t_6), \\ & (n-t_4+t_1+2t_2+t_3+t_6),(n-t_5+t_0+t_2+2t_3+t_7), \\ & (n-t_6+2t_0+t_1+t_3+t_4),(n-t_7+t_0+2t_1+t_2+t_5)\} \\ \leq& \frac{8n-n+5n}{8} \leq \frac{3n}{2}, \end{aligned}$$

so then $r_L(C_5) = r_L(C_7) \leq \frac{3n}{2}$.

If

$$\begin{aligned} x =& (\overbrace{00\cdots00}^{\frac{n}{8}}\overbrace{01\cdots01}^{\frac{n}{8}}\overbrace{0u\cdots0u}^{\frac{n}{8}}\overbrace{01+u\cdots01+u}^{\frac{n}{8}} \\ & \overbrace{10\cdots10}^{\frac{n}{8}}\overbrace{11\cdots11}^{\frac{n}{8}}\overbrace{1u\cdots1u}^{\frac{n}{8}}\overbrace{11+u\cdots11+u}^{\frac{n}{8}})\ \epsilon\ \mathbb{R}^n, \end{aligned}$$

then $d_L(x,\overline{00}) = d_L(x,\overline{01}) = d_L(x,\overline{0u}) = d_L(x,\overline{01+u}) = d_L(x,\overline{10}) = d_L(x,\overline{11}) = d_L(x,\overline{1u}) = d_L(x,\overline{11+u}) = \frac{n}{16}+2(\frac{n}{16})+\frac{n}{16}+\frac{n}{16}+\frac{n}{16}+\frac{n}{16}+\frac{n}{16}+2(\frac{n}{16})+\frac{n}{16}+\frac{n}{16} = \frac{3n}{4}$. Thus $r_L(C_5) = r_L(C_7) \geq \frac{3n}{4}$, and so $\frac{3n}{4} \leq r_L(C_5) = r_L(C_7) \leq \frac{3n}{2}$,

$$\begin{aligned} d_L(x,C_6) =& \min\{(n-t_0+t_2+t_5+2t_6+t_7),(n-t_2+t_0+2t_4+t_5+t_7),\\ &(n-t_4+t_1+2t_2+t_3+t_6),(n-t_6+2t_0+t_1+t_3+t_4)\}\\ \leq& \frac{4n+2n}{4} \leq \frac{3n}{2}, \end{aligned}$$

so then $r_L(C_6) \leq \frac{3n}{2}$.

If

$$x = (\overbrace{00\cdots00}^{\frac{n}{4}}\overbrace{0u\cdots0u}^{\frac{n}{4}}\overbrace{10\cdots10}^{\frac{n}{4}}\overbrace{1u\cdots1u}^{\frac{n}{4}})\ \epsilon\ \mathbb{R}^n,$$

then $d_L(x,\overline{00}) = d_L(x,\overline{0u}) = d_L(x,\overline{10}) = d_L(x,\overline{1u}) = 2(\frac{n}{8}) + \frac{n}{8} + \frac{n}{8} + 2(\frac{n}{8}) = \frac{3n}{4}$. Thus $r_L(C_6) \geq \frac{3n}{4}$, and hence $\frac{3n}{4} \leq r_L(C_6) \leq \frac{3n}{2}$. □

Example 2. *Consider the repetition codes of length $n = 4$ over $\mathbb{R} = \mathbb{Z}_2R^*$, where $R^* = \mathbb{Z}_2 + u\mathbb{Z}_2, u^2 = 0$. Then, we have the seven basic repetition codes of length $n = 4$ respectively.*

$C_1 = \{k(01010101) \mid k\ \epsilon\ \mathbb{R}\}$, $C_2 = \{k(0u0u0u0u) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_3 = \{k(01+u01+u01+u01+u) \mid k\ \epsilon\ \mathbb{R}\}$, $C_4 = \{k(10101010) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_5 = \{k(11111111) \mid k\ \epsilon\ \mathbb{R}\}$, $C_6 = \{k(1u1u1u1u) \mid k\ \epsilon\ \mathbb{R}\}$,
$C_7 = \{k(11+u11+u11+u11+u) \mid k\ \epsilon\ \mathbb{R}\}$.

Further we have, $r_L(C_1) = r_L(C_3) = 8$, $r_L(C_2) = 8$, $r_L(C_4) = 8$, $r_L(C_5) = r_L(C_7) = 6$ and $r_L(C_6) = 6$. The Covering radius of these repetition codes with respect to the Lee weight have achieved the upper bound that appeared in Theorem 4.2.

Theorem 4.3. *The covering radius of the block repetition code C^n have the following properties;*

$$r_E(C^n) \leq \tfrac{1}{2}[5(n_1+n_3+n_6)+3n_2+9n_4]+2(n_5+n_7),$$

and if $n_1 = \cdots = n_7 = n$,

$$r_L(C^{7n}) = 2n.$$

Proof. By Proposition 3.2, Theorem 4.1 and Theorem 4.2, we have

$$x = x_1x_2x_3x_4x_5x_6x_7 \, \epsilon \, \mathbb{R}^n$$

with $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ given by

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7), (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7),$
$(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7), (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7),$
$(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7), (f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7),$
$(g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7),$

respectively, such that

$$n_1 = \sum_{j=0}^{7} a_j,\ n_2 = \sum_{j=0}^{7} b_j,\ n_3 = \sum_{j=0}^{7} c_j,\ n_4 = \sum_{j=0}^{7} d_j,$$
$$n_5 = \sum_{j=0}^{7} e_j,\ n_6 = \sum_{j=0}^{7} f_j,\ n_7 = \sum_{j=0}^{7} g_j.$$

Then

$d_E(x, \overline{00}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_0 + 3e_2 + e_5 + 4e_6 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_0 + 3g_2 + g_5 + 4g_6 + g_7,$

where

$$\overline{00} = \overbrace{00\cdots00}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{00\cdots00}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{00\cdots00}^{n_5}\overbrace{00\cdots00}^{n_6}\overbrace{00\cdots00}^{n_7},$$

is the first vector of C^n, where $n = n_1$,

$d_E(x, \overline{y_1}) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + c_4 + 4c_5 + c_6 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_5 + e_0 + e_2 + 4e_3 + 3e_7 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_7 + g_0 + 4g_1 + g_2 + 3g_5,$

where

$$\overline{y_1} = \overbrace{01\cdots01}^{n_1}\overbrace{0u\cdots0u}^{n_2}\overbrace{01+u\cdots01+u}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{11\cdots11}^{n_5}\overbrace{1u\cdots1u}^{n_6}\overbrace{11+u\cdots11+u}^{n_7},$$

is the second vector of C^n, where $n = n_2$,

$d_E(x, \overline{y_2}) = n_1 - a_1 + 3a_3 + a_4 + a_6 + 4a_7 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_3 + 3c_1 + c_4 + 4c_5 + c_6 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_1 + 3e_3 + e_4 + e_6 + 4e_7 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_3 + 3g_1 + g_4 + 4g_5 + g_6,$

where

$$\overline{y_2} = \overbrace{01\cdots01}^{n_1}\overbrace{0u\cdots0u}^{n_2}\overbrace{01+u\cdots01+u}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{01\cdots01}^{n_5}\overbrace{0u\cdots0u}^{n_6}\overbrace{01+u\cdots01+u}^{n_7},$$

is the third vector of C^n, where $n = n_3$,
$d_E(x,\overline{y_3}) = n_1 - a_2 + 3a_0 + 4a_4 + a_5 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_2 + 3e_0 + 4e_4 + e_5 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_2 + 3g_0 + 4g_4 + g_5 + g_7$,
where

$$\overline{y_3} = \overbrace{0u\cdots0u}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{0u\cdots0u}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{0u\cdots0u}^{n_5}\overbrace{00\cdots00}^{n_6}\overbrace{0u\cdots0u}^{n_7},$$

is the fourth vector of C^n, where $n = n_4$,
$d_E(x,\overline{y_4}) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_3 + 3e_1 + e_4 + 4e_5 + e_6 + n_6 - f_2 + 3f_0 + 4f_4 + f_5 + f_7 + n_7 - g_1 + 3g_3 + g_4 + g_6 + 4g_7$,
where

$$\overline{y_4} = \overbrace{01+u\cdots01+u}^{n_1}\overbrace{0u\cdots0u}^{n_2}\overbrace{01\cdots01}^{n_3}\overbrace{00\cdots00}^{n_4}\overbrace{01+u\cdots01+u}^{n_5}$$
$$\overbrace{0u\cdots0u}^{n_6}\overbrace{01+u\cdots01+u}^{n_7},$$

is the fifth vector of C^n, where $n = n_5$.
$d_E(x,\overline{y_5}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_4 + e_1 + 4e_2 + e_3 + 3e_6 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_4 + g_1 + 4g_2 + g_3 + 3g_6$,
where

$$\overline{y_5} = \overbrace{00\cdots00}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{00\cdots00}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{10\cdots10}^{n_5}\overbrace{10\cdots10}^{n_6}\overbrace{10\cdots10}^{n_7},$$

is the sixth vector of C^n, where $n = n_6$.
$d_E(x,\overline{y_6}) = n_1 - a_2 + 3a_0 + 4a_4 + a_5 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_2 + 3c_0 + 4c_4 + c_5 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_6 + 4e_0 + e_1 + e_3 + 3e_4 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_6 + 4g_0 + g_1 + g_3 + 3g_4$,
where

$$\overline{y_6} = \overbrace{0u\cdots0u}^{n_1}\overbrace{00\cdots00}^{n_2}\overbrace{0u\cdots0u}^{n_3}\overbrace{10\cdots10}^{n_4}\overbrace{1u\cdots1u}^{n_5}\overbrace{10\cdots10}^{n_6}\overbrace{1u\cdots1u}^{n_7},$$

is the seventh vector of C^n, where $n = n_7$.
$d_E(x, \overline{y_7}) = n_1 - a_3 + 3a_1 + a_4 + 4a_5 + a_6 + n_2 - b_2 + 3b_0 + 4b_4 + b_5 + b_7 + n_3 - c_1 + 3c_3 + c_4 + c_6 + 4c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_7 + e_0 + 4e_1 + e_2 + 3e_5 + n_6 - f_6 + 4f_0 + f_1 + f_3 + 3f_4 + n_7 - g_5 + g_0 + g_2 + 4g_3 + 3g_7$,
where

$$\overline{y_7} = \overbrace{01+u\cdots 01+u}^{n_1}\overbrace{0u\cdots 0u}^{n_2}\overbrace{01\cdots 01}^{n_3}\overbrace{10\cdots 10}^{n_4}\overbrace{11+u\cdots 11+u}^{n_5}\overbrace{1u\cdots 1u}^{n_6}\overbrace{11\cdots 11}^{n_7},$$

is the eighth vector of C^n, where $n = n_8$.

Thus

$$\begin{aligned} r_E(C^n) \leq & \frac{8n_1 + 4(a_0 + a_1 + a_2 + a_3) + 12(a_4 + a_5 + a_6 + a_7) + 8n_2}{8} \\ & + \frac{8(b_0 + b_2 + b_5 + b_7) + 16(b_4 + b_6) + 8n_3 + 4(c_0 + c_1 + c_2 + c_3)}{8} \\ & + \frac{12(c_4 + c_5 + c_6 + c_7) + 8n_4 - 4(d_0 + d_4) + 4(d_1 + d_3 + d_5 + d_7)}{8} \\ & + \frac{28(d_2 + d_6) + 8n_5 + 8(e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7)}{8} \\ & + \frac{8n_6 + 12(f_0 + f_2 + f_4 + f_6) + 4(f_1 + f_3 + f_5 + f_7)}{8} \\ & + \frac{8n_7 + 8(g_0 + g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7)}{8}. \end{aligned}$$

Hence

$$r_E(C^n) \leq \tfrac{1}{2}[5(n_1 + n_3 + n_6) + 3n_2 + 9n_4] + 2(n_5 + n_7).$$

For the second part, we have that $\phi(C^{7n})$ is the set given by

$\{000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000000\cdots000,$
$001\cdots001011\cdots011010\cdots010100\cdots100101\cdots101111\cdots111110\cdots110,$
$001\cdots001011\cdots011010\cdots010000\cdots000001\cdots001011\cdots011010\cdots010,$
$011\cdots011000\cdots000011\cdots011000\cdots000011\cdots011000\cdots000011\cdots011,$
$010\cdots010011\cdots011001\cdots001000\cdots000010\cdots010011\cdots011001\cdots001,$
$000\cdots000000\cdots000000\cdots000100\cdots100100\cdots100100\cdots100100\cdots100,$
$011\cdots011000\cdots000011\cdots011100\cdots100111\cdots111100\cdots100111\cdots111,$
$010\cdots010011\cdots011001\cdots001100\cdots100110\cdots110111\cdots111101\cdots101\}.$

Then $r_L(c^{7n}) = r(\phi(C^{7n})) = 2n$. $\square$

5. Simplex codes of types α and β

In this section, we consider the construction of simplex codes of types α and β over $\mathbb{R}$. Let $m^{\alpha}_{2,k}$ be the generator matrix of $S^{\alpha}_{2,k}$, the binary simplex code of type α, defined as

$$\left[\begin{array}{c|c} 00\cdots0 & 11\cdots1 \\ \hline m^{\alpha}_{2,k-1} & m^{\alpha}_{2,k-1} \end{array}\right], \text{ for } k \geq 2,$$

where

$$m^{\alpha}_{2,1} = [0, 1].$$

In [6], the simplex codes $S^{\alpha}_{4,k}$ of type α over R^* were defined. The generator matrix $G^{\alpha}_{R^*,k}$ of $S^{\alpha}_{R^*,k}$ is

$$\left[\begin{array}{c|c|c|c} 00\cdots0 & 11\cdots1 & uu\cdots u & 1+u1+u\cdots1+u \\ \hline G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1} & G^{\alpha}_{R^*,k-1} \end{array}\right], \text{ for } k \geq 2,$$

where

$$G^{\alpha}_{R^*,1} = [0\ 1\ u\ 1+u].$$

We define the generator matrix of S^{α}_{k}, the simplex code of type α over $\mathbb{R}$, as the concatenation of 2^{2k} copies of the generator matrix of $S^{\alpha}_{2,k}$ and 2^{k} copies of the generator matrix of $S^{\alpha}_{R^*,k}$ given by

$$\Theta^{\alpha}_{k} = \left[\begin{array}{c|c|c|c|c|c|c|c} m^{\alpha}_{2,k} & m^{\alpha}_{2,k} & \cdots & m^{\alpha}_{2,k} & G^{\alpha}_{R^*,k} & G^{\alpha}_{R^*,k} & \cdots & G^{\alpha}_{R^*,k} \end{array}\right], \text{ for } k \geq 1,$$

The standard form of Θ^{α}_{k} of the generator matrix of S^{α}_{k} is given by

$$\Theta^{\alpha}_{k} = \left[\begin{array}{c|c|c|c} 00\ 00\cdots00 & 01\ 01\cdots01 & \cdots & 11+u1\ 1+u\cdots11+u \\ \hline \Theta^{\alpha}_{k-1} & \Theta^{\alpha}_{k-1} & \cdots & \Theta^{\alpha}_{k-1} \end{array}\right], \text{ for } k \geq 2,$$

where

$$\Theta^{\alpha}_{1} = [00\ 01\ 0u\ 01+u\ 10\ 11\ 1u\ 11+u].$$

The length of the simplex code of type α over $\mathbb{R}$ is equal to 2^{3k+1}, and the number of codewords is equal to $2^{k_0}{R^*}^{k_1}$ for some k_0 and k_1. In the case where $k = 1$ with $k_0 = 0$ and $k_1 = 1$, we have that, all of the codewords of the simplex code S^{α}_{1} are generated by Θ^{α}_{1}, and are given by

$00\ 00\ 00\ 00\ 00\ 00\ 00\ 00$

$00\ 01\ 0u\ 01+u\ 10\ 11\ 1u\ 11+u$

$00\ 0u\ 00\ 0u\ 00\ 0u\ 00\ 0u$

$00\ 01+u\ 0u\ 01\ 10\ 11+u\ 1u\ 11$

The type β simplex code S_k^β is a punctured version of S_k^α. The number of codewords is $2^{k_0}R^{*k_1}$ for some k_0 and k_1, and its length is $2^k(2^{k-2}+1)(2^k-1)$.

The generator matrix of S_k^β is the concatenation of 2^k copies of the generator matrix of $S_{2,k}^\beta$ and 2^{k-1} copies of the generator matrix of $S_{R^*,k}^\beta$ given by

$$\Theta_k^\beta = \left[\; m_{2,k}^\beta \;\middle|\; m_{2,k}^\beta \;\middle|\; \cdots \;\middle|\; m_{2,k}^\beta \;\middle|\; G_{R^*,k}^\beta \;\middle|\; G_{R^*,k}^\beta \;\middle|\; \cdots \;\middle|\; G_{R^*,k}^\beta \;\right], \text{ for } k \geq 2, \tag{5.1}$$

where $m_{2,k}^\beta$ is the generator matrix of the binary simplex code of type β given by

$$\left[\begin{array}{c|c} 11\cdots1 & 00\cdots0 \\ \hline m_{2,k-1}^\alpha & m_{2,k-1}^\beta \end{array}\right], \text{ for } k \geq 3,$$

with

$$m_{2,2}^\beta = \left[\begin{array}{c|c} 11 & 0 \\ \hline 01 & 1 \end{array}\right],$$

and $G_{R^*,k}^\beta$ is a generator matrix of the simplex code over R^* of type β defined as

$$\left[\begin{array}{c|c|c} 11\cdots1 & 00\cdots0 & uu\cdots u \\ \hline G_{R^*,k-1}^\alpha & G_{R^*,k-1}^\beta & G_{R^*,k-1}^\beta \end{array}\right], \text{ for } k \geq 3,$$

with

$$G_{R^*,2}^\beta = \left[\begin{array}{c|c|c} 1111 & 0 & u \\ \hline 01u1+u & 1 & 1 \end{array}\right].$$

The following theorems provide upper bounds on the covering radius of simplex codes over $\mathbb{R}$ with respect to the Lee and Euclidean weights.

Theorem 5.1. *The covering radius of the* $\mathbb{R}-$*simplex codes of type* α *are upper bounded as follows:*

$$r_L(S_k^\alpha) \leq 3.2^{3k-1} \textit{ and } r_E(S_k^\alpha) \leq 2^k(\tfrac{7.2^{2k}-1}{3}).$$

Proof. $\mathbb{R}$-simplex codes of type α have a Lee weight equal to 2^{3k} or 3.2^{3k-1}. Hence, from (3.1), Proposition 3.2 and Theorem 4.2, we have

$$\begin{aligned} r_L(S_k^\alpha) &\leq r_L(2^{2k}S_{2,\,k}^\alpha) + r_L(2^kS_{R^*,\,k}^\alpha) \\ &\leq 2^{2k}r_L(S_{2,\,k}^\alpha) + 2^kr_L(S_{R^*,\,k}^\alpha) \\ &\leq 2^{2k}r_H(S_{2,\,k}^\alpha) + 2^kr_L(S_{R^*,\,k}^\alpha) \\ &\leq 2^{2k}(2^{k-1}) + 2^k[(3.2^{2(k-1)} + 3.2^{2(k-2)} + \cdots + 3.2^{2.1}) + r_L(S_{R^*,\,1}^\alpha)] \\ &\leq 2^{3k-1} + 2^k[(2^{2k}-1)+1] \\ &\leq 2^{3k-1} + 2^k.2^{2k} \\ &\leq 2^{3k-1} + 2^{3k} \end{aligned}$$

$$\leq 3.2^{3k-1}.$$

Thus $r_L(S_k^\alpha) \leq 3.2^{3k-1}$. Similar arguments using (3.1), Proposition 3.2 and Theorem 4.2 give that

$$\begin{aligned}
r_E(S_k^\alpha) &\leq r_E(2^{2k} S_{2,\ k}^\alpha) + r_E(2^k S_{R^*,\ k}^\alpha) \\
&\leq 2^{2k} r_E(S_{2,\ k}^\alpha) + 2^k r_E(S_{R^*,\ k}^\alpha) \\
&\leq 2^{2k} r_H(S_{2,\ k}^\alpha) + 2^k r_E(S_{R^*,\ k}^\alpha) \\
&\leq 2^{2k}.2^{k-1} + 2^k(\frac{11(2^{2k}-1)+9}{6}) \\
&\leq 2^k[2^{2k-1} + (\frac{11(2^{2k}-1)+9}{6})] \\
&\leq 2^k(\frac{7.2^{2k}-1}{3}).
\end{aligned}$$

□

Theorem 5.2. *The covering radius of the* ℝ*−simplex codes of type* β *are given by*

(i) $r_L(S_k^\beta) \leq 2^{k-1}[(2^{k-1}+1)(2^k-1)-2]$,

(ii) $r_E(S_k^\beta) \leq 2^{k-1}(\dfrac{14.2^{2k}-449}{6})$.

Proof. From (3.1), (5.1), Proposition 3.2 and Theorem 4.2, we have

$$\begin{aligned}
r_L(S_k^\beta) &\leq r_L(2^k S_{2,\ k}^\beta) + r_L(2^{k-1} S_{R^*,\ k}^\beta) \\
&\leq 2^k r_L(S_{2,\ k}^\beta) + 2^{k-1} r_L(S_{R^*,\ k}^\beta) \\
&\leq 2^k r_H(S_{2,\ k}^\beta) + 2^{k-1} r_L(S_{R^*,\ k}^\beta) \\
&\leq 2^k(\frac{2^k-1)}{2} + 2^{k-1}[2^{k-1}(2^k-1)-2] \\
&\leq 2^{k-1}(2^k-1) + 2^{k-1}[2^{k-1}(2^k-1)-2] \\
&\leq 2^{k-1}[(2^{k-1}+1)(2^k-1)-2].
\end{aligned}$$

Similar arguments using (3.1), Proposition 3.2 and Theorem 4.2 give that

$$\begin{aligned}
r_E(S_k^\beta) &\leq r_E(2^k S_{2,\ k}^\beta) + r_E(2^{k-1} S_{R^*,\ k}^\beta) \\
&\leq 2^k r_E(S_{2,\ k}^\beta) + 2^{k-1} r_E(S_{R^*,\ k}^\beta) \\
&\leq 2^k r_H(S_{2,\ k}^\beta) + 2^{k-1} r_E(S_{R^*,\ k}^\beta) \\
&\leq 2^k(\frac{2^k-1)}{2} + 2^{k-1}[2^k(2^{k+1}-1) + \frac{1}{3}(2^{2k}-1) - \frac{147}{2}] \text{ (since } r_E(S_2^\beta) \leq 25) \\
&\leq 2^{k-1}(2^k-1) + 2^{k-1}[2^k(2^{k+1}-1) + \frac{1}{3}(2^{2k}-1) - \frac{147}{2}] \\
&\leq 2^{k-1}[(2^k(2^{k+1}-1) + \frac{1}{3}(2^{2k}-1) + (2^k-1) - \frac{147}{2}] \\
&\leq 2^{k-1}(\frac{14.2^{2k}-449}{6}).
\end{aligned}$$

□

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