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BETWEEN NANO CLOSED SETS AND NANO GENERALIZED CLOSED SETS IN NANO TOPOLOGICAL SPACES

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ABSTRACT. The aim of this paper, we offer a new class of sets called N$\ddot{g}$-closed sets in Nano topological spaces and we study some of its basic properties. It turns out that this class lies between the class of Nano closed sets and the class of Nano generalized closed sets. As applications of N$\ddot{g}$-closed sets, we introduce T$_{N\ddot{g}}$-spaces, $_g$T$_{N\ddot{g}}$-spaces and $_\alpha$T$_{N\ddot{g}}$-spaces. Moreover, we obtain certain new characterizations for the T$_{N\ddot{g}}$-spaces, $_g$T$_{N\ddot{g}}$-spaces and $_\alpha$T$_{N\ddot{g}}$-spaces.

1. INTRODUCTION

Lellis Thivagar and Carmel Richard [6] introduced and studied Nano semi-open, Nano α-open, Nano-preopen and Nano regular open respectively. Revathy and Illango [7] introduced and studied Nano β-open sets. Bhuvaneswari and Mythili Gnanapriya [1] introduced Nano generalised closed sets. Lalitha and Francina Shalini [5] introduced Nano generalized $\wedge$-closed and open sets in Nano topological spaces. Bhuvaneshwari and Ezhilarasi [3] , Thanga Nachiyar and Bhunaneswari [9] defined Nano semi generalized closed sets and Nano generalized semi closed sets, Nano αg-closed sets respectively.
S.Ganesan et al [4] studied Nano *g-closed sets. The aim of this paper, we introduce and study some basic properties of Nano $\ddot{g}$-closed sets. It turns out that

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this class lies between the class of Nano closed sets and the class of Nano generalized closed sets. As applications of N$\ddot{g}$-closed sets, we introduce and study three new spaces, namely T$_{N\ddot{g}}$-spaces, $_g$T$_{N\ddot{g}}$-spaces and $_\alpha$T$_{N\ddot{g}}$-spaces. Moreover, we obtain their properties and characterizations.

2. Preliminaries

Let (U, $\tau_R(X)$) (or U) represent Nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset M of a space (U, $\tau_R(X)$), clo(M) $\&$ inte(M) denote the closure of M $\&$ the interior of M respectively.

Definition 2.1. *[6] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let X $\subseteq$ U.*

(1) *The lower approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup x \in U\{R(X) : R(X) \subseteq X\}$ where $R(x)$ denotes the equivalence class determined by X.*
(2) *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup$ x $\in$ U {R(X): R(X) $\cap$ X $\neq \phi$}.*
(3) *The boundary region of X with respect to R is the set of all objects, which can be neither in nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Proposition 2.1. *[6] If (U, R) is an approximation space and X, Y $\subseteq$ U, then*

(1) $L_R(X) \subseteq X \subseteq U_R(X)$.
(2) L_R (ϕ) =U_R (ϕ) = ϕ, $L_R(U) = U_R(U)$ = U.
(3) U_R (X $\cup$ Y) = $U_R(X) \cup U_R(Y)$.
(4) U_R (X $\cap$ Y) $\subseteq U_R(X) \cap U_R(Y)$.
(5) L_R (X $\cup$ Y) $\subseteq L_R(X) \cup L_R(Y)$.
(6) L_R (X $\cap$ Y) = $L_R(X) \cap L_R(Y)$.
(7) $L_R(X) \subseteq L_R(Y)$ *and* $U_R(X) \subseteq U_R(Y)$ *whenever* X $\subseteq$ Y.
(8) U_R (X^c) = L_R (X^c)] *and* $L_R(X^c)$ = [U_R (X^c)].

(9) $U_R(U_R(X)) = L_R(U_R(X)) = U_R(X)$.
(10) $L_R(L_R(X)) = U_R(L_R(X)) = L_R(X)$.

Definition 2.2. *[6] Let U be an universe, R be an equivalence relation on U and* $\tau_R(X)$ *=*$U, \phi, L_R(X), U_R(X), B_R(X)$ *where X ⊆ U. Then by Proposition 2.1,* $\tau_R(X)$ *satisfies the following axioms:*

(1) *U ,* $\phi \in \tau_R(X)$.
(2) *The union of the elements of any sub-collection of* $\tau_R(X)$ *is in* $\tau_R(X)$.
(3) *The intersection of the elements of any finite sub collection of* $\tau_R(X)$ *is in* $\tau_R(X)$.

Then $\tau_R(X)$ *is called the Nano topology on U with respect to X.*

The space (U, $\tau_R(X)$*) is the Nano topological space. The elements of are called Nano open sets.*

Definition 2.3. *[6] If (U,* $\tau_R(X)$*) is the Nano topological space with respect to X where X ⊆ U and if M ⊆ U, then*

(1) *The Nano interior of the set M is defined as the union of all Nano open subsets contained in M and it is denoted by NInte(M). That is, NInte(M) is the largest Nano open subset of M.*
(2) *The Nano closure of the set M is defined as the intersection of all Nano closed sets containing M and it is denoted by NClo(M). That is, NClo(M) is the smallest Nano closed set containing M.*

Definition 2.4. *A subset M of a space (U,* $\tau_R(X)$*) is called:*

(1) *Nano* α*-open set [6] if M ⊆ Ninte(Nclo(Ninte(M))).*
(2) *Nano semi-open set [6] if M ⊆ Nclo(Ninte(M)).*
(3) *Nano pre-open set [6] if M ⊆ Ninte(Nclo(M)).*
(4) *Nano* β*-open set [7] if M ⊆ Nclo(Ninte(Nclo(M))).*
(5) *Nano regular-open set [6]if M = Ninte(Nclo(M)).*

The complements of the above mentioned Nano open sets are called their respective Nano closed sets.

The Nano α-closure (resp. Nano semi-closure, Nano pre-closure [2], Nano semi-pre-closure) of a subset M of U, denoted by Nαclo(M) (resp.Nsclo(M), Npclo(M) , Nβclo(M)) is defined to be the intersection of all Nano α-closed (resp. Nano semi-closed, Nano pre-closed, Nano β-closed) sets of (U, $\tau_R(X)$)

containing M.
The Nano α-interior(resp. Nano semi-interior, Nano pre-interior [2], Nano semi-pre-interior) of a subset M of U, denoted by Nαinte(M) (resp.Nsinte(M), Npinte(M), Nβinte(M)) is defined to be the union of all Nano α-open (resp. Nano semi-open, Nano pre-open, Nano β-open) sets of (U, $\tau_R(X)$) containing M.

Definition 2.5. *A subset M of a space (U, $\tau_R(X)$) is called:*

(1) *a Nano generalized closed (briefy Ng-closed) set [1] if Nclo(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano open in (U, $\tau_R(X)$).*
(2) *a Nano generalized semi-closed (briefly Ngs-closed) set [3] if Nsclo(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano open in (U, $\tau_R(X)$).*
(3) *a Nano semi generalized closed (briefly Nsg-closed) set [3] if Nsclo(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano semi-open in (U, $\tau_R(X)$).*
(4) *an Nano α-generalized closed (briefly Nαg-closed) set [9] if Nαclo(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano open in (U, $\tau_R(X)$).*
(5) *a Nano $\hat{g}$-closed (briefly N$\hat{g}$-closed) set [5] if Nclo(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano semi-open in (U, $\tau_R(X)$).*
(6) *a Nano generalized semi pre-closed (briefly Ngsp-closed) set [8] Nspcl(M) $\subseteq$ T whenever M $\subseteq$ T and T is Nano open in (U, $\tau_R(X)$).*
The complements of above Nano closed sets is called Nano open sets.

Remark 2.1. *The collection of all Ng-closed (resp. Ngs-closed, Nsg-closed, Nαg-closed, N$\hat{g}$-closed, Ngsp-closed, Nano semi-closed, Nano pre-closed, Nano β-closed) sets is denoted by Ngc($\tau_R(X)$) (resp. Ngsc($\tau_R(X)$), Nsgc(($\tau_R(X)$), Nαgc($\tau_R(X)$), N$\hat{g}$c($\tau_R(X)$), Ngspc(($\tau_R(X)$), Nsc(($\tau_R(X)$), Npc(($\tau_R(X)$), Nβc(($\tau_R(X)$)).*

We denote the power set of U by P(U).

Definition 2.6. *[4] A space (U, $\tau_R(X)$) is called:*

(i) *$T_{N1/2}$-space if every Ng-closed set is Nano closed.*
(ii) *T_{Nb}-space if every Ngs-closed set is Nano closed.*
(iii) *NαT_b-space if every Nαg-closed set is Nano closed.*
(iv) *$T_N\hat{g}$-space if every Nano semi-closed set is Nano closed.*
(v) *Nsemi-$T_{1/2}$-space if every Nsg-closed set is Nano semi closed.*

3. N$\ddot{g}$-CLOSED AND N$\ddot{g}$-OPEN SETS

We introduce the following definitions.

Definition 3.1. *A subset M of a space (U, $\tau_R(X)$) is called*
(i) *Nano $\ddot{g}$-closed (briefly N$\ddot{g}$-closed) set if Nclo(M) ⊆ T whenever M ⊆ T and T is Nsg-open in (U, $\tau_R(X)$). The complement of N$\ddot{g}$-closed set is called N$\ddot{g}$-open set.*
(ii) *Nano $\ddot{g}_\alpha$-closed (briefly N$\ddot{g}_\alpha$-closed) set if Nαclo(M) ⊆ T whenever M ⊆ T and T is Nsg-open in (U, $\tau_R(X)$). The complement of N$\ddot{g}_\alpha$-closed set is called N$\ddot{g}_\alpha$-open set.*

The collection of all $N\ddot{g}$- closed (resp. $N\ddot{g}_\alpha$- closed) sets is denoted by $N\ddot{g}c((\tau_R(X))$ (resp. $N\ddot{g}_\alpha c(\tau_R(X))$.

Proposition 3.1. *Every Nano closed set is N$\ddot{g}$-closed.*

Proof. Let M be a Nano closed set and T be any Nsg-open set containing M. Since M is nano closed, we have Nclo(M) = M ⊆ T. Hence M is N$\ddot{g}$-closed. □

The converse of Proposition 3.1 need not be true as seen from the following example.

Example 1. *Let U = {p, q, r} with U/ R= {{r}, {p, q} {q, p} and X= {p, q}. The Nano topology $\tau_R(X)$= {ϕ, {p, q}, U}. Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {r}, {p, r}, {q, r}, U}. Here, H = {p, r} is N$\ddot{g}$-closed set but not Nano closed.*

Proposition 3.2. *Every N$\ddot{g}$-closed set is Ng-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nano open set containing M. Since every Nano open set is Nsg-open, we have Nclo(M) ⊆ T. Hence M is Ng-closed. □

The converse of Proposition 3.2 need not be true as seen from the following example.

Example 2. *Let U = {p, q, r, s}, with U/ R= {{p}, {r}, {q, s} and X= {p, q}. Then the Nano topology $\tau_R(X)$ = {ϕ, {p}, {q, s}, {p, q, s}, U} . Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {r}, {p, r}, {q, r, s}, U} and Ngc($\tau_R(X)$) = {ϕ, {r}, {p, r},{q, r},{r, s}, {p, q, r}, {p, r, s}, {q, r, s}, U}. Here, H = {q, r} is Ng-closed set but not N $\ddot{g}$ -closed.*

Proposition 3.3. *Every N$\ddot{g}$-closed set is N$\hat{g}$-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nano semi-open set containing M. Since every Nano semi-open set is Nsg-open, we have Nclo(M) $\subseteq$ T. Hence M is N$\hat{g}$-closed. □

The converse of Proposition 3.3 need not be true as seen from the following example.

Example 3. *Let U = {p, q, r}, with U/ R= {{p}, {q, r}} and X= {p, q}. Then the Nano topology $\tau_R(X)$ = {ϕ, {p}, {q, r}, U}. Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {p}, {q, r}, U} and N$\hat{g}$ c ($\tau_R(X)$) = P(U). Here, H = {p, q} is N$\hat{g}$-closed set but not N$\ddot{g}$-closed.*

Proposition 3.4. *Every N$\ddot{g}$-closed set is Ngs-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nano-open set containing M. Since every Nano open set is Nsg-open, we have Nsclo(M) $\subseteq$ Nclo(M) $\subseteq$ T. Hence M is Ngs-closed. □

The converse of Proposition 3.4 need not be true as seen from the following example.

Example 4. *Let U and $\tau_R(X)$ as in the Example 3. Then Ngsc($\tau_R(X)$) = P(U). Here, H = {p, q} is Ngs-closed set but not N$\ddot{g}$-closed.*

Proposition 3.5. *Every N$\ddot{g}$-closed set is Ngsp-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nano-open set containing M. Since every Nano open set is Nsg-open, we have Nspclo(M) $\subseteq$ Nclo(M) $\subseteq$ T. Hence M is Ngsp-closed. □

The converse of Proposition 3.5 need not be true as seen from the following example.

Example 5. *Let U and $\tau_R(X)$ as in the Example 3. Then Ngspc($\tau_R(X)$) = P(U). Here, H = {r} is Ngsp-closed set but not N$\ddot{g}$-closed.*

Proposition 3.6. *Every N$\ddot{g}$-closed set is Nαg-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nano-open set containing M. Since every Nano open set is Nsg-open, we have Nαclo(M) $\subseteq$ Nclo(M) $\subseteq$ T. Hence M is Nαg-closed. □

The converse of Proposition 3.6 need not be true as seen from the following example.

Example 6. *Let U and $\tau_R(X)$ as in the Example 3. Then Nα gc($\tau_R(X)$) = P(U). Here, H = {q} is Nα g-closed set but not N$\ddot{g}$-closed.*

Proposition 3.7. *Every N$\ddot{g}$-closed set is N$\ddot{g}_\alpha$-closed.*

Proof. Let M be an N$\ddot{g}$-closed set and T be any Nsg-open set containing M. We have Nαclo(M) $\subseteq$ Nclo(M) $\subseteq$ T. Hence M is N$\ddot{g}_\alpha$-closed. □

The converse of Proposition 3.7 need not be true as seen from the following example.

Example 7. *Let U = {p, q, r}, with U/ R= {{p}, {q, r} and X= {p}. Then the Nano topology $\tau_R(X)$ = {ϕ, {p}, U}. Here N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {q, r}, U}, $N\ddot{g}$ α c ($\tau_R(X)$) = {ϕ, {q}, {r}, {q, r}, U}. Here, H = {q} is N$\ddot{g}_\alpha$-closed but not N$\ddot{g}$-closed.*

Remark 3.1. *we obtain the following diagram where A$\rightarrow$B represents A implies B, but not conversely.*

Nano closed set $\longrightarrow$ N$\ddot{g}$-closed set $\longrightarrow$ N$\hat{g}$-closed $\longrightarrow$ Ng-closed
$\downarrow$
Nαg-closed

Remark 3.2. *If P and Q are N$\ddot{g}$-closed sets, then P $\cup$ Q is also a N$\ddot{g}$-closed set.*

Proof. It follows from the fact that Nclo(P $\cup$ Q) = Nclo(P) $\cup$ Nclo(Q) . □

Example 8. *Let U = {p, q, r, s, t}, with U/ R= {{s}, {p, q}, {r, t}} and X= {p, s}. Then the Nano topology $\tau_R(X)$ = {ϕ, {s}, {p, q}, {p, q, s}, U}. Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {r}, {t}, {p, r}, {p, t}, {q, s}, {q, t}, {r, s}, {r, t}, {s, t}, {p, q, r}, {p, q, t}, {p, r, s}, {p, r, t}, {p, s, t}, {q, r, s}, {q, r, t}, {q, s, t}, {r, s, t}, {p, q, r, s}, {p, q, r, t}, {p, q, s, t}, {p, r, s, t}, {q, r, s, t}, U}. Here, P = {r} and Q= {t} are N$\ddot{g}$-closed sets but P $\cup$ Q = {r, t} is also a N$\ddot{g}$-closed sets.*

Remark 3.3. *If K and L are N$\ddot{g}$-closed sets, then K $\cap$ L is a N$\ddot{g}$-closed set.*

Example 9. *Let U and $\tau_R(X)$ as in the Example 1. Here, k = {p, r} and L = {q, r} are N$\ddot{g}$-closed sets but K $\cap$ L = {r} is a N$\ddot{g}$-closed set.*

Proposition 3.8. *If A subset M of (U, $\tau_R(X)$) is a N$\ddot{g}$-closed if and only if Nclo(A) - M does not contain any nonempty Nsg-closed set.*

Proof. Necessity. Suppose that M is N$\ddot{g}$-closed. Let S be a Nsg-closed subset of Nclo(M) - M. Then M $\subseteq$ S^c . Since M is N$\ddot{g}$-closed, we have Nclo(M) $\subseteq$ S^c . Consequently, S $\subseteq$ $(Nclo(M))^c$. Hence, S $\subseteq$ Nclo(M) $\cap$ $(Nclo(M))^c$ = ϕ. Therefore S is empty.
Sufficiency. Suppose that Nclo(M) - M contains no nonempty Nsg-closed set. Let M $\subseteq$ G and G be Nsg-closed If Nclo(M) $\neq$ G, then Nclo(M) $\subseteq$ G^c $\neq \phi$. Since Nclo(M) is a Nano closed set and G^c is a Nsg-closed set, Nclo(M) $\cap$ G^c is a nonempty Nsg-closed subset of Nclo(M) - M. This is a contradiction. Therefore, Nclo(M) $\subseteq$ G and hence M is N$\ddot{g}$-closed. □

Proposition 3.9. *If A is N$\ddot{g}$-closed in (U, $\tau_R(X)$) such that A $\subseteq$ B $\subseteq$ Nclo(A), then B is also a N$\ddot{g}$-closed set of (U, $\tau_R(X)$).*

Proof. Let W be a Nsg-open set of (U, $\tau_R(X)$) such that B $\subseteq$ W. Then A $\subseteq$ W. Since A is N$\ddot{g}$-closed, we get, Nclo(A) $\subseteq$ W. Now Nclo(B) $\subseteq$ Nclo(Nclo(A)) = Nclo(A) $\subseteq$ W. Therefore, B is also a N$\ddot{g}$-closed set of (U, $\tau_R(X)$). □

Definition 3.2. *The intersection of all Nsg-open subsets of (U, $\tau_R(X)$) containing A is called the Nano sg-kernel of A and denoted by Nsg-ker(A).*

Lemma 3.1. *A subset A of (U, $\tau_R(X)$) is N$\ddot{g}$-closed if and only if Ncl(A) $\subseteq$ Nsg-ker(A).*

Proof. Suppose that A is N$\ddot{g}$-closed. Then Ncl(A) $\subseteq$ U whenever A $\subseteq$ U and U is Nsg-open. Let x $\in$ Ncl(A). If x $\notin$ Nsg-ker(A), then there is a Nsg-open set U containing A such that x $\notin$ U. Since U is a Nsg-open set containing A, we have x $\notin$ Ncl(A) and this is a contradiction.

Conversely, let Ncl(A) $\subseteq$ Nsg-ker(A). If U is any Nsg-open set containing A, then Ncl(A) $\subseteq$ Nsg-ker(A) $\subseteq$ U. Therefore, A is N$\ddot{g}$-closed. □

Definition 3.3. *A subset M of a space U is said to be N$\ddot{g}$-open if M^C is N$\ddot{g}$-closed.*

The class of all N$\ddot{g}$-open subsets of U is denoted by N$\ddot{g}$o($\tau_R(X)$).

Proposition 3.10. (1) *Every Nano open set is N$\ddot{g}$-open set but not conversely.*
(2) *Every N$\ddot{g}$-open set is Ng-open set but not conversely.*

(3) *Every Ng-open set is N$\hat{g}$-open set but not conversely.*
(4) *Every N$\ddot{g}$-open set is Ngs-open set but not conversely.*
(5) *Every N$\ddot{g}$-open set is Nαg-open set but not conversely.*
(6) *Every N$\ddot{g}$-open set is N$\ddot{g}_\alpha$-open set but not conversely.*

Proposition 3.11. *A subset M of a Nano topological space U is said to N$\ddot{g}$-open if and only if P $\subseteq$ Ninto(M) whenever M $\supseteq$ P and P is Nsg-closed in U.*

Proof. Suppose that M is N$\ddot{g}$-open in U and M $\supseteq$ P, where P is Nsg-closed in U. Then $M^c \subseteq P^c$, where P^c is Nsg-open-open in U. Hence we get Nclo (M^c) $\subseteq$ P^c implies ($Ninto(M))^c \subseteq P^c$. Thus, we have Ninto(M) $\supseteq$ P.

Conversely, suppose that M^c $\subseteq$ T and T is Nsg-open-open in U then M $\supseteq$ T^c and T^c is Nsg-closed then by hypothesis Ninto(M) $\supseteq$ T^c implies ($Ninto(M))^c$ $\subseteq$ T. Hence Nclo (M^c) $\subseteq$ T gives M^c is N$\ddot{g}$-closed. □

Proposition 3.12. *In a Nano topological space U, for each u $\in$ U, either {u} is Nsg-closed or N$\ddot{g}$-open in U.*

Proof. Suppose that {u} is not Nsg-closed in U. Then $\{u\}^c$ is not Nsg-open-open and the only Nsg-open set containing $\{u\}^C$ is the space U itself. Therefore, Nclo $(\{u\}^C)$ $\subseteq$ U and so $\{u\}^C$ is N$\ddot{g}$-closed gives {u} is N$\ddot{g}$-open. □

4. Application

We introduce the following definitions.

Definition 4.1. *A space (U, $\tau_R(X)$) is called a $T_{N\ddot{g}}$-space if every N$\ddot{g}$-closed set in it is Nano closed.*

Example 10. *Let U = {p, q, r}, with U/ R= {{q}, {p, r}} and X= {q}. Then the Nano topology $\tau_R(X)$ = {ϕ, {q}, U}. Here N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {p, r}, U}. Thus (U, $\tau_R(X)$) is a $T_{N\ddot{g}}$-space .*

Example 11. *Let U = {p, q, r} with U/ R= {{q}, {p, r} {r, p}} and X= {p, r}. The Nano topology $\tau_R(X)$= {ϕ, {p, r}, U}. Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {q}, {p, q}, {q, r}, U}. Thus (U, $\tau_R(X)$) is not a $T_{N\ddot{g}}$-space.*

Proposition 4.1. *Every $T_{N1/2}$-space is $T_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.2. □

The converse of Proposition 4.1 need not be true as seen from the following example.

Example 12. *Let U and $\tau_R(X)$ as in the Example 10. Ngc($\tau_R(X)$) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, U}. Thus (U, $\tau_R(X)$) is not a $T_{N1/2}$-space.*

Proposition 4.2. *Every $N_\alpha T_b$-space is $T_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.6. □

The converse of Proposition 4.2 need not be true as seen from the following example.

Example 13. *Let U and $\tau_R(X)$ as in the Example 10. Nαgc($\tau_R(X)$) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, U}. Thus (U, $\tau_R(X)$) is not a $N_\alpha T_b$-space.*

Proposition 4.3. *Every T_{Nb}-space is $T_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.4. □

The converse of Proposition 4.3 need not be true as seen from the following example.

Example 14. *Let U and $\tau_R(X)$ as in the Example 10. Ngsc($\tau_R(X)$) = {ϕ, {p}, {r}, {p, q}, {p, r}, {q, r}, U}. Thus (U, $\tau_R(X)$) is not a T_{Nb}-space.*

Proposition 4.4. *Every $T_N\hat{g}$-space is $T_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.3. □

The converse of Proposition 4.4 need not be true as seen from the following example.

Example 15. *Let U = {p, q, r}, with U/ R= {{q}, {p, r} and X= {q, r}. Then the Nano topology $\tau_R(X)$ = {ϕ, {q}, {p, r}, U}. Then N$\ddot{g}$ c ($\tau_R(X)$) = {ϕ, {q}, {p, r}, U} and N$\hat{g}$ c ($\tau_R(X)$) = P(U). Thus (U, $\tau_R(X)$) is $T_{N\ddot{g}}$-space but not a $T_N\hat{g}$-space.*

Definition 4.2. *A space (U, $\tau_R(X)$) is called a $T_{N\alpha}$-space if every Nano α-closed set in it is Nano closed.*

Remark 4.1. *$T_{N\ddot{g}}$-spaces and $T_{N\alpha}$-spaces are independent.*

Example 16. *Let U and $\tau_R(X)$ as in the Example 10, Nαc($\tau_R(X)$) = {ϕ, {p}, {r}, {p, r}, U}. Thus (U, $\tau_R(X)$) is a $T_{N\ddot{g}}$-space but not a $T_{N\alpha}$-space.*

Example 17. *Let U and $\tau_R(X)$ as in the Example 11, Nαc($\tau_R(X)$) = {ϕ, {q}, U}. Thus (U, $\tau_R(X)$) is a $T_{N\alpha}$ but not $T_{N\ddot{g}}$-space.*

Theorem 4.1. *For a space (U, $\tau_R(X)$) the following properties are equivalent:*

(i) *(U, $\tau_R(X)$) is a $T_{N\ddot{g}}$-space.*
(ii) *Every singleton subset of (U, $\tau_R(X)$) is either Nsg-closed or Nano open.*

Proof. (i) $\rightarrow$ (ii). Assume that for some u $\in$ U, the set {u} is not a Nsg-closed in (U, $\tau_R(X)$). Then the only Nsg-open-open set containing $\{u\}^c$ is U and so $\{u\}^c$ is N$\ddot{g}$-closed in (U, $\tau_R(X)$). By assumption $\{u\}^c$ is Nano closed in (U, $\tau_R(X)$) or equivalently {u} is Nano open.

(ii) $\rightarrow$ (i). Let M be a N$\ddot{g}$-closed subset of (U, $\tau_R(X)$) and let u $\in$ Nclo(M). By assumption {u} is either Nsg-closed or Nano open.

Case (a) Suppose that {u} is Nsg-closed. If u $\notin$ M, then Nclo(M) - M contains a nonempty Nsg-closed set {u}, which is a contradiction to Theorem 3.8. Therefore x $\in$ M.

Case (b) Suppose that {u} is Nano open. Since u $\in$ Nclo(M), {u} $\cap$ M $\neq \phi$ and so u $\in$ M. Thus in both case, u $\in$ M and therefore Nclo(M) $\subseteq$ M or equivalently M is a Nano closed set of (U, $\tau_R(X)$). □

5. $_g$T$_{N\ddot{g}}$-SPACES

Definition 5.1. *A space (U, $\tau_R(X)$) is called a $_gT_{N\ddot{g}}$-space if every Ng-closed set in it is N$\ddot{g}$-closed.*

Example 18. *Let X and τ as in the Example 11, is a $_gT_{N\ddot{g}}$-space and the space (U, $\tau_R(X)$) in the Example 10, is not a $_gT_{N\ddot{g}}$-space.*

Proposition 5.1. *Every $T_{N1/2}$-space is $_gT_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.1. □

The converse of Proposition 5.1 need not be true as seen from the following example.

Example 19. *Let X and τ as in the Example 11, is a ${}_gT_{N\ddot{g}}$-space but not a $T_{N1/2}$-space.*

Remark 5.1. *$T_{\ddot{g}}$-space and ${}_gT_{N\ddot{g}}$-space are independent.*

Example 20. *The space (U, $\tau_R(X)$) in the Example 11, is a ${}_gT_{N\ddot{g}}$-space but not a $T_{N\ddot{g}}$-space and the space (U, $\tau_R(X)$) in the Example 10, is a $T_{N\ddot{g}}$-space but not a ${}_gT_{N\ddot{g}}$-space.*

Theorem 5.1. *If (U, $\tau_R(X)$) is a ${}_gT_{N\ddot{g}}$-space, then every singleton subset of (U, $\tau_R(X)$) is either Ng-closed or N$\ddot{g}$-open.*

Proof. Assume that for some x $\in$ X, the set {x} is not a Ng-closed in (U, $\tau_R(X)$). Then {x} is not a Nano closed set, since every Nano closed set is a Ng-closed set. So $\{x\}^c$ is not Nano open and the only Nano open set containing $\{x\}^c$ is X itself. Therefore $\{x\}^c$ is trivially a Ng-closed set and by assumption, $\{x\}^c$ is an N$\ddot{g}$-closed set or equivalently {x} is N$\ddot{g}$-open. □

The converse of Theorem 5.1 need not be true as seen from the following example.

Example 21. *Let X and τ as in the Example 10. The sets {a} and {c} are Ng-closed in (U, $\tau_R(X)$) and the set {b} is N$\ddot{g}$-open. But the space (U, $\tau_R(X)$) is not a ${}_gT_{N\ddot{g}}$-space.*

Theorem 5.2. *A space (U, $\tau_R(X)$) is $T_{N1/2}$ if and only if it is both $T_{N\ddot{g}}$ and ${}_gT_{N\ddot{g}}$.*

Proof. Necessity. Follows from Propositions 4.1 and 5.1.

Sufficiency. Assume that (U, $\tau_R(X)$) is both $T_{N\ddot{g}}$ and ${}_gT_{N\ddot{g}}$. Let A be a Ng-closed set of (U, $\tau_R(X)$). Then A is N$\ddot{g}$-closed, since (U, $\tau_R(X)$) is a ${}_gT_{N\ddot{g}}$. Again since (U, $\tau_R(X)$) is a $T_{N\ddot{g}}$, A is a Nano closed set in (U, $\tau_R(X)$) and so (U, $\tau_R(X)$) is a $T_{1/2}$. □

6. ${}_\alpha T_{N\ddot{g}}$-SPACES

Definition 6.1. *A space (U, $\tau_R(X)$) is called*

(1) *a ${}_\alpha T_{N\ddot{g}}$-space if every Nαg-closed set in it is N$\ddot{g}$-closed.*
(2) *a $N_\alpha T_d$-space if every Nαg-closed set in it is Ng-closed.*

Example 22. *Let X and $\tau_R(X)$ as in the Example 11, is a ${}_{\alpha}T_{N\ddot{g}}$-space and the space (U, $\tau_R(X)$) in the Example 10, is not a ${}_{\alpha}T_{N\ddot{g}}$-space.*

Proposition 6.1. *Every $N_{\alpha}T_b$-space is ${}_{\alpha}T_{N\ddot{g}}$-space but not conversely.*

Proof. Follows from Proposition 3.1. □

The converse of Proposition 6.1 need not be true as seen from the following example.

Example 23. *Let X and $\tau_R(X)$ in the Example 11, is a ${}_{\alpha}T_{N\ddot{g}}$-space but not a $N_{\alpha}T_b$-space.*

Proposition 6.2. *Every ${}_{\alpha}T_{N\ddot{g}}$-space is a $N_{\alpha}T_d$-space but not conversely.*

Proof. Let (U, $\tau_R(X)$) be an ${}_{\alpha}T_{N\ddot{g}}$-space and let A be an Nαg-closed set of (U, $\tau_R(X)$). Then A is a N$\ddot{g}$-closed subset of (U, $\tau_R(X)$) and by Proposition 3.2, A is Ng-closed. Therefore (U, $\tau_R(X)$) is an $N_{\alpha}T_d$-space. □

The converse of Proposition 6.2 need not be true as seen from the following example.

Example 24. *Let X and $\tau_R(X)$ in the Example 11, is a $N_{\alpha}T_d$-space but not a ${}_{\alpha}T_{N\ddot{g}}$-space.*

Theorem 6.1. *If (U, $\tau_R(X)$) is a ${}_{\alpha}T_{N\ddot{g}}$-space, then every singleton subset of (U, $\tau_R(X)$) is either Nαg-closed or N$\ddot{g}$-open.*

Proof. Similar to Theorem 5.1. □

The converse of Theorem 6.1 need not be true as seen from the following example.

Example 25. *Let X and $\tau_R(X)$ as in the Example 10. The sets {a} and {c} are Nαg-closed in (U, $\tau_R(X)$) and the set {b} is N$\ddot{g}$-open. But the space (U, $\tau_R(X)$) is not a ${}_{\alpha}T_{N\ddot{g}}$-space.*

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