ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.3, 1259–1279
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.3.51

CONTINUITY OF THE BLOW-UP TIME FOR A NONLOCAL DIFFUSION PROBLEM WITH NEUMANN BOUNDARY CONDITION AND A REACTION TERM

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ABSTRACT. In this paper, we address the following initial value problem

$$u_t = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u) \quad \text{in} \quad \overline{\Omega}\times(0,T),$$
$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $f:[0,\infty)\to[0,\infty)$ is a C^1 nondecreasing function, $\int^\infty \frac{d\sigma}{f(\sigma)} < \infty$, $J:\mathbb{R}^N\to\mathbb{R}$ is a kernel which is nonnegative and bounded in $\mathbb{R}^N$. Under some conditions, we show that the solution of a perturbed form of the above problem blows up in a finite time and estimate its blow-up time. We also prove the continuity of the blow-up time as a function of the initial datum. Finally, we give some numerical results to illustrate our analysis.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u) \quad \text{in} \quad \overline{\Omega}\times(0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega}, \tag{1.2}$$

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2010 *Mathematics Subject Classification.* 35B40, 45A07, 45G10, 65M06.

Key words and phrases. Nonlocal diffusion, blow-up, numerical blow-up time, continuity.

where $f : [0,\infty) \to [0,\infty)$ is a C^1 nondecreasing function, $\displaystyle\int^{\infty} \frac{d\sigma}{f(\sigma)} < \infty$, $J : \mathbb{R}^N \to \mathbb{R}$ is a kernel which is nonnegative and bounded in $\mathbb{R}^N$. In addition, J is symmetric ($J(z) = J(-z)$) and $\int_{\mathbb{R}^N} J(z)dz = 1$. The initial datum $u_0 \in C^0(\overline{\Omega})$, $u_0(x) \geq 0$ in $\overline{\Omega}$, and $\sup_{x\in\Omega} u_0(x) < b$.

Here, $(0,T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} |u(x,t)|$. In this last case, we say that the solution u blows up in a finite time, and the time T is called the blow-up time of the solution u. Recently, nonlocal diffusion has been the subject of investigation of many authors (see [1], [2], [5]- [7], [10], [16], [17], [18], [20], [29] and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy,$$

and variations of it, have been used by several authors to model diffusion processes (see [5], [6], [15], [28]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at the time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then the convolution $(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which they are leaving location x to travel to any other site (see [22]). For the problem described in (1.1)-(1.2), the integral is taken over Ω. Consequently, there is no individuals that arrive or leave the domain Ω. It is the reason why in the title of the paper, we have added Neumann boundary condition. On the other hand, the term of the source $f(u)$ can be rewritten as follows

$$f(u(x,t)) = \int_{\mathbb{R}^N} J(x-y)f(u(x,t))dy.$$

Therefore, the term $f(u)$ can be interpreted as a force that increases the rate of individuals which are arriving to position x from all other places. Thus, the general rate of individuals which are arriving to position x from all other places increases, provoking as we shall see later, the blow-up of the solution. For local diffusion, solutions which blows up in a finite time have been the subject of investigation of many authors (see [3], [9], [11], [14], [19], [23], [24], [26], and the references cited therein). In addition, by standard methods, one may easily prove the time-local existence and uniqueness of the classical solution (see [8], [9]). In this paper, we are interested in the blow-up of the solution, and the continuity of the blow-up time for the problem described in (1.1)-(1.2). More precisely, consider the following initial value problem

$$v_t = \int_{\Omega} J(x-y)(v(y,t)-v(x,t))dy + f(v) \quad \text{in} \quad \overline{\Omega}\times(0,T_h), \tag{1.3}$$

$$v(x,0) = u_0^h(x) \quad \text{in} \quad \overline{\Omega}, \tag{1.4}$$

where $u_0^h \in C^0(\overline{\Omega})$, $0 \leq u_0^h(x) \leq u_0(x)$ in $\overline{\Omega}$, $\lim_{h\to 0} u_0^h = u_0$. Here $(0,T_h)$ is the maximal time interval of existence of the solution v. In the current paper, under some hypotheses, we show that the solution v of (1.3)-(1.4) blows up in a finite time and estimate its blow-up time. We demonstrate in passing that, when the norm of the initial datum is large, then the solution v of (1.3)-(1.4) blows up in a finite time, and its blow-up time goes to that of the solution of a certain differential equation when $\|u_0^h(x)\|_\infty$ is large enough (see Theorem 3.4). A similar result is obtain in [8] where the authors are considered the phenomenon of quenching. In addition, we show under some hypotheses that, the solution v of (1.3)-(1.4) blows up in a finite time and its blow-up time goes to that of the solution u of (1.1)-(1.2) when h goes to zero. In [9] Boni and N'gohisse are considered the equation (1.1)-(1.2) with a potential and studied the asymptotic behavior. The present work was motivated by the paper [12] of the same authors, where they considered a parabolic equation and proved the continuity of the quenching time (we say that a solution quench in a finite time if it develops a singularity in a finite time).

The remainder of the paper is organized in the following manner. In the next section, we prove that $T_h \geq T$. In the third section, under some conditions, we show that the solution v of (1.3)-(1.4) blows up in a finite time and estimate its blow-up time. We also show that the blow-up time goes to that of the solution u of (1.1)-(1.2) as h tends to zero. Finally, in the last section, we give some computational results to illustrate our analysis.

2. Local existence

This section begins by recalling a version of the maximum principle for non-local problems. Let us notice that results on maximum principle are know for our problem (see, for instance [8]).

Lemma 2.1. *Let $a \in C^0(\overline{\Omega} \times [0,T))$ and let u, $v \in C^{0,1}(\overline{\Omega} \times [0,T))$ satisfying the following inequalities*

$$u_t - \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + a(x,t)u(x,t) \geq$$

$$v_t - \int_\Omega J(x-y)(v(y,t)-v(x,t))dy + a(x,t)v(x,t) \quad \textit{in} \quad \overline{\Omega} \times (0,T),$$

$$u(x,0) \geq v(x,0) \quad \textit{in} \quad \overline{\Omega}.$$

Then, we have $u(x,t) \geq v(x,t) \quad$ in $\overline{\Omega} \times (0,T)$.

Proof. For the proof see [8]. □

Remark 2.1. *Invoking the mean value theorem and Lemma 2.1, it is not hard to see that $v(x,t) \leq u(x,t)$ as long as all of them are defined. We infer that $T_h \geq T$.*

This remark is important for the proof of Theorem 4.1.

3. Blow-up times

In this section, under some conditions, we show that the solution v of (1.3)-(1.4) blows up in a finite time and estimate its blow-up time. We demonstrate in passing that, when the norm of the initial datum is large enough, then the solution v blows up in a finite time and its blow-up time goes to that of the solution of a differential equation as $\|u_0^h\|_\infty$ goes to infinity.
Our first result which is stated in the following theorem says that the solution v of (1.3)-(1.4) blows up in a finite time when if $\int_\Omega u_0^h(x)dx$ is positive.

Theorem 3.1. *Assume that the initial datum at (1.4) satisfies $\int_\Omega u_0^h(x)dx > 0$. Then, the solution v of (1.3)-(1.4) blows up in a finite time and its blow-up time T_h satisfies the following estimate*

$$T_h \leq \int_A^\infty \frac{ds}{f(s)},$$

where $A = \frac{1}{|\Omega|}\int_\Omega u_0^h(x)dx$.

Proof. Since $(0, T_h)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h is finite and satisfies the above inequality. Integrating both sides of (1.3) over $(0, t)$, we find that

$$v(x,t) - u_0^h(x) = \int_0^t \int_\Omega J(x-y)(v(y,s) - v(x,s))dyds$$

$$+ \int_0^t f(v(x,s))ds \quad \text{for} \quad t \in (0,T).$$

Integrate again in the x variable and apply Fubini's theorem to obtain

$$\int_\Omega v(x,t)dx - \int_\Omega u_0^h(x)dx = \int_0^t \left(\int_\Omega f(v(x,s)\right) ds \quad \text{for} \quad t \in (0,T_h). \tag{3.1}$$

Set

$$w(t) = \frac{1}{|\Omega|}\int_\Omega v(x,t)dx \quad \text{for} \quad t \in [0,T_h).$$

Taking the derivative of w in t and using (3.1), we arrive at

$$w'(t) = \int_\Omega \frac{1}{|\Omega|} f(v(x,t))dx, \quad \text{for} \quad t \in (0,T_h).$$

It follows from Jensen's inequality that $w'(t) \geq f(w(t))$ for $t \in (0, T_h)$, or equivalently

$$\frac{dw}{f(w)} \geq dt \quad \text{for} \quad t \in (0,T_h).$$

Integrate the above inequality over $(0, T)$ to obtain

$$T_h \leq \int_{w(0)}^\infty \frac{ds}{f(s)}.$$

Since the quantity on the right hand side of the above inequality is finite, we deduce that v quenches in a finite time T_h which obeys the above inequality. Use the fact that $w(0) = A$ to complete the rest of the proof. □

Remark 3.1. *Let us notice that the above result remains valid when* $\int_\Omega u_0^h(x)dx = 0$ *and* $f(0) > 0$.

The above theorem has the particularity that the solution v blows up in a finite time with a weak condition. However, the estimation of the blow-up time involves the norm L^1 which is not interesting to obtain the continuity of the blow-up time. In the following theorem, we get an estimation of the blow-up time which takes into account the norm of the initial datum.

Theorem 3.2. *Let $f(0) > 0$ and let $A = \int_0^\infty \frac{d\sigma}{f(\sigma)}$. If $A < 1$, then the solution v of (1.3)-(1.4) blows up in a finite time, and its blow-up time T_h obeys the following estimate*

$$T_h \leq \frac{1}{1-A} \int_{\|u_0^h\|_\infty}^\infty \frac{d\sigma}{f(\sigma)}.$$

Proof. Since $(0, T_h)$ is the maximal time interval on which u exists, then our goal is to prove that T_h is finite and obeys the above inequality. Due to the fact that $J(z)$ is nonnegative for $z \in \mathbb{R}^N$, and

$$\int_\Omega J(x-y)dy \leq \int_{\mathbb{R}^N} J(x-y)dy = 1 \quad \text{for} \quad x \in \overline{\Omega},$$

we note that

$$v_t(x,t) \geq -v(x,t) + f(v(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T_h),$$

which implies that

$$v_t(x,t) \geq f(v(x,t)) \left(1 - \frac{v(x,t)}{f(v(x,t))}\right) \quad \text{in} \quad \overline{\Omega} \times (0, T_h).$$

It is not hard to see that

$$\int_0^\infty \frac{d\sigma}{f(\sigma)} \geq \sup_{0 \leq t < \infty} \int_0^t \frac{d\sigma}{f(\sigma)} \geq \sup_{0 \leq t < \infty} \frac{t}{f(t)},$$

because $f(s)$ is nondecreasing for $s \in (0, b)$. We infer that

$$v_t(x,t) \geq (1-A) f(v(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T_h),$$

or equivalently

$$\frac{dv}{f(v)} \geq (1-A)dt \quad \text{in} \quad \overline{\Omega} \times (0, T_h). \tag{3.2}$$

Integrate the above inequality over $(0, T_h)$ to obtain

$$(1-A)T_h \leq \int_{u_0^h(x)}^\infty \frac{d\sigma}{f(\sigma)} \quad \text{in} \quad \overline{\Omega}.$$

It follows that

$$T_h \leq \frac{1}{1-A}\int_{\|u_0^h\|_\infty}^{\infty}\frac{d\sigma}{f(\sigma)}.$$

We conclude that the solution v of (1.3)-(1.4) blows up in a finite time, because the quantity on the right hand side of the above inequality is finite. This finishes the proof. □

Remark 3.2. *Let $t_0 \in (0, T_h)$. Integrating the inequality (3.2) over (t_0, T_h), we find that*

$$(1-A)(T_h - t_0) \leq \int_{v(x,t_0)}^{\infty}\frac{d\sigma}{f(\sigma)} \quad \text{for} \quad x \in \overline{\Omega},$$

which implies that

$$T_h - t_0 \leq \frac{1}{1-A}\int_{\|v(\cdot,t_0)\|_\infty}^{\infty}\frac{d\sigma}{f(\sigma)}.$$

It is worth noting that the above estimate is crucial to obtain the continuity of the blow-up time. Let us notice that the above theorem is very restrictive in certain situations. For instance, we need $f(0) > 0$. In the theorem below, we avoid this condition and also obtain an estimation of the blow-up time which takes into account the norm of the initial datum.

Theorem 3.3. *Let v be the solution of (1.3)–(1.4), and suppose that the initial datum at (1.4) obeys the following condition*

$$f(\|u_0^h\|_\infty) > \|u_0^h\|_\infty > 0.$$

Assume that $\frac{f(s)}{s}$ is a nondecreasing function for $s \geq \|u_0^h\|_\infty$. Then, the solution v blows up in a finite time, and its blow-up time T_h is estimated as follows

$$T_h \leq \frac{f(\|u_0^h\|_\infty)}{f(\|u_0^h\|_\infty) - \|u_0^h\|_\infty}\int_{\|u_0^h\|_\infty}^{\infty}\frac{d\sigma}{f(\sigma)}.$$

Proof. Since $(0, T_h)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h is finite and satisfies the above inequality. We note that

$$\int_\Omega J(x-y)dy \leq \int_{R^N} J(x-y)dy = 1 \quad \text{for} \quad x \in \overline{\Omega},$$

which implies that

$$v_t(x,t) \geq -v(x,t) + f(v(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T).$$

Let $x_0(t) \in \overline{\Omega}$ be such that

$$U(t) = \max_{x\in\overline{\Omega}} v(x,t) = v(x_0(t),t) \quad \text{for} \quad t \in (0,T).$$

Consequently, we have

$$U'(t) \geq -U(t) + f(U(t)) \quad \text{for} \quad t \in (0,T),$$

or equivalently

$$U'(t) \geq f(U(t))\left(1 - \frac{U(t)}{f(U(t))}\right) \quad \text{for} \quad t \in (0,T). \tag{3.3}$$

We note that $U'(0) > 0$, and we claim that $U'(t) > 0$ for $t \in (0,T)$. To prove the claim, we argue by contradiction. Let t_0 be the first $t \in (0,T)$ such that $U'(t) > 0$ for $t \in (0,t_0)$, but $U'(t_0) = 0$. This implies that $U(t_0) \geq U(0) = \|u_0^h\|_\infty$. Therefore, we get

$$0 = U'(t_0) \geq f(\|u_0^h\|_\infty)\left(1 - \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}\right) > 0,$$

which is a contradiction, and the claim is proved. In view of the claim, we find that $U(t) \geq \|u_0^h\|_\infty$ for $t \in (0,T_h)$, and making use of (3.3), we arrive at

$$U'(t) \geq \left(1 - \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}\right) f(U(t)) \quad \text{for} \quad t \in (0,T_h),$$

or equivalently

$$\frac{dU}{f(U)} \geq \left(1 - \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}\right) dt \quad \text{for} \quad t \in (0,T_h).$$

Integrate the above estimate over $(0,T_h)$ to obtain

$$\left(1 - \frac{b}{f(\|u_0^h\|_\infty)}\right) T_h \leq \int_{\|u_0^h\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)},$$

which implies that

$$T_h \leq \frac{f(\|u_0^h\|_\infty)}{f(\|u_0^h\|_\infty) - \|u_0^h\|_\infty} \int_{\|u_0^h\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

Remark 3.3. *If $f(s)$ is a convex function for nonnegative values of s and $f(0) = 0$, then it is well known that $\frac{f(s)}{s}$ is a nondecreasing function for $s > 0$.*

Up to now, the results obtained allow us to see some upper bounds of the blow-up time. In the theorem below, we derive a lower bound of the blow-up time.

Theorem 3.4. *Suppose that the solution v of (1.3)–(1.4) blows up in a finite time T_h. Then, we have*

$$T_h \geq \int_{\|u_0^h\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Proof. Let $\alpha(t)$ be the solution of the following ordinary differential equation

$$\alpha'(t) = f(\alpha(t)), \quad t \in (0, T_e), \quad \alpha(0) = \|u_0^h\|_\infty,$$

where $(0, T_e)$ is the maximal time interval of existence of the solution $\alpha(t)$. By a routine computation, one easily sees that $T_e = \int_{\|u_0^h\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}$. Now, let us introduce the function z defined as follows

$$z(x,t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times [0, T_e).$$

A straightforward calculation yields

$$z_t(x,t) = \int_\Omega J(x-y)(z(y,t) - z(x,t))dy + f(z(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T_e),$$

$$z(x,0) \geq v(x,0) \quad \text{in} \quad \overline{\Omega}.$$

Set

$$w(x,t) = z(x,t) - v(x,t) \quad \text{in} \quad \overline{\Omega} \times [0, T_*),$$

where $T_* = \min\{T_h, T_e\}$. Making use of the mean value theorem, we find that

$$w_t(x,t) \geq \int_\Omega J(x-y)(w(y,t) - w(x,t))dy + f'(\xi(x,t))w(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, T_*),$$

$$w(x,0) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where $\xi(x,t)$ is an intermediate value between $v(x,t)$ and $z(x,t)$. It follows from Lemma 2.1 that

$$w(x,t) \geq 0 \quad \text{in} \quad \overline{\Omega} \times (0, T_*),$$

or equivalently

$$v(x,t) \leq \alpha(t) \quad \text{in} \quad \overline{\Omega} \times (0, T_*). \tag{3.4}$$

We claim that $T_h \geq T_e$. To prove the claim, we argue by contradiction. Suppose that $T_h < T_e$. In view of (3.4), we see that $\|v(\cdot, T_h)\|_\infty \leq \alpha(T_h) < \infty$, which contradicts the fact that $(0, T_h)$ is the maximum time interval of the existence of the solution v. This demonstrates the claim, and the proof is complete. □

With the aid of Theorem 3.2 and 3.3, we can derive the following interesting result.

Theorem 3.5. *Let v be the solution of (1.3)–(1.4), and suppose that the initial datum (1.4) obeys the following condition*

$$f(\|u_0^h\|_\infty) > \|u_0^h\|_\infty > 0.$$

Then, the solution v blows up in a finite time, and its blow-up time T_h obeys the following estimates

$$0 \leq T_h - T_e \leq \frac{\|u_0^h\|_\infty T_e}{f(\|u_0^h\|_\infty)} + o\left(\frac{\|u_0^h\|_\infty T_e}{f(\|u_0^h\|_\infty)}\right) \quad \textit{as} \quad \|u_0^h\|_\infty \to \infty,$$

where $T_e = \displaystyle\int_{\|u_0^h\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}$.

Proof. Since $(0, T_h)$ is the maximal time interval on which u exists, then our goal is to prove that T_h is finite and obeys the above estimates. Making use of Theorem 3.3 and 3.4, we find that T_e is finite and obeys the following estimates

$$T_e \leq T_h \leq \frac{T_e}{1 - \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}}. \tag{3.5}$$

Apply Taylor's expansion to obtain

$$\frac{1}{1 - \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}} = 1 + \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)} + o\left(\frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}\right) \quad \text{as} \quad \|u_0^h\|_\infty \to \infty.$$

Use (3.5) and the above relation to complete the rest of the proof. □

Remark 3.4. *The estimates of Theorem 3.4 can be rewritten as follows*

$$0 \leq \frac{T_h}{T_e} - 1 \leq \frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)} + o\left(\frac{\|u_0^h\|_\infty}{f(\|u_0^h\|_\infty)}\right) \quad \textit{as} \quad \|u_0^h\|_\infty \to \infty.$$

We infer that

$$\lim_{\|u_0^h\|_\infty \to \infty} \frac{T_h}{T_e} = 1.$$

Thus results permit to generalize the known result on the quenching time which converge to that of the solution of a differential equation (see, [8]) .

4. CONTINUITY OF THE BLOW-UP TIME

In this section, under some assumptions, we show that the solution v of (1.3)-(1.4) blows up in a finite time, and its blow-up time goes to that of the solution u of (1.1)-(1.2) when the parameter h goes to zero. In order to obtain the above result, we firstly reveal that the solution v approaches the solution u in any interval $\overline{\Omega} \times [0, T-\tau]$ where $\tau \in (0, T)$. This result is stated in the following theorem.

Theorem 4.1. *Assume that the problem (1.1)-(1.2) has a solution* $u \in C^{0,1}(\overline{\Omega} \times [0, T))$. *Suppose that the initial datum at (1.4) satisfies the following condition*

$$\|u_0^h - u_0\|_\infty = o(1) \quad \textit{as} \quad h \to 0. \tag{4.1}$$

Then, the problem (1.3)-(1.4) admits a unique solution $v \in C^{0,1}(\overline{\Omega} \times [0, T_h))$*, and the following relation holds*

$$\sup_{t \in [0, T-\tau]} \|v(\cdot, t) - u(\cdot, t)\|_\infty = O(\|u_0^h - u_0\|_\infty) \quad \textit{as} \quad h \to 0,$$

where $\tau \in (0, T)$.

Proof. The problem (1.3)-(1.4) admits a unique solution $v \in C^{0,1}(\overline{\Omega} \times [0, T_h))$. In Remark 2.1, we have mentioned that $T_h \geq T$. Let $t(h) \leq T - \tau$ be the first t such that

$$\|v(\cdot, t) - u(\cdot, t)\|_\infty < 1 \quad \text{for} \quad t \in (0, t(h)). \tag{4.2}$$

We know from (4.2) that $t(h) > 0$ for h small enough. Due to the fact that $u \in C^{0,1}$, there exists a positive constant M such that $\|u(\cdot, t)\|_\infty \leq M$ for $t \in (0, t(h))$. An application of the triangle inequality yields

$$\|v(\cdot, t)\|_\infty \leq \|u(\cdot, t)\|_\infty + \|v(\cdot, t) - u(\cdot, t)\|_\infty \quad \text{for} \quad t \in (0, t(h)),$$

which implies that

$$\|v(\cdot, t)\|_\infty \leq M + 1 \quad \text{for} \quad t \in (0, t(h)).$$

Introduce the error e defined as follows

$$e(x, t) = v(x, t) - u(x, t) \quad \text{in} \quad \overline{\Omega} \times [0, t(h)).$$

Making use of the mean value theorem, we find that

$$e_t(x,t) = \int_\Omega J(x-y)(e(y,t)-e(x,t))dy + f'(\xi(x,t))e(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,t(h)),$$

$$e(x,0) = u_0^h(x) - u_0(x) \quad \text{in} \quad \overline{\Omega},$$

where $\xi(x,t)$ is an intermediate value between $v(x,t)$ and $u(x,t)$. Set

$$z(x,t) = e^{(L+1)t}\|u_0^h - u_0\|_\infty \quad \text{in} \quad \overline{\Omega}\times[0,T],$$

where $L = f'(M+1)$. A straightforward computation reveals that

$$z_t(x,t) \geq \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + f'(\xi(x,t))z(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,t(h)),$$

$$z(x,0) \geq e(x,0) \quad \text{in} \quad \overline{\Omega}.$$

Invoking Lemma 2.1, we obtain

$$z(x,t) \geq e(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,t(h)).$$

In the same way, we also prove that

$$z(x,t) \geq -e(x,t) \quad \text{in} \quad \overline{\Omega}\times(0,t(h)),$$

which implies that

$$\|v(\cdot,t)-u(\cdot,t)\|_\infty \leq e^{(L+1)t}\|u_0^h - u_0\|_\infty \quad \text{for} \quad t\in(0,t(h)). \tag{4.3}$$

Now, we claim that $t(h) = T$. To prove the claim, we argue by contradiction. Suppose that $t(h) < T-\tau$. In view of (4.2) and (4.3), it is easy to check that

$$1 = \|v(\cdot,t(h)) - u(\cdot,t(h))\|_\infty \leq e^{(L+1)T}\|u_0^h - u_0\|_\infty.$$

Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we infer that $1 \leq 0$, which is a contradiction. This demonstrates the claim, and the proof is complete. □

At the moment, we are in a position to prove the main result of this section.

Theorem 4.2. *Assume that the problem (1.1)–(1.2) has a solution u which blows up in a finite time T such that $u \in C^{0,1}(\overline{\Omega}\times[0,T))$. Suppose that the initial datum at (1.4) satisfies the condition (4.1). Then, the problem (1.3)–(1.4) admits a unique solution v which blows up in a finite time, and the following relation holds*

$$\lim_{h\to 0} T_h = T.$$

Proof. Let $0 < \varepsilon < T/2$. There exists a positive constant R such that

$$\frac{1}{1-A}\int_R^\infty \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{4.4}$$

Since u blows up at the time T, then there exists a time $T_0 \in (T-\varepsilon/2, T)$ such that $\|u(\cdot,t)\|_\infty \geq 2R$ for $t \in [T_0, T)$. Invoking Theorem 4.1, we note that the problem (1.3)–(1.4) admits a unique solution v, and the following estimate holds $\|v(\cdot,T_0) - u(\cdot,T_0)\|_\infty \leq R$. Making use of the triangle inequality, we find that

$$\|v(\cdot,T_0)\|_\infty \geq \|u(\cdot,T_0)\|_\infty - \|v(\cdot,T_0) - u(\cdot,T_0)\|_\infty,$$

which implies that

$$\|v(\cdot,T_0)\|_\infty \geq 2R - R = R.$$

In Remark 2.1 of the paper, we have revealed that $T_h \geq T$. We infer from (4.4) and Remark 3.2 that

$$0 \leq T_h - T \leq T_h - T_0 + T_0 - T \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. $\square$

5. NUMERICAL RESULTS

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the problem (1.1)–(1.2) in the case where $\Omega = (-1,1)$, $f(u) = u^p$ with $p > 1$,

$$J(x) = \begin{cases} \frac{3}{2}x^2 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

$u_0(x) = \gamma\left(\frac{2-(\varepsilon\cos(\pi x))^2}{4}\right)$ with $\gamma > 0$, $\varepsilon \in (0,1]$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer, and let $h = 2/I$. Define the grid $x_i = -1 + ih$, $0 \leq i \leq I$, and approximate the solution u of (1.1)–(1.2) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)}) + (U_i^{(n)})^p, \quad 0 \leq i \leq I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $\varphi_i = \gamma \left(\frac{2 - (\varepsilon \cos(\pi x_i))^2}{4} \right)$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min \left\{ h^2, \frac{h^2}{\|U_h^{(n)}\|_\infty^{p-1}} \right\},$$

with $\|U_h^{(n)}\|_\infty = \max_{0 \le i \le I} U_i^{(n)}$. Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution. We also approximate the solution u of (1.1)–(1.2) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n+1)} - U_i^{(n+1)}) + (U_i^{(n)})^p, \quad 0 \le i \le I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I.$$

As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = \frac{h^2}{\|U_h^{(n)}\|_\infty^{p-1}}.$$

Let us again remark that for the above implicit scheme, existence and nonnegativity of the discrete solution are also guaranteed using standard methods (see, for instance [4] and [13]).

We need the following definition.

Definition 5.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = \infty$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical blow-up time of the discrete solution $U_h^{(n)}$.*

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \le 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $p = 2$, $\gamma = 1$.

First case: $\varepsilon = 1$

TABLE 1. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.314164	1064	1	-
32	2.312775	3889	4	-
64	2.311608	14107	54	1.81
128	2.310101	60208	920	1.91

TABLE 2. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.318321	2499	1	-
32	2.314167	9592	3	-
64	2.311934	36882	24	0.89
128	2.310588	141835	356	0.91

Second case: $\varepsilon = 1/10$

TABLE 3. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.041179	2426	0.8	-
32	2.032652	9296	3	-
64	2.030421	35713	144	1.93
128	2.030123	137062	785	2.76

Third case: $\varepsilon = 1/100$

TABLE 4. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.044887	2427	0.8	-
32	2.033687	9297	2	-
64	2.030689	35714	23	1.9
128	2.030124	137127	347	2.41

TABLE 5. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.014937	1133	1	-
32	2.005966	4155	4	-
64	2.003642	15178	58	1.94
128	2.002991	55012	927	1.84

TABLE 6. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.018631	2420	1	-
32	2.007017	9267	2	-
64	2.004010	35593	23	1.95
128	2.003268	136638	340	2.02

Numerical experiments for $p = 2$, $\gamma = 10$.

First case: $\varepsilon = 1$

TABLE 7. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.206370	2009	1	-
32	0.204147	7633	7	-
64	0.203626	29067	113	2.09
128	0.203513	90101	1516	2.20

TABLE 8. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.206517	2009	-	-
32	0.204188	7633	2	-
64	0.203637	29068	19	2.08
128	0.203515	110548	276	2.17

Second case: $\varepsilon = 1/10$

TABLE 9. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.203541	2007	0.8	-
32	0.201216	7625	8	-
64	0.200639	29033	115	2.01
128	0.200497	90121	1535	2.02

TABLE 10. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.203686	2007	1	-
32	0.201256	7625	2	-
64	0.200650	29033	19	2.00
128	0.200500	110408	276	2.01

Third case: $\varepsilon = 1/100$

TABLE 11. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.203169	2007	1	-
32	0.200827	7623	8	-
64	0.200242	29025	113	2.00
128	0.200097	90110	1523	2.01

TABLE 12. Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	0.203314	2007	1	-
32	0.200868	7623	2	-
64	0.200253	29025	19	1.99
128	0.200099	110379	279	2.00

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