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APPLICATION OF HOMOTOPY PERTURBATION METHOD TO AN SIR MUMPS MODEL

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ABSTRACT. Mumps is one of the diseases that pose global threat to children well-being. In this paper, the problem of the spread of mumps in a closed population is investigated using a SIR compartmental model. Mathematical interpretation of the problem generates nonlinear first-order differential equations. The method of Homotopy Perturbation is adopted to derive the theoretical solutions of the system. Numerical simulations of the analytical results are carried out with the help of Maple 18 software and the solutions are presented in graphical form. The solutions show that Homotopy Perturbation Method (HPM) is an appropriate technique for solving epidemic models.

1. INTRODUCTION

Mumps is a serious deadly viral disease triggered by paramyxovirus which usually attacks children [1]. The mumps virus is the agent of mumps and the virus transmits simply from human-to-human via infected saliva. Susceptible individuals can be infected with mumps by inhaling droplets of infectious individuals. Mumps can also spread from an infected individual to a susceptible individual through sharing of utensils.

A good number of individuals who are infected with mumps experience either no symptoms or very mild signs. Mumps has an incubation period of about 14

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to 21 days. The signs and symptoms of mumps may include headache, loss appetite, fever and fatigue. At the advanced stage, about 20-30% males and 5% females experience orchitis and oophoritis respectively [2]. Mumps can bring about aseptic meningitis for more than 10% of mumps patients which is capable of resulting in encephalitis that can induce disability or death. Other undesirable effects of mumps are pancreatitis and deafness though the major symptom of mumps disease is swollen salivary gland that makes the cheeks to swell [3].

The method of Homotopy Perturbation was developed by He [4,5]. The HPM has been proved efficient in obtaining the solutions of many nonlinear systems with approximations such as the singular IVPs problem involving Lane-Emden-type [6], differential-difference equations [7], bifurcation of nonlinear problems [8] and nonlinear oscillator problems with discontinuities [9]. In most cases, the Homotopy Perturbation Method produces rapid convergence of solutions after some iteration. The technique of homotopy in topology can be used to construct a homotopy with a parameter p in [0, 1] that is regarded a 'small parameter' and by this technique, large computational work, linearization, round-off errors and discretization are avoided. The HPM is amenable to Biomathematics since mathematical modelling of phenomena often results in nonlinear problems. Rafei et al. [10] employed the HPM to work out estimation to the solutions of the epidemic model. The HPM is a powerful mathematical technique that has attracted interests of mathematicians globally. Some of the recent applications of the HPM can be found in [15–18]. It is on this ground that we adopt a basic mathematical model that captures the essential transmission dynamics of mumps disease with the HPM as a means of obtaining the analytical solutions of the generating differential equations.

2. Model Formulation

There are several compartmental models (e.g SIS, SIR, SIRS, MSIR, SEIR) but the appropriate compartment to adopt in a particular study is a function of the features of the disease under study and also, the essence of the study. The SIR epidemiological model has the feature that an individual receives permanent immunity as soon as he recovers from the illness. The SIR compartmental

model was introduced by Kermack and Mckendrick in 1927 to study the population of individuals who were infected with a communicable disease in a closed population. The SIR model is appropriate for viral diseases such as measles, mumps and rubella. The SIR model is partitioned into compartments of susceptible individuals *S(t),* infectious individuals *I(t)* and removed individuals *R(t)*. Each compartment is a function of time meaning that the number of individuals in the compartments may fluctuate with time. The susceptible population increases due to the coming of individuals as a result of birth at a rate β. The susceptible population however reduces due to natural death at a rate μ and infection at a rate α. The population dynamics of the infectious class grows with the incidence rate of αSI . This class however reduces by the natural death rate μ, death rate due to the disease δ, and successful cure of infectious patients at a rate γ. Lastly, the dynamics of the removed class increases with successful cure of infectious patients at the rate γ but decreases by the natural death rate μ. Each member of the population typically progresses from susceptible to infectious to recovered. This formulation can be expressed mathematically as

$$\frac{dS}{dt} = \beta - \alpha SI - \mu S\,, \tag{2.1}$$

$$\frac{dI}{dt} = \alpha SI - (\mu + \gamma + \delta)\, I\,, \tag{2.2}$$

$$\frac{dR}{dt} = \gamma I - \mu R\,. \tag{2.3}$$

The numerical values assigned to the parameters are presented in table 1.

TABLE 1. Parameters Description, Symbols, Values and Sources

Parameter	Symbol	Value	Source
Human recruitment rate	β	10	[10]
Contact rate	α	0.1	[12]
Recovery rate	γ	0.1	Assumed
Disease mortality rate	δ	0.4	[12]
Natural death rate	μ	0.1	[14]

3. The Basic Idea of Homotopy Perturbation Method

To demonstrate the ideas of the homotopy perturbation method (HPM) for solving nonlinear differential equations, the following differential equation was considered by He [19].

$$A(u) - f(r) = 0, r \in \Omega, \tag{3.1}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ represents differentiation along the normal vector drawn outwards from Ω. The operator A can be divided into two parts of L and N, where L is the linear part and N is nonlinear. Equation (3.1) can therefore be rewritten as

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega, \tag{3.2}$$

He [15] formulated a homotopy $v(r,p) : \Omega \text{ x } [0,1] \to \Re$, which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{3.3}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{3.4}$$

where $L(u)$ is the linear part and $N(u)$ is the nonlinear part.
$L(u) = L(v) - L(u_0) + pL(u_0)$ and $N(u) = pN(v)$.
In equation (3.3), $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of equation (3.1) that satisfied the boundary condition.
From equation (3.4),

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0 \\ H(v,1) &= L(v) + N(v) - f(r) = 0\,. \end{aligned}$$

The changing process of p from zero to unity is just that of $H(v,p)$ from $L(v) - L(u_0)$ to $L(v) + N(v) - f(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, we can first use the embedding parameter

p as a small parameter and assume that the solution of equation (3.3) can be written in power series in p:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots$$

As $\mathrm{p} \to 1$, equation (3.3) becomes equation (3.2) and the approximate solution to the problem is

$$\lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \dots \tag{3.5}$$

The series (3.5) is convergent for most cases. However, the convergence rate depends on the nonlinear part of operator A.

4. Implementation of the Method

We shall apply the homotopy perturbation method to obtain the solution of the proposed SIR mumps model in this section. Consider the system of equations (2.1)-(2.3):

$$\frac{dS}{dt} - \beta + \alpha SI + \mu S = 0 \tag{4.1}$$

$$\frac{dI}{dt} - \alpha SI + (\mu + \gamma + \delta) I = 0 \tag{4.2}$$

$$\frac{dR}{dt} - \gamma I + \mu R = 0\,, \tag{4.3}$$

with the initial conditions $S(0) = S_0, \quad I(0) = I_0$ and $R(0) = R_0$.
Applying HPM to equation (4.1) we obtain:

$$(1-p)\frac{dS}{dt} + p\left[\frac{dS}{dt} - \beta + \alpha SI + \mu S\right] = 0\,. \tag{4.4}$$

Suppose

$$S = x_0 + px_1 + p^2x_2 + \dots \tag{4.5}$$

$$I = y_0 + py_1 + p^2y_2 + \dots \tag{4.6}$$

$$R = z_0 + pz_1 + p^2z_2 + \dots \tag{4.7}$$

Substituting equations (4.5) and (4.6) into equation (4.4) and expanding, we have:

$$x_0' + px_1' + p^2x_2' + \dots + p\alpha\left(x_0y_0 + px_0y_1 + px_1y_0 + \dots\right) + p\mu\left(x_0 + px_1 + \dots\right) - p\beta = 0\,.$$

Collecting the coefficients of powers of p, we have:

$$p^0 : x_0' = 0\,,$$

from which $x_0 = A$. Therefore,

$$x_0 = S_0, \text{ since } x(0) = S_0 \tag{4.8}$$

$$p^1 : x_1' + \alpha x_0 y_0 + \mu x_0 - \beta = 0 \tag{4.9}$$

$$p^2 : x_2' + \alpha(x_0 y_1 + x_1 y_0) + \mu x_1 = 0$$

Applying HPM to equation (4.2), substitute equations (4.5) and (4.6) in the result and repeat the same process as in above then

$$y_0 = I_0 \tag{4.10}$$

$$p^1 : y_1' - \alpha x_0 y_0 + (\mu+\gamma+\delta) y_0 = 0 \tag{4.11}$$

$$p^2 : y_2' - \alpha(x_0 y_1 + x_1 y_0) + (\mu+\gamma+\delta) y_1 = 0$$

Also, applying HPM to equation (4.3), substitute equations (4.6) and (4.7) in the result and follow the same procedure then

$$z_0 = R_0 \tag{4.12}$$

$$p^1 : z_1' + \mu z_0 - \gamma y_0 = 0 \tag{4.13}$$

$$p^2 : z_2' + \mu z_1 - \gamma y_1 = 0 \tag{4.14}$$

Substituting equations (4.8) and (4.10) into equation (4.9)

$$\begin{aligned} x_1' &= \beta - \alpha S_0 I_0 - \mu S_0 \\ \int dx_1 &= (\beta - \alpha S_0 I_0 - \mu S_0) \int dt \\ x_1 &= (\beta - \alpha S_0 I_0 - \mu S_0)\, t + C\,. \end{aligned}$$

But $x_1(0) = 0$ then, $C = 0$. Therefore,

$$x_1 = (\beta - \alpha S_0 I_0 - \mu S_0)\, t\,. \tag{4.15}$$

Substitute equations (4.8) and (4.10) into equation (4.11) and repeat the same process as in above then

$$y_1 = I_0[\alpha S_0 - (\mu+\gamma+\delta)]\ \mathrm{t} \tag{4.16}$$

Substituting equations (4.8), (4.10), (4.15) and (4.16) into equation (4.6), we have:

$$\begin{aligned} x_2' &= -\{\,\alpha S_0 I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)]\,\mathrm{t}{+}\,(\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,(\alpha I_0 + \,\mu)t\} = 0 \\ \int dx_2 &= -\int \{\,\alpha S_0 I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)]\,\mathrm{t}{+}\,(\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,(\alpha I_0 + \,\mu)t\}\,dt \\ x_2 &= \{\,\alpha S_0 I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)] + (\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,(\alpha I_0 + \,\mu)\}\,\frac{\mathrm{t}^2}{2} + D\,, \end{aligned}$$

$x_2(0) = 0$ hence, $D = 0$. Therefore,

$$x_2 = \{\,\alpha S_0 I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)] + (\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,(\alpha I_0 + \,\mu)\}\,\frac{\mathrm{t}^2}{2}\,.$$

As $\mathrm{p} \to 1$, the result of equation (4.5) is obtained as

$$S(t) = \lim_{p\to 1} x = x_0 + x_1 + x_2 + x_3 + \ldots$$

Therefore,

$$\begin{aligned} S(t) = S_0 + [(\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)t] - \{\,\alpha S_0 I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)] + \\ (\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,(\alpha I_0 + \,\mu)\}\frac{\mathrm{t}^2}{2}\,. \end{aligned} \tag{4.17}$$

Substituting equation (4.8), (4.10), (4.15) and (4.16) into equation (4.6) and perform the same process as above, we have:

$$\begin{aligned} I(t) = I_0 + I_0\,[\alpha S_0 - \,(\mu{+}\gamma{+}\delta)]\,\mathrm{t}{+}I_0[\alpha S_0{-}\,(\mu{+}\gamma{+}\delta)]^2{+} \\ \alpha I_0\,(\beta{-}\,\alpha S_0 I_0 - \,\mu S_0)\,\frac{\mathrm{t}^2}{2}\,. \end{aligned} \tag{4.18}$$

Substituting equation (4.10) and (4.12) into equations (4.13) to obtain

$$z_1 = (\gamma I_0 - \,\mu R_0)\,t\,. \tag{4.19}$$

Also substituting equations (4.16) and (4.19) into equation (4.14) and carry out the same process as in equations (4.17), we have:

(4.20)

$$R(t) = R_0 + [(\gamma I_0 - \,\mu R_0)]t + \{\gamma I_0\,[\alpha S_0 - \,(\,\mu{+}\gamma{+}\delta\,)] - \mu\,(\gamma I_0 - \,\mu R_0)\}\,\frac{\mathrm{t}^2}{2}\,.$$

Hence, the general solutions of the model (4.1) – (4.3) are given by equations (4.17), (4.18) and (4.20).

5. Simulation and Discussion

To verify the validity and accuracy of homotopy perturbation method in solving the proposed model, parameters values in table 1 with the initial value for each state variable i.e. S_0, I_0 and R_0 starting from zero are used to simulate the analytical solutions given in equations (4.17), (4.18) and (4.20) with emphasis on the values of the parameters β, α and γ, the result of which is presented in figure 1. Emphasis is placed on the values of the parameters β, α and γ because epidemiological studies are built around the recruitment rate into susceptibility, transmission rate of the disease and recovery rate. The values of these key parameters are shown under figure 1. In figure 1, it is observed that the S(t)

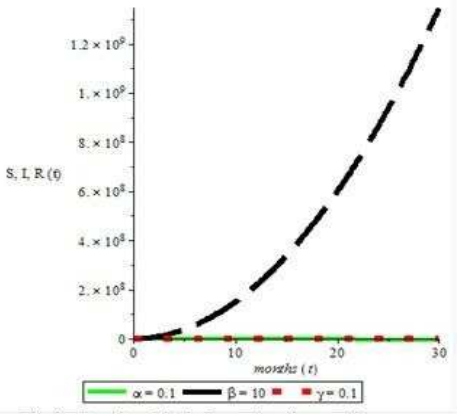

Fig.1. Graph of S, I, R against time with variations in recruitment, contact and treatment

curve slopes upward which indicates direct relationship between susceptibility and recruitment rate. On the other hands, the I(t) and R(t) curves do not get off the horizontal axis which implies that contact rate is not effective for the disease to spread in the population and in epidemiology, $R = 0$ when $I = 0$. We expect the transmission rate to be lower and the recovery rate to be higher with the adoption of effective prevention and control measures. This claim, which is supported in [20] where they argued that vaccination programs have resulted in a drop of more than 99% in the number of mumps reported cases in the United States and Canada, will reduce the number of susceptible and infected individuals but improve the number of individuals in the recovered compartment. However, findings due to [21] and [22] have shown that there may be treatment failure and inadequate protection as mumps vaccine strains (Jery1-Lynn, Urabe and Rubini) have different level of efficacy. In that sense we expect the

transmission rate to be higher and the recovery rate to be lower which in turn increase mumps infection and susceptibility but lower the number of individuals in compartment *R*.

In order to investigate the convergence rate of the method (HPM), we plot each state variable against time by varying the key parameters (β, α and γ) for the susceptible, infectious and recovered individuals. The parameters values in table 1 are made use of with $S_0 = 70$; I_0 and R_0 begin from zero, the results of which are displayed in figures 2–4.

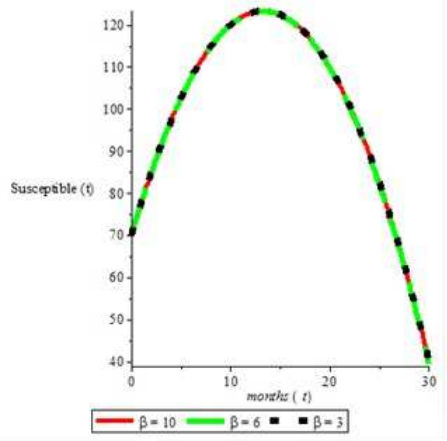

Fig. 2. Graph of the susceptible against time with variation in recruitment rate

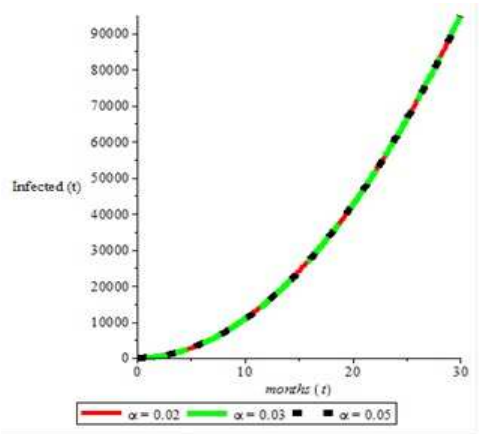

Fig.3. Graph of the infectious against time with variation in contact rate

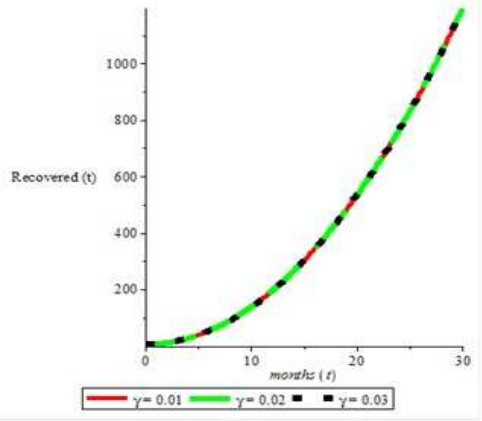

Fig.4. Graph of the recovered against time with variation in treatment rate

Figures 2–4 demonstrate the excellent convergence of homotopy perturbation method. In Figure 2, by decreasing the recruitment rate, the number of susceptible individuals firstly rises before it falls. The solutions of the three cases

of decrease in recruitment rate follow the same pattern and direction as shown in Figure 2. Likewise in Figure 3 and 4, increase in the contact rate and the recovery rates results in increase in the infection and the recovery rates respectively and the solutions of the increases follow the same pattern and behaviour as well.

6. Conclusion

HPM is a well known technique for obtaining the solutions of nonlinear differential equations. In this work, we have adopted SIR compartmental model to analyse the transmission dynamics of mumps disease and have solved the model analytically by applying HPM. The theoretical results are then solved numerically with the help of Mathematical software (Maple) and the solutions are biologically and mathematically meaningful which confirm the ability and appropriateness of the HPM in solving epidemic models.

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