ON A CERTAIN SUBCLASS OF MEROMORPHICALLY UNIFORMLY CONVEX FUNCTIONS AT INFINITY

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ABSTRACT. In this paper, we introduce and study a new subclass of meromorphically uniformly convex functions at infinity defined by a differential operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\sigma_\xi^+(\alpha)$.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S subclass of A consisting of functions f which are all univalent in E. A function $f \in A$ is starlike function of order α, $0 \leq \alpha < 1$ if it satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in E).$$

We denote this class with $S^*(\alpha)$. A function $f \in A$ is a convex function of the order α, $0 \leq \alpha < 1$ if it satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (z \in E).$$

We denote this class $K(\alpha)$. Let T denote the class of functions analytic in E that are of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0, \ (z \in E),$$

and let $T^*(\alpha) = T \cap S^*(\alpha)$, $C(\alpha) = T \cap K(\alpha)$. The class $T^*(\alpha)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [10] and others.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and denoted by $UCV(\gamma)$, if

$$(1.1) \qquad \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > \left|\frac{zf''(z)}{f'(z)}\right|,$$

where $\gamma \in [-1, 1)$ and is said to be in the class corresponding class denoted by $SP(\gamma)$ if

$$(1.2) \qquad \Re\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|,$$

where $\gamma \in [-1, 1)$.

Indeed it follows from (1.1) and (1.2) that

$$f \in UCV(\gamma) \Leftrightarrow zf' \in SP(\gamma).$$

Further Ahuja et al. [1], Bharathi et al. [2], Murugusundaramoorthy and Magesh [6] and others have studied and investigated interesting properties for the classes $UCV(\gamma)$ and $SP(\gamma)$.

Let Σ denote the class the class of functions of the form:

$$(1.3) \qquad f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

which are regular in domain $E = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. Let Σ_s, $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of Σ that are univalent, moromorphically starlike of order α and

meromorphically convex of order α respectively. Analytically f of the form (1.3) is in $\Sigma^*(\alpha)$ if and only if

$$\Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \alpha \ (z \in E).$$

Similarly, $f \in \Sigma_k(\alpha)$ if and only if, f is of the form (1.3) and satisfies

$$\Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \ (z \in E).$$

It being understood that if $\alpha = 1$ then $f(z) = \dfrac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$. The classes $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ have been extensively studied by Pommerenke [7], Clunie [3], Royster [8] and Venkateswarlu et al. [11, 12]. Let Σ_ξ represents all of $f(z)$ such that

$$(1.4) \qquad f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^{mq}, \ q = 1 - \frac{1}{\xi}, \ \xi \in N,$$

and are analytic in E. When ξ goes to infinity then $1 - \dfrac{1}{\xi}$ approaches to 1; hence $\Sigma_\xi = \Sigma$, which was studied by Faisal et al. [4].

Σ_ξ^+ also denote functions such as

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^{mq}, \ q = 1 - \frac{1}{\xi}, \ \xi \in N, \ a_m \geq 0,$$

and analytic in E.

For functions f in the class Σ_ξ^+, we define a differential operator D^n by the following form

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} (mq + 2) a_m z^{mq} = \frac{(z^2 f(z))'}{z}$$
$$D^2 f(z) = D(D^1 f(z))$$

and for $n = 1, 2, 3, \cdots$.

$$(1.5) \quad D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (mq + 2)^n a_m z^{mq} = \frac{(z^2 D^{n-1} f(z))'}{z}.$$

Now, we define a new subclass $\sigma_\xi^+(\alpha)$ of Σ_ξ.

Definition 1.1. *For* $-1 \leq \alpha < 1$, *we let* $\sigma_\xi^+(\alpha)$ *be the subclass of* Σ_ξ *consisting of functions of the form* (1.4) *and satisfying the analytic criterion.*

$$\Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right\} > \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|.$$

$D^n f(z)$ *is given by* (1.5).

The object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and δ−neighbourhoods for the class $\sigma_\xi^+(\alpha)$.

2. COEFFICIENT INEQUALITY

Theorem 2.1. *A function* f *of the form* (1.4) *is in* $\sigma_\xi^+(\alpha)$ *if*

$$\sum_{m=1}^{\infty}(mq + 2)^n[2mq + 3 - \alpha]\,|a_m| \leq (1 - \alpha), \ -1 \leq \alpha < 1 \ .$$

Proof. It sufficient to show that

$$\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right\} - \Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| - \Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right\}$$

$$\leq 2\left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right|$$

$$\leq \frac{2\sum_{m=1}^{\infty}(mq + 2)^n(mq + 1)|a_m||z^m|}{\frac{1}{|z|} - \sum_{m=1}^{\infty}(mq + 2)^n|a_m||z|^m}.$$

Letting $z \to 1$ along the real axis, we obtain

$$\leq \frac{2\sum_{m=1}^{\infty}(mq + 2)^n(mq + 1)|a_m|}{1 - \sum_{m=1}^{\infty}(mq + 2)^n|a_m|}.$$

This last expression is bounded by $(1 - \alpha)$ if

$$\sum_{m=1}^{\infty} (mq + 2)^n [2mq + 3 - \alpha]|a_m| \leq (1 - \alpha).$$

Hence the theorem is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.1. *Let the function f defined by (1.4) be in the class $\sigma_\xi^+(\alpha)$. Then*

$$a_m \leq \frac{(1 - \alpha)}{\sum\limits_{m=1}^{\infty} (mq + 2)^n [2mq + 3 - \alpha]}, \quad (m \geq 1).$$

Equality holds for the functions of the form

(2.1) $$f_m(z) = \frac{1}{z} + \frac{(1 - \alpha)}{(mq + 2)^n [2mq + 3 - \alpha]} z^m.$$

3. DISTORTION THEOREMS

Theorem 3.1. *Let the function f defined by (1.4) be in the class $\sigma_\xi^+(\alpha)$. Then for $0 < |z| = r < 1$,*

(3.1) $$\frac{1}{r} - \frac{(1 - \alpha)}{(q + 2)^n [2q + 3 - \alpha)]} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1 - \alpha)}{(q + 2)^n [2q + 3 - \alpha)]} r$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(1 - \alpha)}{(q + 2)^n [2mq + 3 - \alpha]} z.$$

Proof. Suppose f is in $\sigma_\xi^+(\alpha)$. In view of Theorem 2.1, we have

$$(q + 2)^n [2q + 3 - \alpha] \sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty} (mq + 2)^n [2mq + 3 - \alpha] \leq (1 - \alpha),$$

which evidently yields

$$\sum_{m=1}^{\infty} a_m \leq \frac{(1 - \alpha)}{(q + 2)^n [2q + 3 - \alpha]}.$$

Consequently, we obtain

$$
\begin{aligned}
|f(z)| &= \left| \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \right| \\
&\leq |\frac{1}{z}| + \sum_{m=1}^{\infty} a_m |z|^m \leq \frac{1}{r} + r \sum_{m=1}^{\infty} a_m \\
&\leq \frac{1}{r} + \frac{(1-\alpha)}{(q+2)^n [2q+3-\alpha]} r.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|f(z)| &= \left| \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \right| \\
&\geq |\frac{1}{z}| - \sum_{m=1}^{\infty} a_m |z|^m \geq \frac{1}{r} - r \sum_{m=1}^{\infty} a_m \\
&\geq \frac{1}{r} - \frac{(1-\alpha)}{(q+2)^n [2mq+3-\alpha]} r.
\end{aligned}
$$

Hence the results (3.1) follow. □

Theorem 3.2. *Let the function f defined by (1.4) be in the class $\sigma_\xi^+(\alpha)$. Then for $0 < |z| = r < 1$,*

$$
\frac{1}{r^2} - \frac{(1-\alpha)}{(q+2)^n [2q+3-\alpha]} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\alpha)}{(q+2)^n [2q+3-\alpha]}.
$$

The result is sharp, the external function being of the form (2.1).

Proof. From the Theorem 2.1, we have

$$
(q+2)^n [2q+3-\alpha] \sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} (m+2)^n [2(m+1)+1-\alpha] \leq (1-\alpha).
$$

which evidently yields

$$
\sum_{m=1}^{\infty} m a_m \leq \frac{(1-\alpha)}{(q+2)^n [2q+3-\alpha]}.
$$

Consequently, we obtain

$$|f(z)| \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}$$

$$\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m$$

$$\leq \frac{1}{r^2} + \frac{(1-\alpha)}{(q+2)^n[2q+3-\alpha]}.$$

Also,

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}$$

$$\geq |\frac{1}{r^2}| - \sum_{m=1}^{\infty} m a_m$$

$$\geq \frac{1}{r^2} - \frac{(1-\alpha)}{(q+2)^n[2mq+3-\alpha]}.$$

This completes the proof. □

4. CLASS PRESERVING INTEGRAL OPERATOR

In this section we consider the class preserving integral operators of the form (1.4).

Theorem 4.1. *Let the function f be defined by (1.4) be in the class $\sigma_\xi^+(\alpha)$. Then*

$$F(z) = cz^{-c-1} \int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \; c > 0,$$

belongs to the class $\sigma[\delta(\alpha, n, c)]$, where

$$\delta(\alpha, n, c) = \frac{3^n(5-\alpha)(c+2) - (1-\alpha)c}{3^n(5-\alpha)(c+2) + (1-\alpha)c}.$$

The result sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{(q+2)^n[2q+3-\alpha]} \; z.$$

Proof. Suppose $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_\xi^+(\alpha)$. We have

$$F(z) = cz^{-c-1} \int\limits_0^z t^c f(t) dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1} a_m z^m, \ c > 0.$$

It is sufficient to show that

(4.1)
$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta} \cdot \frac{ca_m}{(m+c+1)} \leq 1 .$$

Since f is in $\sigma_\xi^+(\alpha)$, we have

$$\frac{\sum\limits_{m=1}^{\infty} (mq+2)^n [2mq + 3 - \alpha]|a_m|}{1-\alpha} \leq 1 .$$

Thus (4.1) will be satisfied if

$$\frac{(m+\delta)c}{(1-\delta)(m+c+1)} \leq \frac{(mq+2)^n [2mq + 3 - \alpha]}{1-\alpha} \text{ for each } m ,$$

or

$$\delta \leq \frac{(mq+2)^n [2mq + 3 - \alpha](c+m+1) - mc(1-\alpha)}{(mq+2)^n [2mq + 3 - \alpha](c+m+1) + c(1-\alpha)}$$

$$G(m) = \frac{(mq+2)^n [2mq + 3 - \alpha](c+m+1) - mc(1-\alpha)}{(mq+2)^n [2mq + 3 - \alpha](c+m+1) + c(1-\alpha)} .$$

Then $G(m+1) - G(m) > 0$ for each m. Hence $G(m)$ is an increasing function of m. Since

$$G(1) = \frac{3^n(5-\alpha)(c+2) - (1-\alpha)c}{3^n(5-\alpha)(c+2) + (1-\alpha)c} .$$

The result follows. $\square$

5. CONVEX LINEAR COMBINATIONS AND CONVOLUTION PROPERTIES

Theorem 5.1. *If the function* $f(z) = \dfrac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *is in* $\sigma_\xi^+(\alpha)$ *then* f *is mero-morphically convex of order* $\delta(0 \leq \delta < 1)$ *in* $|z| < r = r(\alpha, \delta)$ *where*

$$r(\alpha, \delta) = \inf_{n \geq 1} \left\{ \frac{(1-\delta)(mq+2)^n [2mq + 3 - \alpha]}{(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}} .$$

The result is sharp.

Proof. Let f is in $\sigma_\xi^+(\alpha)$. Then by Theorem 2.1, we have

(5.1)
$$\sum_{m=1}^{\infty}(mq+2)^n[2mq+3-\alpha]|a_m| \leq (1-\alpha).$$

It is sufficient to show that
$$\left|2 + \frac{zf''(z)}{f'(z)}\right| \leq 1-\delta$$

for $|z| < r = r(\alpha,\delta)$, where $r(\alpha,\delta)$ is specified in the statement of the theorem. Then

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2} + \sum\limits_{m=1}^{\infty} ma_m z^{m-1}}\right|$$

$$\leq \sum_{m=1}^{\infty} \frac{m(m+1)a_m|z|^{m+1}}{1 - \sum\limits_{m=1}^{\infty} ma_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

(5.2)
$$\sum_{m=1}^{\infty} \frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \leq 1.$$

By (5.1), it follow that (5.2) is true if
$$\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \leq \frac{(mq+2)^n[2mq+3-\alpha]}{1-\alpha}, \ m \geq 1$$

or

(5.3)
$$|z| \leq \left\{\frac{(1-\delta)(mq+2)^n[2mq+3-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}.$$

Setting $|z| = r(\alpha,\delta)$ in (5.3), the result follows. The result is sharp for the function
$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(mq+2)^n[2mq+3-\alpha]}z^m, \ (m \geq 1).$$

$\square$

Theorem 5.2. *Let* $f_0(z) = \dfrac{1}{z}$ *and*

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(mq+2)^n[2mq+3-\alpha]}z^m, \ (m \geq 1).$$

Then $f(z) = \dfrac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in the class $\sigma_\xi^+(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z),$$

where $\lambda_0 \geq 0$, $\lambda_m \geq 0$ $(m \geq 1)$ and $\lambda_0 + \sum\limits_{m=1}^{\infty} \lambda_m = 1$.

Proof. Let $f(z) = \lambda_0 f_0(z) + \sum\limits_{m=1}^{\infty} \lambda_m f_m(z)$, where $\lambda_0 \geq 0$, $\lambda_m \geq 0$ $(m \geq 1)$ and

$$\lambda_0 + \sum_{m=1}^{\infty} \lambda_m = 1.$$

Then

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} \lambda_m \frac{(1-\alpha)}{(mq+2)^n[2mq+3-\alpha]} z^m.$$

Since

$$\sum_{m=1}^{\infty} \frac{(mq+2)^n[2mq+3-\alpha]}{(1-\alpha)} \lambda_m \frac{(1-\alpha)}{(mq+2)^n[2mq+3-\alpha]}$$

$$= \sum_{m=1}^{\infty} \lambda_m = 1 - \lambda_0 \leq 1.$$

By Theorem 2.1, f is in the class $\sigma_\xi^+(\alpha)$.

Conversely suppose that the function $f(z)$ is in the class $\sigma_\xi^+(\alpha)$.

Since

$$a_m \leq \frac{(1-\alpha)}{(mq+2)^n[2mq+3-\alpha]}, \quad (m \geq 1)$$

$$\lambda_m = \frac{(m+2)^n[2mq+3-\alpha]}{(1-\alpha)} a_m,$$

and $\lambda_0 = 1 - \sum\limits_{m=1}^{\infty} \lambda_m$, it follows that $f(z) = \lambda_0 f_0(z) + \sum\limits_{m=1}^{\infty} \lambda_m f_m(z)$.

This completes the proof of the theorem. $\qquad\qquad\square$

For the functions $f(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} b_m z^m$ belongs to Σ_ξ we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ or $(f * g)(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} a_m b_m z^m$.

Theorem 5.3. *If the functions $f(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} b_m z^m$ are in the class $\sigma_\xi^+(\alpha)$ then $(f * g)(z) = \dfrac{1}{z} + \displaystyle\sum_{m=1}^{\infty} a_m b_m z^m$ is in the class $\sigma_\xi^+(\alpha)$.*

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_\xi^+(\alpha)$. By Theorem 2.1, we have

$$\sum_{m=1}^{\infty} \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} a_m \leq 1 \text{ and}$$

$$\sum_{m=1}^{\infty} \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} b_m \leq 1.$$

Since $f(z)$ and $g(z)$ are regular in E, so is $(f * g)(z)$. Furthermore,

$$\sum_{m=1}^{\infty} \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} a_m b_m$$

$$\leq \sum_{m=1}^{\infty} \left\{ \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} \right\}^2 a_m b_m$$

$$\leq \left(\sum_{m=1}^{\infty} \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} a_m \right) \left(\sum_{m=1}^{\infty} \frac{(mq+2)^n [2mq+3-\alpha]}{(1-\alpha)} b_m \right)$$

$$\leq 1.$$

Hence by Theorem 2.1, $(f * g)(z)$ is in the class $\sigma_\xi^+(\alpha)$. $\qquad\qquad \square$

6. Neighborhoods for the class $\sigma_\xi^+(\alpha)$

Definition 6.1. *A function $f \in \Sigma_\xi$ is said to in the class $\sigma_\xi^+(\alpha, \gamma)$ if there exists a function $g \in \sigma_\xi^+(\alpha)$ such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma, \ z \in E, \ (0 \leq \gamma < 1).$$

Following the earlier works on neighborhoods of analytic functions by Goodman [5] and Ruschweyh [9], we define the $\delta-$neighborhood of a function $f \in \Sigma_\xi$ by

$$(6.1) \qquad N_\delta(f) := \left\{ g \in \Sigma_\xi : g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \right\}.$$

Theorem 6.1. *If $g \in \sigma_\xi^+(\alpha)$ and*

$$(6.2) \qquad\qquad\qquad \gamma = 1 - \frac{\delta(5-\alpha)}{4}.$$

Then $N_\delta(g) \subset \sigma_\xi^+(\alpha, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.1) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta,$$

which implies the coefficient in equality

$$\sum_{m=1}^{\infty} |a_m - b_m| \leq \delta, (m \in N).$$

Since $g \in \sigma_\xi^+(\alpha)$, we have

$$\sum_{m=1}^{\infty} b_m \ < \ \frac{1-\alpha}{5-\alpha},$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta(5-\alpha)}{4} = 1 - \gamma.$$

provided γ is given by (6.2). Hence by definition, $f \in \sigma_\xi^+(\alpha, \gamma)$ for γ given by (6.2), which completes the proof. □

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