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GENERALIZED DISCRETE FINITE HALF RANGE FOURIER SERIES

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ABSTRACT. In this paper, we obtain a discrete finite half-range Fourier series for odd and even functions by using generalized difference operator. Suitable numerical results and examples are verified by MATLAB.

1. INTRODUCTION

The Fourier Series is the most widely used series expansion in mathematics modeling of engineering systems. It serves as the basis for the Fourier integral, the Laplace transform, the solution of autonomous linear differential equations, frequency response methods and many engineering applications. There are many good treatments on the subject; too many to mention in a comprehensive manner. However, the treatment by Tolstov [7] is classical. Jerri [4] provides an excellent overview on convergence of the Fourier series and discusses Gibbs-like phenomena in continuous and discrete wavelet representations. Sidi [5] reviews the state of the art of extrapolation methods giving applied scientists and engineers a practical guide to accelerating convergence in difficult computational problems. Also, accelerated convergence by means of periodic bridge functions was developed by Anguelov [1].

In Fourier analysis, a signal is decomposed into its constituent sinusoids. In the reverse by operating the inverse Fourier transform, the signal can be synthesized by adding up its constituent frequencies. Many signals that we encounter

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in daily life such as speech, automobile noise, chirps of birds, music etc. have a periodic or quasi-periodic structure, and that the cochlea in the human hearing system performs a kind of harmonic analysis of the input audio signals in biological and physical systems [8]. The Fourier series decomposes the given input signals into a sum of sinusoids. By removing the high frequency terms(noise) of Fourier series and then adding the remaining terms can yield better signals [2].

Finite Fourier series is a powerful tool for attacking many problems in the theory of numbers. It is related to certain types of exponential and trigonometric sums. It may therefore be expanded into a finite Fourier series of the form

$$f(\alpha^{\mu}) = \sum_{j=0}^{m-1} g(j)\alpha^{\mu j}(\mu = 0, 1, \cdots, m-1).$$

The orthogonality relation

$$\sum_{j=0}^{m-1} \alpha^{aj}\alpha^{-bj} = \begin{cases} m & (a \equiv b(mod\, m)), \\ 0 & (a \not\equiv b(mod\, m)), \end{cases}$$

enables us to determine the finite Fourier coefficients $g(k)$ explicitly by means of the formula $g(k) = \frac{1}{m}\sum_{\mu=0}^{m-1} f(\alpha^{\mu})\alpha^{-\mu k}$. If we are given k distinct complex numbers $z_0, z_1, \cdots, z_{k-1}$, then there is one and only one polynomial $P(x) = \zeta_0 + \zeta_1 x + \cdots + \zeta_{k-1}x^{k-1}$ satisfying the equations $P(\omega_\nu) = z_\nu$ $(\nu = 0, 1, \cdots, k-1)$, [9].

A finite Fourier series: $\eta(t) = A_0 + \sum_{q=1}^{N/2} A_q \cos(q\sigma_1 t) + \sum_{q=1}^{N/2-1} B_q \sin(q\sigma_1 t)$, where $\eta =$ sea surface elevation (m), $t =$ time (s), $A_0 =$ recond mean (m), $N =$ total number of sampling points, A_q and $B_q =$ Fourier coefficients (m), $q =$ harmonic component index (in the frequency domain) and $\sigma_1 =$ fundamental radian frequency, is used in [3]. The sum of N sine waves defined over the time interval, $0 \leq t \leq T : y = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n)$, $0 \leq t_n \leq T$, $a_n \geq 0$, $0 \leq \phi_n < 2\pi$, where a_n is amplitude and t is time, is also a finite Fourier series. In [6], the authors describe an efficient formulation, based on a discrete Fourier series expansion, of the analysis of axi-symmetric solids subjected to non-symmetric loading. They have discussed the Fourier series approach, the discrete Fourier series representation problems such as the presence of Gibb's phenomenon and the lack of conformity of elements. Here we arrive a new type of finite Fourier series for functions (signals) by defining discrete orthonormal family of functions using inverse generalized difference operator Δ_ℓ^{-1}.

In this paper we mainly concentrate on the discrete finite half-range Fourier series of odd and even functions and examples are verified using MATLAB.

2. Preliminaries

In this section, we present some basic definitions and results.

Definition 2.1. *Let $u(k),\ k \in [0,\infty)$, be a real or complex valued function and $\ell > 0$ be a fixed shift value. Then, the ℓ-difference operator Δ_ℓ on $u(k)$ is defined as*

$$\Delta_\ell u(k) = u(k+\ell) - u(k),$$

and its infinite $\ell-$ difference sum is defined by

$$\Delta_\ell^{-1} u(k) = \sum_{r=0}^{\infty} u(k + r\ell).$$

Lemma 2.1. *Let s_r^β and S_r^β are Stirling numbers of first kind and second kinds, $k_\ell^{(\beta)} = k(k-\ell)(k-2\ell)\cdots(k-(\beta-1)\ell)$, $\ell > 0$*

$$(i)\ k_\ell^{(\beta)} = \sum_{r=1}^{\beta} s_r^\beta \ell^{\beta-r} k^r,\ (ii)\ k^\beta = \sum_{r=1}^{\beta} S_r^\beta \ell^{\beta-r} k_\ell^{(r)}.$$

Definition 2.2. *Let $u(k)$ and $v(k)$ are the two real valued functions defined on $(-\infty,\infty)$ and if $\Delta_\ell v(k) = u(k)$, then the finite inverse principle law is given by*

$$v(k) - v(k-\beta\ell) = \sum_{r=1}^{\beta} u(k - r\ell), \beta \in Z^+ .$$

Applying the Definition 2.1, we get the modified identities as follows:

$$(i)\ \Delta_\ell k_\ell^{(\beta)} = (\beta\ell) k_\ell^{(\beta-1)},\ \ (ii)\ \Delta_\ell^{-1} k_\ell^{(\beta)} = \frac{k_\ell^{(\beta+1)}}{\ell\beta+1}\ (iii)\ \Delta_\ell^{-1} k^\beta = \sum_{r=1}^{\beta} \frac{S_r^\beta \ell^{\beta-r} k_\ell^{(r+1)}}{(r+1)\ell}.$$

Lemma 2.2. *Let $\ell > 0$ and $u(k)$, $w(k)$ are real valued bounded functions. Then*

$$\Delta_\ell^{-1}(u(k)w(k)) = u(k)\Delta_\ell^{-1} w(k) - \Delta_\ell^{-1}(\Delta_\ell^{-1} w(k+\ell)\Delta_\ell u(k)). \tag{2.1}$$

Lemma 2.3. *Let p be real $\ell > 0$, $k \in (\ell,\infty)$ and $p\ell \neq \beta 2\pi$. Then we have*

$$\Delta_\ell^{-1} \sin pk = \frac{(\sin p(k-\ell) - \sin pk)}{2(1-\cos p\ell)} + c_j$$

and

$$\Delta_\ell^{-1}\cos pk = \frac{(\cos p(k-\ell) - \cos pk)}{2(1-\cos p\ell)} + c_j. \tag{2.2}$$

Definition 2.3. *The generalized finite Fourier series is defined by*

$$u(k) = \frac{a_0}{2} + \sum_{n=1}^{\eta-1}(a_n\cos nk + b_n \sin nk) + \frac{a_\eta}{2}\cos\eta k, \quad k\in\{a+r\ell\}_{r=0}^{2\eta-1}, \tag{2.3}$$

where $a_n = \frac{\ell}{\pi}\Delta_\ell^{-1}u(k)\cos nk\Big|_a^{a+2\pi}$,$b_n = \frac{\ell}{\pi}\Delta_\ell^{-1}u(k)\sin nk\Big|_a^{a+2\pi}$.

Definition 2.4. ***(Odd and Even function)*** *Even functions are the functions for which the left half of the plane looks like the mirror image of the right of the plane. Odd functions are the functions where the left half of the plane looks like the mirror image of the right of the plane,only upside-down. The function* $u(k)$ *is an even function if* $u(-k) = u(k)$ *and odd if* $u(-k) = -u(k)$.

Definition 2.5. *The generalized finite half-range Fourier series (GFHFS) of odd and even functions are is defined as If* $u(k)$ *is even function, then* (2.3) *becomes,*

$$u(k) = \frac{a_0}{2} + \sum_{n=1}^{\eta-1} a_n\cos nk + \frac{a_\eta}{2}\cos\eta k, \tag{2.4}$$

where $a_n = \frac{\ell}{\pi}\Delta_\ell^{-1}u(k)\cos nk\Big|_0^{\pi}$. *If* $u(k)$ *is odd function, then* (2.3) *becomes,*

$$u(k) = \sum_{n=1}^{\eta-1} b_n \sin nk, \tag{2.5}$$

where $b_n = \frac{2\ell}{\pi}\Delta_\ell^{-1}u(k)\sin nk\Big|_0^{\pi}$.

3. Finite Half Range Fourier Series for Odd functions

Theorem 3.1. *Let* $k\in(-\infty,\infty)$, $\ell>0$ *and* $n\ell \neq \beta 2\pi$*, then we have*

$$\Delta_\ell^{-1}k^p\sin nk = \sum_{m=1}^{p}\sum_{t=0}^{\beta}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_\beta^p(\beta)_1^{(t)}k_\ell^{(\beta-t)}\sin n(k-\ell+r\ell)}{(-1)^{r-1}\ell^{\beta-t-p}(2(\cos n\ell-1))^{t+1}}. \tag{3.1}$$

Proof. Taking, $u(k) = k$ and $w(k) = \sin k$ in (2.1), we get,

$$\Delta_\ell^{-1}k\sin k = \frac{k\sin(k-\ell)}{2(1-\cos\ell)} - \frac{k\sin k}{2(1-\cos\ell)} - \frac{\ell\sin(k-\ell)}{(2(1-\cos\ell))^2} + \frac{2\ell\sin k}{(2(1-\cos\ell))^2} - \frac{\ell\sin(k+\ell)}{(2(1-\cos\ell))^2}.$$

Taking $u(k) = k$ and $w(k) = \sin 2k$ in (2.1), we have,

$$\Delta_\ell^{-1} k \sin 2k = \frac{k \sin 2(k-\ell)}{2(1-\cos 2\ell)} - \frac{k \sin 2k}{2(1-\cos 2\ell)} - \frac{\ell \sin 2(k-\ell)}{(2(1-\cos 2\ell))^2} + \frac{2\ell \sin 2k}{(2(1-\cos 2\ell))^2} - \frac{\ell \sin 2(k+\ell)}{(2(1-\cos 2\ell))^2}$$

Taking $u(k) = k^2$ and $w(k) = \sin k$ in (2.1), we have,

$$\begin{aligned}\Delta_\ell^{-1} k^2 \sin k = {} & \frac{k^2 \sin(k-\ell)}{2(1-\cos \ell)} - \frac{k^2 \sin k}{2(1-\cos \ell)} - \frac{\ell^2 \sin(k-\ell)}{(2(1-\cos \ell))^2} + \frac{2\ell^2 \sin k}{(2(1-\cos \ell))^2} \\ & - \frac{\ell^2 \sin(k+\ell)}{(2(1-\cos \ell))^2} - \frac{2k\ell \sin(k-\ell)}{(2(1-\cos \ell))^2} \\ & + \frac{2\ell^2 \sin(k-\ell)}{(2(1-\cos \ell))^3} - \frac{6\ell^2 \sin k}{(2(1-\cos \ell))^3} \\ & + \frac{6\ell^2 \sin(k+\ell)}{(2(1-\cos \ell))^3} + \frac{4\ell k \sin k}{(2(1-\cos \ell))^2} - \frac{2\ell k \sin(k+\ell)}{(2(1-\cos \ell))^2} - \frac{2\ell^2 \sin(k+2\ell)}{(2(1-\cos \ell))^3}.\end{aligned}$$

Taking $u(k) = k^2$ and $w(k) = \sin 2k$ in (2.1) we get,

$$\begin{aligned}\Delta_\ell^{-1}(k^2 \sin 2k) = {} & \frac{k^2 \sin 2(k-\ell)}{2(1-\cos 2\ell)} - \frac{k^2 \sin 2k}{2(1-\cos 2\ell)} - \frac{\ell^2 \sin 2(k-\ell)}{(2(1-\cos 2\ell))^2} + \frac{2\ell^2 \sin 2k}{(2(1-\cos 2\ell))^2} \\ & - \frac{\ell^2 \sin 2(k+\ell)}{(2(1-\cos 2\ell))^2} - \frac{2k\ell \sin 2(k-\ell)}{(2(1-\cos 2\ell))^2} + \frac{2\ell^2 \sin 2(k-\ell)}{(2(1-\cos 2\ell))^3} - \frac{6\ell^2 \sin 2k}{(2(1-\cos 2\ell))^3} \\ & + \frac{6\ell^2 \sin 2(k+\ell)}{(2(1-\cos 2\ell))^3} + \frac{4\ell k \sin 2k}{(2(1-\cos 2\ell))^2} - \frac{2\ell k \sin 2(k+\ell)}{(2(1-\cos 2\ell))^2} - \frac{2\ell^2 \sin 2(k+2\ell)}{(2(1-\cos 2\ell))^3}.\end{aligned}$$

By repeating the above process, we obtain (3.1). □

Corollary 3.2. *If p is odd, $I = [0, \pi]$, $\ell = \frac{\pi}{N}$, then the GFHF coefficients b_n for $n = 0, 1, 2, ..., N$, for the polynomial k^p is*

$$b_n = \frac{2\ell}{\pi} \Delta_\ell^{-1} k^p \sin nk \Big|_0^\pi = \sum_{\beta=1}^{p-1} \sum_{t=0}^{\beta} \sum_{r=0}^{t+1} \binom{t+1}{r} \frac{S_\beta^p(\beta)_1^t(\pi)_\ell^{(\beta-t)} \sin n(r-1)\ell}{(-1)^{r-1}\eta \ell^{\beta-t-p}(2(\cos n-\ell))^{t+1}}. \tag{3.2}$$

Proof. The proof follows by applying the limit 0 to π for (3.1). □

Corollary 3.3. *If p is odd, $k \in r\ell$, $r = 0, 1, ...2N-1$, then the GFHFS for k^p is*

$$k^p = \frac{2\ell}{\pi} \sum_{n=1}^{\eta-1} \Delta_\ell^{-1} k^p \sin nk \sin nk. \tag{3.3}$$

Proof. Taking $u(k) = k^p$ and using (3.2), we get (3.3). □

Example 1. *Taking* $p = 5$, $\eta = 2$, $\ell = \frac{\pi}{2}$ *in* (3.3), *we obtain*

$$\begin{aligned}
k^5 = {} & \frac{\pi^5 \sin(\pi-\ell)}{2(1-\cos\ell)} - \frac{\pi^5 \sin\pi}{2(1-\cos\ell)} + \frac{2\ell^5 \sin\pi}{(2(1-\cos\ell))^2} - \frac{\ell^5 \sin(\pi-\ell)}{(2(1-\cos\ell))^2} \\
& - \frac{\ell^5 \sin(\pi+\ell)}{(2(1-\cos\ell))^2} - \frac{5\pi\ell^4 \sin(\pi-\ell)}{(2(1-\cos\ell))^2} + \frac{10\pi\ell^4 \sin\pi}{(2(1-\cos\ell))^2} - \frac{5\pi\ell^4 \sin(\pi+\ell)}{(2(1-\cos\ell))^2} \\
& + \frac{30\ell^5 \sin(\pi-\ell)}{(2(1-\cos\ell))^3} - \frac{90\ell^5 \sin\pi}{(2(1-\cos\ell))^3} + \frac{90\ell^5 \sin(\pi+\ell)}{(2(1-\cos\ell))^3} - \frac{30\ell^5 \sin(\pi+2\ell)}{(2(1-\cos\ell))^3} \\
& - \frac{10\pi^2\ell^3 \sin(\pi-\ell)}{(2(1-\cos\ell))^2} + \frac{20\pi^2\ell^3 \sin\pi}{(2(1-\cos\ell))^2} - \frac{10\pi^2\ell^3 \sin(\pi+\ell)}{(2(1-\cos\ell))^2} + \frac{70\pi\ell^4 \sin(\pi-\ell)}{(2(1-\cos\ell))^3} \\
& - \frac{210\pi\ell^4 \sin\pi}{(2(1-\cos\ell))^3} + \frac{210\pi\ell^4 \sin(\pi+\ell)}{(2(1-\cos\ell))^3} - \frac{70\pi\ell^4 \sin(\pi+2\ell)}{(2(1-\cos\ell))^3} - \frac{150\ell^5 \sin(\pi-\ell)}{(2(1-\cos\ell))^4} \\
& + \frac{600\ell^5 \sin\pi}{(2(1-\cos\ell))^4} - \frac{900\ell^5 \sin(\pi+\ell)}{(2(1-\cos\ell))^4} + \frac{600\ell^5 \sin(\pi+2\ell)}{(2(1-\cos\ell))^4} - \frac{150\ell^5 \sin(\pi+3\ell)}{(2(1-\cos\ell))^4} \\
& - \frac{10\pi^3\ell^2 \sin(\pi-\ell)}{(2(1-\cos\ell))^2} + \frac{20\pi^3\ell^2 \sin\pi}{(2(1-\cos\ell))^2} - \frac{10\pi^3\ell^2 \sin(\pi+\ell)}{(2(1-\cos\ell))^2} + \frac{60\ell^3\pi^2 \sin(\pi-\ell)}{(2(1-\cos\ell))^3} \\
& - \frac{180\ell^3\pi^2 \sin\pi}{(2(1-\cos\ell))^3} + \frac{180\ell^3\pi^2 \sin(\pi+\ell)}{(2(1-\cos\ell))^3} - \frac{60\ell^3\pi^2 \sin(\pi+2\ell)}{(2(1-\cos\ell))^3} - \frac{180\ell^4\pi \sin(\pi-\ell)}{(2(1-\cos\ell))^4} \\
& + \frac{720\ell^4\pi \sin\pi}{(2(1-\cos\ell))^4} - \frac{1080\ell^4\pi \sin(\pi+\ell)}{(2(1-\cos\ell))^4} - \frac{720\ell^4\pi \sin(\pi+2\ell)}{(2(1-\cos\ell))^4} - \frac{180\ell^4\pi \sin(\pi+3\ell)}{(2(1-\cos\ell))^4} \\
& + \frac{240\ell^5 \sin(\pi-\ell)}{(2(1-\cos\ell))^5} - \frac{1200\ell^5 \sin\pi}{(2(1-\cos\ell))^5} + \frac{2400\ell^5 \sin(\pi+\ell)}{(2(1-\cos\ell))^5} - \frac{2400\ell^5 \sin(\pi+2\ell)}{(2(1-\cos\ell))^5} \\
& + \frac{1200\ell^5 \sin(\pi+3\ell)}{(2(1-\cos\ell))^5} - \frac{240\ell^5 \sin(\pi+4\ell)}{(2(1-\cos\ell))^5} - \frac{5\ell\pi^4 \sin(\pi-\ell)}{(2(1-\cos\ell))^2} + \frac{10\ell\pi^4 \sin\pi}{(2(1-\cos\ell))^2} \\
& + \frac{110\ell^3\pi^2 \sin\pi}{(2(1-\cos\ell))^2} - \frac{5\ell\pi^4 \sin(\pi+\ell)}{(2(1-\cos\ell))^2} + \frac{20\ell^2\pi^3 \sin(\pi-\ell)}{(2(1-\cos\ell))^3} - \frac{60\ell^2\pi^3 \sin\pi}{(2(1-\cos\ell))^3} \\
& + \frac{60\ell^2\pi^3 \sin(\pi+\ell)}{(2(1-\cos\ell))^3} - \frac{20\ell^2\pi^3 \sin(\pi+2\ell)}{(2(1-\cos\ell))^3} - \frac{60\ell^3\pi^2 \sin(\pi-\ell)}{(2(1-\cos\ell))^4} + \frac{240\ell^3\pi^2 \sin\pi}{(2(1-\cos\ell))^4} \\
& - \frac{360\ell^3\pi^2 \sin(\pi+\ell)}{(2(1-\cos\ell))^4} + \frac{240\ell^3\pi^2 \sin(\pi+2\ell)}{(2(1-\cos\ell))^4} - \frac{60\ell^3\pi^2 \sin(\pi+3\ell)}{(2(1-\cos\ell))^4} + \frac{120\ell^4\pi \sin(\pi-\ell)}{(2(1-\cos\ell))^5} \\
& - \frac{600\ell^4\pi \sin\pi}{(2(1-\cos\ell))^5} + \frac{1200\ell^4\pi \sin(\pi+\ell)}{(2(1-\cos\ell))^5} - \frac{1200\ell^4\pi \sin(\pi+2\ell)}{(2(1-\cos\ell))^5} + \frac{600\ell^4\pi \sin(\pi+3\ell)}{(2(1-\cos\ell))^5} \\
& - \frac{120\ell^4\pi \sin(\pi+4\ell)}{(2(1-\cos\ell))^5} - \frac{120\ell^5 \sin(\pi-\ell)}{(2(1-\cos\ell))^6} + \frac{720\ell^5 \sin\pi}{(2(1-\cos\ell))^6} - \frac{1800\ell^5 \sin(\pi+\ell)}{(2(1-\cos\ell))^6} \\
& + \frac{2400\ell^5 \sin(\pi+2\ell)}{(2(1-\cos\ell))^6} - \frac{1800\ell^5 \sin(\pi+3\ell)}{(2(1-\cos\ell))^6} + \frac{720\ell^5 \sin(\pi+4\ell)}{(2(1-\cos\ell))^6} - \frac{120\ell^5 \sin(\pi+5\ell)}{(2(1-\cos\ell))^6} \\
& - \frac{2\ell^5 \sin 0}{(2(1-\cos\ell))^2} + \frac{\ell^5 \sin(0-\ell)}{(2(1-\cos\ell))^2} + \frac{\ell^5 \sin(0+\ell)}{(2(1-\cos\ell))^2} - \frac{30\ell^5 \sin(0-\ell)}{(2(1-\cos\ell))^3} + \frac{90\ell^5 \sin 0}{(2(1-\cos\ell))^3} \\
& - \frac{90\ell^5 \sin(0+\ell)}{(2(1-\cos\ell))^3} + \frac{30\ell^5 \sin(0+2\ell)}{(2(1-\cos\ell))^3} + \frac{150\ell^5 \sin(0-\ell)}{(2(1-\cos\ell))^4} - \frac{600\ell^5 \sin 0}{(2(1-\cos\ell))^4} \\
& + \frac{900\ell^5 \sin(0+\ell)}{(2(1-\cos\ell))^4} - \frac{600\ell^5 \sin(0+2\ell)}{(2(1-\cos\ell))^4} + \frac{150\ell^5 \sin(0+3\ell)}{(2(1-\cos\ell))^4} - \frac{240\ell^5 \sin(0-\ell)}{(2(1-\cos\ell))^5}
\end{aligned}$$

$$+\frac{1200\ell^5\sin 0}{(2(1-\cos\ell))^5}-\frac{2400\ell^5\sin(0+\ell)}{(2(1-\cos\ell))^5}+\frac{2400\ell^5\sin(0+2\ell)}{(2(1-\cos\ell))^5}-\frac{1200\ell^5\sin(0+3\ell)}{(2(1-\cos\ell))^5}$$
$$+\frac{240\ell^5\sin(0+4\ell)}{(2(1-\cos\ell))^5}+\frac{120\ell^5\sin(0-\ell)}{(2(1-\cos\ell))^6}-\frac{720\ell^5\sin 0}{(2(1-\cos\ell))^6}+\frac{1800\ell^5\sin(0+\ell)}{(2(1-\cos\ell))^6}$$
$$-\frac{2400\ell^5\sin(0+2\ell)}{(2(1-\cos\ell))^6}+\frac{1800\ell^5\sin(0+3\ell)}{(2(1-\cos\ell))^6}-\frac{720\ell^5\sin(0+4\ell)}{(2(1-\cos\ell))^6}+\frac{120\ell^5\sin(0+5\ell)}{(2(1-\cos\ell))^6}\sin k$$

In particular taking $k=\frac{\pi}{2}$, we obtain $k^5=9.5631$.

Theorem 3.4. *Let $k\in(-\infty,\infty)$ and $\ell>0$, then we have GFHFS for the function $\sin k$ as*

$$\sin k=\frac{2\ell}{\pi}\sum_{n=1}^{\eta-1}\Delta_\ell^{-1}\sin nk\sin nk. \tag{3.4}$$

Proof. Taking $\ell=\frac{\pi}{3}$, and $N=3$ in (3.4) we get, $\sin k=\sum_{n=1}^{3-1}b_n\sin nk$, $k\in\{r\ell\}_{r=0}^{6-1}$,

$$\sin k=b_1\sin k+b_2\sin 2k, k\in\{r\ell\}_{r=0}^{6-1},$$

where $b_1=\frac{2\ell}{\pi}\Delta_\ell^{-1}\sin k\sin k\Big|_0^\pi$ and $b_2=\frac{2\ell}{\pi}\Delta_\ell^{-1}\sin k\sin 2k\Big|_0^\pi$. By using 2.2 we get,

$$\Delta_\ell^{-1}\sin k\sin k=\Delta_\ell^{-1}\sin^2 k=\Delta_\ell^{-1}(1-\cos 2k)$$
$$=\Delta_\ell^{-1}(1)-\Delta_\ell^{-1}\cos 2k=\frac{k}{\ell}-\frac{\cos 2(k-\ell)}{2(1-\cos 2\ell)}+\frac{\cos 2k}{2(1-\cos 2\ell)}$$
$$b_1=\frac{2\ell}{\pi}\frac{k}{\ell}-\frac{\cos 2(k-\ell)}{2(1-\cos 2\ell)}+\frac{\cos 2k}{2(1-\cos 2\ell)}$$
$$b_1=\frac{2\ell}{\pi}\frac{k}{\ell}-\frac{\cos 2(k-\ell)}{2(1-\cos 2\ell)}+\frac{\cos 2k}{2(1-\cos 2\ell)}\Big|_0^\pi$$
$$b_1=1$$
$$b_2=\frac{2\ell}{\pi}\Delta_\ell^{-1}\sin k\sin 2k\Big|_0^\pi=\frac{\ell}{\pi}\Delta_\ell^{-1}(\cos k-\cos 3k)\Big|_0^\pi$$
$$b_2=0$$
□

Example 2. *In particular $k=\frac{\pi}{3}$ in (3.4), which gives $\sin k=b_1\sin k+b_2\sin 2k=0.8660$.*

Theorem 3.5. *Let $\ell>0$, $k\in\{r\ell\}_{r=0}^{2N-1}$ then we have the GFHFS for the function $k^2\sin 3k$ as*

$$k^2\sin 3k=\sum_{n=1}^{\eta-1}b_n\sin nk. \tag{3.5}$$

Proof. The proof follows by taking $u(k)=k\cos 2k$ in (2.5). □

Example 3. *Taking $\ell=\frac{\pi}{2}$, and $\eta=2$ in (3.5) we get,*

$$k^2\sin 3k=b_1\sin k, k\in\{r\ell\}_{r=0}^{4-1}, \tag{3.6}$$

where $b_1=\frac{2\ell}{\pi}\Delta_\ell^{-1}k^2\sin 3k\sin k\Big|_0^\pi$. By using (2.1) we get,

$$\Delta_\ell^{-1}k^2\sin 3k\sin k = \tfrac{1}{2}\frac{k^2\cos 2(k-\ell)}{2(1-\cos 2\ell)} - \frac{k^2\cos 2k}{2(1-\cos 2\ell)} - \frac{2\ell k\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2}$$
$$+\frac{4\ell k\cos 2k}{(2(1-\cos 2\ell))^2} + \frac{2\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^3} - \frac{6\ell^2\cos 2k}{(2(1-\cos 2\ell))^3} + \frac{6\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^3}$$
$$-\frac{2\ell k\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2} - \frac{2\ell^2\cos 2(k+2\ell)}{(2(1-\cos 2\ell))^3} + \frac{2\ell^2\cos 2k}{(2(1-\cos 2\ell))^2} - \frac{\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2}$$
$$-\frac{\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2} - \frac{k^2\cos 4(k-\ell)}{2(1-\cos 4\ell)} + \frac{k^2\cos 4k}{2(1-\cos 4\ell)} + \frac{2\ell k\cos 4(k-\ell)}{(2(1-\cos 4\ell))^2}$$
$$-\frac{4\ell k\cos 4k}{(2(1-\cos 4\ell))^2} - \frac{2\ell^2\cos 4(k-\ell)}{(2(1-\cos 4\ell))^3} + \frac{6\ell^2\cos 4k}{(2(1-\cos 4\ell))^3} - \frac{6\ell^2\cos 4(k+\ell)}{(2(1-\cos 4\ell))^3}$$
$$+\frac{2\ell k\cos 4(k+\ell)}{(2(1-\cos 4\ell))^2} + \frac{2\ell^2\cos 4(k+2\ell)}{(2(1-\cos 4\ell))^3} - \frac{2\ell^2\cos 4k}{(2(1-\cos 4\ell))^2} + \frac{\ell^2\cos 4(k-\ell)}{(2(1-\cos 4\ell))^2}$$
$$+\frac{\ell^2\cos 4(k+\ell)}{(2(1-\cos 4\ell))^2}$$

$$b_1 = \tfrac{\ell}{\pi}\frac{k^2\cos 2(k-\ell)}{2(1-\cos 2\ell)} - \frac{k^2\cos 2k}{2(1-\cos 2\ell)} - \frac{2\ell k\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2} + \frac{4\ell k\cos 2k}{(2(1-\cos 2\ell))^2}$$
$$+\frac{2\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^3} - \frac{6\ell^2\cos 2k}{(2(1-\cos 2\ell))^3} + \frac{6\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^3} - \frac{2\ell k\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2}$$
$$-\frac{2\ell^2\cos 2(k+2\ell)}{(2(1-\cos 2\ell))^3} + \frac{2\ell^2\cos 2k}{(2(1-\cos 2\ell))^2} - \frac{\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2} - \frac{\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2}$$
$$-\frac{k^2\cos 4(k-\ell)}{2(1-\cos 4\ell)} + \frac{k^2\cos 4k}{2(1-\cos 4\ell)} + \frac{2\ell k\cos 4(k-\ell)}{(2(1-\cos 4\ell))^2} - \frac{4\ell k\cos 4k}{(2(1-\cos 4\ell))^2}$$
$$-\frac{2\ell^2\cos 4(k-\ell)}{(2(1-\cos 4\ell))^3} + \frac{6\ell^2\cos 4k}{(2(1-\cos 4\ell))^3} - \frac{6\ell^2\cos 4(k+\ell)}{(2(1-\cos 4\ell))^3} + \frac{2\ell k\cos 4(k+\ell)}{(2(1-\cos 4\ell))^2}$$
$$+\frac{2\ell^2\cos 4(k+2\ell)}{(2(1-\cos 4\ell))^3} - \frac{2\ell^2\cos 4k}{(2(1-\cos 4\ell))^2} + \frac{\ell^2\cos 4(k-\ell)}{(2(1-\cos 4\ell))^2} + \frac{\ell^2\cos 4(k+\ell)}{(2(1-\cos 4\ell))^2}\Big|_0^\pi.$$

Particularly, taking $k = \dfrac{\pi}{2}$ *in (3.6), which gives* $b_1 = -1.5708$*, we get* $b_1\sin k = k^2\sin 3k = -1.5708$.

Theorem 3.6. *Let* $\ell > 0$, $k \in \{r\ell\}_{r=0}^{2N-1}$*, then we have the GFHFS for the function* $k\cos 2k$ *as*

$$k\cos 2k = \sum_{n=1}^{\eta-1} b_n\sin nk. \tag{3.7}$$

Proof. The proof follows by taking $u(k) = k\cos 2k$ in (2.5).

□

Example 4. *Taking* $\ell = \frac{\pi}{2}$,$\eta = 2$ *in* (3.7) *we get,*

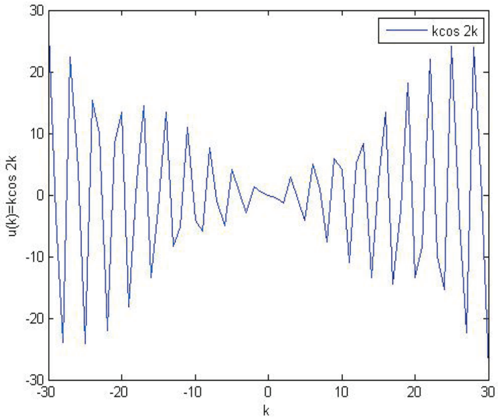
$$k\cos 2k = b_1\sin k, k \in \{r\ell\}_{r=0}^{4-1}, \tag{3.8}$$

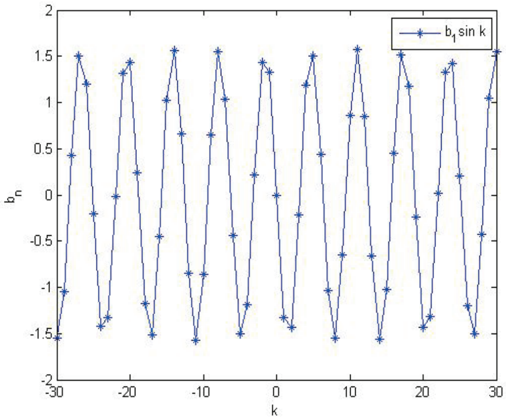
where, $b_1 = \dfrac{2\ell}{\pi}\Delta_\ell^{-1} k\cos 2k \sin k\Big|_0^\pi$*. By using (2.1) we get,*

$$\Delta_\ell^{-1} k\cos 2k\sin k = \tfrac{1}{2}\frac{k\sin 3(k-\ell)}{2(1-\cos\ell)} - \frac{k\sin 3k}{2(1-\cos 3\ell)} - \frac{\ell\sin 3(k-\ell)}{(2(1-\cos 3\ell))^2} + \frac{2\ell\sin 3k}{(2(1-\cos 3\ell))^2} - \frac{\ell\sin 3(k+\ell)}{(2(1-\cos 3\ell))^2} - \frac{k\sin(k-\ell)}{2(1-\cos\ell)} + \frac{k\sin k}{2(1-\cos\ell)} + \frac{\ell\sin(k-\ell)}{(2(1-\cos\ell))^2} - \frac{2\ell\sin k}{(2(1-\cos\ell))^2} + \frac{\ell\sin(k+\ell)}{(2(1-\cos\ell))^2}$$

$$b_1 = \frac{k\sin 3(k-\ell)}{2(1-\cos\ell)} - \frac{k\sin 3k}{2(1-\cos 3\ell)} - \frac{\ell\sin 3(k-\ell)}{(2(1-\cos 3\ell))^2} + \frac{2\ell\sin 3k}{(2(1-\cos 3\ell))^2} - \frac{\ell\sin 3(k+\ell)}{(2(1-\cos 3\ell))^2} - \frac{k\sin(k-\ell)}{2(1-\cos\ell)} + \frac{k\sin k}{2(1-\cos\ell)} + \frac{\ell\sin(k-\ell)}{(2(1-\cos\ell))^2} - \frac{2\ell\sin k}{(2(1-\cos\ell))^2} + \frac{\ell\sin(k+\ell)}{(2(1-\cos\ell))^2}\Big|_0^\pi.$$

Particularly, taking $k = \dfrac{\pi}{2}$ *in* (3.8)*, which gives,*$b_1 = -1.5708$*, we get,* $b_1 \sin k = k\cos 2k = -1.5708$*. Here, we decompose the function* $k\cos 2k$ *which are given below:*





4. Half Range Fouries Series for Even Functions

Theorem 4.1. *Let* $k \in (-\infty,\infty)$*,* $\ell > 0$ *and* $n\ell \neq \beta 2\pi$*, then we have*

$$\Delta_\ell^{-1} k^p \cos nk = \sum_{\beta=1}^{p}\sum_{t=0}^{\beta}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_\beta^p(\beta)_1^{(t)} k_\ell^{(\beta-t)}\cos n(k-\ell+r\ell)}{(-1)^{r-1}\ell^{\beta-t-p}(2(\cos n\ell - 1))^{t+1}}. \tag{4.1}$$

Proof. Taking, $u(k) = k^2$, and $w(k) = \cos k$ in (2.1) we get,

$$\Delta_\ell^{-1}(k^2\cos k) = \frac{k^2\cos(k-\ell)}{2(1-\cos\ell)} - \frac{k^2\cos k}{2(1-\cos\ell)} - \frac{\ell^2\cos(k-\ell)}{(2(1-\cos\ell))^2} + \frac{2\ell^2\cos k}{(2(1-\cos\ell))^2} - \frac{\ell^2\cos(k+\ell)}{(2(1-\cos\ell))^2} - \frac{2k\ell\cos(k-\ell)}{(2(1-\cos\ell))^2} + \frac{2\ell^2\cos(k-\ell)}{(2(1-\cos\ell))^3} - \frac{6\ell^2\cos k}{(2(1-\cos\ell))^3}$$

$$+\frac{6\ell^2\cos(k+\ell)}{(2(1-\cos\ell))^3}+\frac{4\ell k\cos k}{(2(1-\cos\ell))^2}-\frac{2\ell k\cos(k+\ell)}{(2(1-\cos\ell))^2}-\frac{2\ell^2\cos(k+2\ell)}{(2(1-\cos\ell))^3}$$

Let $u(k)=k^2$ and $w(k)=\cos 2k$ in (2.1) we get,

$$\Delta_\ell^{-1}(k^2\cos 2k)=\frac{k^2\cos 2(k-\ell)}{2(1-\cos 2\ell)}-\frac{k^2\cos 2k}{2(1-\cos 2\ell)}-\frac{\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2}+\frac{2\ell^2\cos 2k}{(2(1-\cos 2\ell))^2}$$
$$-\frac{\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2}-\frac{2k\ell\cos 2(k-\ell)}{(2(1-\cos 2\ell))^2}+\frac{2\ell^2\cos 2(k-\ell)}{(2(1-\cos 2\ell))^3}-\frac{6\ell^2\cos 2k}{(2(1-\cos 2\ell))^3}$$
$$+\frac{6\ell^2\cos 2(k+\ell)}{(2(1-\cos 2\ell))^3}+\frac{4\ell k\cos 2k}{(2(1-\cos 2\ell))^2}-\frac{2\ell k\cos 2(k+\ell)}{(2(1-\cos 2\ell))^2}-\frac{2\ell^2\cos 2(k+2\ell)}{(2(1-\cos 2\ell))^3}.$$

By repeating the process, we obtain the proof of (4.1). □

Corollary 4.2. *When $I=[0,\pi]$, $\ell=\frac{\pi}{\eta}$,the GFHF coefficients a_n for $n=0,1,2,...,\eta$ for the polynomial k^p given by*

$$a_n=\frac{2\ell}{\pi}\Delta_\ell^{-1}k^p\cos nk\Big|_0^\pi=\sum_{\beta=1}^{p-1}\sum_{t=0}^{\beta}\sum_{r=0}^{t+1}\binom{t+1}{r}\frac{S_\beta^p(\beta)_1^t(\pi)_\ell^{(\beta-t)}\cos n(r-1)\ell}{(-1)^{r-1}\eta\ell^{\beta-t-p}(2(\cos n-\ell))^{t+1}}.$$

Corollary 4.3. *Let p is even, we get the GFHFS for the polynomial k^p as*

$$k^p=\frac{a_0}{2}+\sum_{n=1}^{\eta-1}a_n\cos nk+\frac{a_\eta}{2}\cos\eta k,\ k\in\{r\ell\}_{r=0}^{2\eta-1}\ . \tag{4.2}$$

Proof. From (2.4) and (4.1), we get the proof of of (4.2). □

Example 5. *Taking $\eta=2$, $\ell=\frac{\pi}{4}$ and $k=\frac{\pi}{2}$ in (4.2) and for $u(k)=k^4$, we obtain*

$$k^4=\frac{a_0}{2}+\sum_{n=1}^{4-1}a_n\cos nk+\frac{a_\eta}{2}\cos\eta k=\frac{a_0}{2}+a_1\cos k+a_2\cos 2k+a_3\cos 3k+\frac{a_4}{2}\cos 4k \tag{4.3}$$

In (4.3) R.H.S=L.H.S=6.0881.

Theorem 4.4. *Let $k\in(-\infty,\infty)$ and $\ell>0$, $k\in\{r\ell\}_{r=0}^{2\eta-1}$ then we have the GFHFS for the function $\cos k$ as*

$$\cos k=\frac{a_0}{2}+\sum_{n=1}^{\eta-1}a_n\cos nk+\frac{a_\eta}{2}\cos\eta k, \tag{4.4}$$

where $a_n=\frac{2\ell}{\pi}\Delta_\ell^{-1}\cos k\cos nk\Big|_0^\pi k\in\{r\ell\}_{r=0}^{2\eta-1}$.

Proof. The proof follows by taking $u(k)=\cos k$ in (2.4). □

Example 6. *Taking, $\ell=\dfrac{\pi}{2}$, $\eta=2$ in (4.4) we get,*

$$\cos k=\frac{a_0}{2}+a_1\cos k+\frac{a_2}{2}\cos 2k, \tag{4.5}$$

where $a_0 = \frac{2\ell}{\pi}\Delta_\ell^{-1}\cos k\Big|_0^\pi$, $a_1 = \frac{2\ell}{\pi}\Delta_\ell^{-1}\cos k\cos k\Big|_0^\pi$ *and* $a_2 = \frac{2\ell}{\pi}\Delta_\ell^{-1}\cos k\cos 2k\Big|_0^\pi$. *By using ((2.2)) we get,* $a_0 = 1$,$a_1 = 1$ *and* $a_2 = 1$.

Particularly taking $k = \frac{\pi}{2}$ *in* (4.5) *which gives,*$a_0 = 1$,$a_1 = 1$ *and* $a_2 = 1$, *we get* $\frac{a_0}{2} + a_1\cos k + \frac{a_2}{2}\cos 2k = \cos k = 0$.

Theorem 4.5. *Let* $k \in (-\infty,\infty)$, $\ell > 0$, *and* $k \in \{r\ell\}_{r=0}^{2\eta-1}$ *then we have the GFHFS for the function* $k\sin k$ *as*

$$k\sin k = \frac{a_0}{2} + \sum_{n=1}^{\eta-1} a_n\cos nk + \frac{a_\eta}{2}\cos\eta k. \tag{4.6}$$

Proof. The proof follows by taking $u(k) = k\sin k$ in (2.4) □

Example 7. *Taking* $\ell = \frac{\pi}{3}$, *and* $\eta = 3$ *in* (4.6) *we get,*

$$k\sin k = \frac{a_0}{2} + a_1\cos k + a_2\cos 2k + \frac{a_3}{2}\cos 3k, \tag{4.7}$$

where $a_0 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k\sin k\Big|_0^\pi$, $a_1 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k\sin k\cos k\Big|_0^\pi$, $a_2 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k\sin k\cos 2k\Big|_0^\pi$ *and* $a_3 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k\sin k\cos 3k\Big|_0^\pi$.

Particularly taking, $k = \frac{\pi}{3}$ *in* (4.7) *which gives,* $a_0 = 1.8138$,$a_1 = -0.3023$, $a_2 = -0.9069$ *and* $a_3 = 0.6045$, *we get* $\frac{a_0}{2} + a_1\cos k + a_2\cos 2k + \frac{a_3}{2}\cos 3k = k\sin k = 0.9069$.

Theorem 4.6. *Let* $k \in (-\infty,\infty)$, $\ell > 0$, *and* $k \in \{r\ell\}_{r=0}^{2\eta-1}$ *then we have the GFHFS for the function* $k^3\sin 2k$ *as*

$$k^3\sin 2k = \frac{a_0}{2} + \sum_{n=1}^{\eta-1} a_n\cos nk + \frac{a_\eta}{2}\cos\eta k.$$

Proof. The proof follows by taking $u(k) = k^3\sin 2k$ in (2.4). □

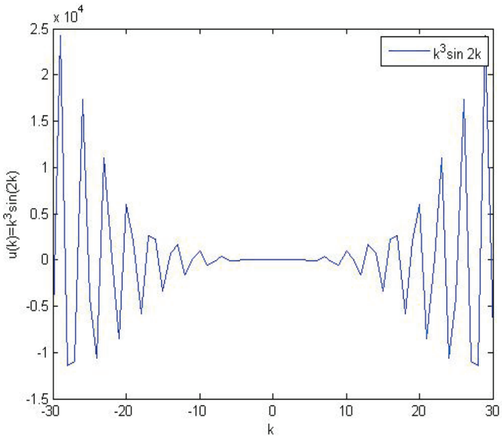
Example 8. *Taking* $\ell = \frac{\pi}{3}$, *and* $\eta = 3$ *in* (4.6) *we get,*

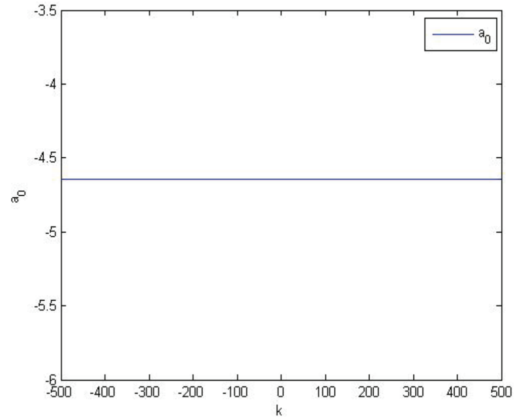
$$k^3\sin 2k = \frac{a_0}{2} + a_1\cos k + a_2\cos 2k + \frac{a_3}{2}\cos 3k, \tag{4.8}$$

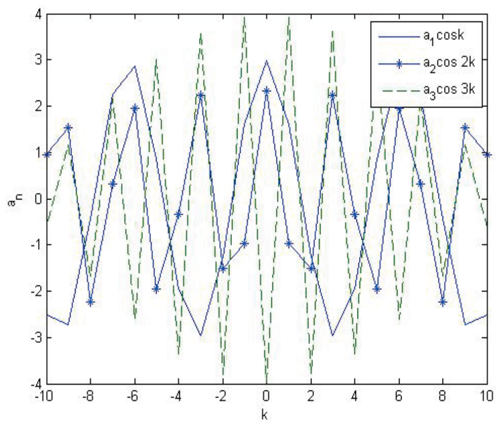
where $a_0 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k^3\sin 2k\Big|_0^\pi$, $a_1 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k^3\sin 2k\cos k\Big|_0^\pi$, $a_2 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k^3\sin 2k\cos 2k\Big|_0^\pi$ *and* $a_3 = \frac{2\ell}{\pi}\Delta_\ell^{-1}k^3\sin 2k\cos 3k\Big|_0^\pi$.

Particularly taking,$k = \frac{\pi}{3}$ *in* (4.8) *which gives,* $a_0 = -4.6411$,$a_1 = 2.9836$, $a_2 = 2.3205$ *and* $a_3 = -5.9672$, *we get* $\frac{a_0}{2} + a_1\cos k + a_2\cos 2k + \frac{a_3}{2}\cos 3k = k^3\sin 2k$. *Here we provide the MATLAB coding for verification of the above relation and also the decomposition diagrams are given below:* $(pi./3).\wedge 3.*(\sin(2.*pi./3)) = (-4.6411./2) + (2.9836).*$

$\cos(pi./3) + (2.3205). * \cos(2. * pi./3) + ((-5.9672)./2). * \cos(3. * pi./3)$







5. Conclusion

The Fourier series and its transforms have wide range of applications specially in the field of digital signal processing. For the odd and even functions no usual finite half range Fourier series expression, we are able to find finite half range Fourier series expression(decomposition) using generalized inverse difference operator. Furthermore, the results were analyzed by MATLAB to validate our findings.

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