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TWO-STAGE MULTIDERIVATIVE EXPLICIT RUNGE-KUTTA METHOD FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In dynamical systems, Delay-Differential equations (DDEs) are more important in dealing with natural or technological control problems. Among which the Runge-Kutta methods are familiar for ODEs it can also extend to DDEs. In this work, we develop a two-stage multiderivative clarifying Runge Kutta method of order 4. The primary goal of this article is providing a numerical solution for delay differential equations. Here the Lagrange interpolation is applied for estimating the delay term. We have used Stability polynomial of the method for obtaining the corresponding stability region. Our proposed methods efficiency is represented using numerical examples. On this observation, obtained explicit of Runge Kutta methods of order four is compared with classical four-stage fourth by using the numerical results of two-stage multiderivative.

1. INTRODUCTION

Delay differential equations (DDEs) arise in control systems, traffic models, population dynamics, chemical kinetics and in many fields. Bellman and Cooke [5] developed DDEs general theory. In recent days, to provide a better numerical method providing a solution of DDEs and stiff DDEs several research works are carried out. Some well-known methods for solving DDEs are Rung-Kutta

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method [8], Variational iteration method [6], Chebyshev method [7], Predictor-corrector method [9], Direct Lyapunov method [2].

This paper provides a prominent numerical solution of delay differential equations using two-stage fourth-order multiderivative explicating Runge Kutta (MERK) method. Next, the Lagrange interpolation is applied for determining the delay term. The entire work is represented as follows: In Section 2, the MERK method for solving delay differential equations has been proposed. In Section 3, the Stability analysis of this method has been discussed. Finally, in Section 4 the MERK method overall efficiency is determined by the linear and non-linear numerical examples.

2. MERK METHOD FOR DELAY DIFFERENTIAL EQUATIONS

Consider the ordinary differential equation (ODE) of the form

$$y'(t) = y(t, y(t))$$

with the initial condition $y(t_0) = y_0$.

The expression of MERK methods is stated by adding of f's higher-order derivatives to the k_r terms for $r > 1$. The ODE solution of s-stage pth order MERK method [1] is expressed by its general form such as:

$$y_{n+1} = y_n + \sum_{r=1}^{s} b_r k_r,$$

where

$$k_r = hf\left(t_n + c_r h, y_n + \sum_{u=1}^{r-1}\sum_{v=0}^{p-2} a_{r(u+v)} \frac{h^{v+1}}{v!}\left(f\frac{\partial}{\partial y}\right)^v f\right)$$

$$c_r = \sum_{u=1}^{r-1} a_{ru}, \quad r = 2, 3, \cdots, s.$$

The MERK method family of 2-stage of order 4's parameters are given in Table 1.

We adapt this MERK method to solve DDEs as follows:

Consider the first-order DDEs of the form

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t > t_0,$$
$$y(t) = \phi(t)$$

TABLE 1. Parameters for MERK methods

Methods	c_2	b_1	b_2	a_{21}	a_{22}	a_{23}
MERK 1	$\frac{1}{3}$	$\frac{-1}{2}$	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{18}$
MERK 2	$\frac{5}{8}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{5}{8}$	$\frac{5}{24}$	$\frac{5}{48}$
MERK 3	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
MERK 4	$\frac{75}{64}$	$\frac{43}{75}$	$\frac{32}{75}$	$\frac{75}{64}$	$\frac{25}{64}$	$\frac{25}{128}$
MERK 5	$\frac{4}{9}$	$\frac{-1}{8}$	$\frac{9}{8}$	$\frac{4}{9}$	$\frac{4}{27}$	$\frac{2}{27}$

where the delay is positive constant and $\phi(t)$ is the initial function. The DDEs representation using MERK method's general form of s-stage and p^{th} order is given by

$$y_{n+1} = y_n + \sum_{r=1}^{s} b_r k_r,$$

where

$$k_r = hf\left(t_n + c_r h, y_n + \sum_{u=1}^{r-1}\sum_{v=0}^{p-2} a_{r(u+v)}\frac{h^{v+1}}{v!}\left(f\frac{\partial}{\partial y}\right)^v f, y(t_n + c_r h - \tau)\right)$$

$$c_r = \sum_{u=1}^{r-1} a_{ru}, \quad r = 2, 3, \cdots, s.$$

Here the delay term is $y(t_n + c_r h - \tau)$ and for obtaining the approximate value interpolation is required. There are several techniques for obtaining approximations. To get the approximate delay term the Lagrange interpolation is applied. This newly proposed scheme involves first and second derivatives f.

3. STABILITY ANALYSIS OF MERK METHOD

According to the delay terms and different test equations, there are several concepts regarding the numerical methods of DDEs. The stability has been considered with the following linear DDEs,

$$y'(t) = \lambda y(t) + \mu y(t - \tau), \quad t \geq t_0 \tag{3.1}$$
$$y(t) = g(t), \qquad -\tau \leq t \leq 0,$$

where the complex numbers λ are and the positive constant delay is μ with a positive constant delay τ and a specified initial function of g.

It is discussed in Al-Mutib [3], that if

(i) $g(t)$ is continuous

(3.2) (ii) $|\mu| < -\mathrm{Re}(\lambda)$

By the above equation (3.1) a unique solution is obtained for the initial value problem. The stability investigations are based on the test equation (3.1) and the concept of P-stability was discussed by Barewell [4].

For satisfying the numerical solution, for P-stable a numerical method for solving (3.1) for all λ and μ satisfying (3.2)

$$y(t_n) \to 0 \ \text{ as } \ t \to \infty,$$

with step size $h = \dfrac{\tau}{m}$, m is a positive integer.

By approximating the delay term using Lagrange interpolation, we get

$$y(t_n + c_i h - \tau) = y(t_{n-m} + c_i h) = \sum_{l=r1}^{s1} L_l(c_i) y_{n-m+l},$$

where y_{n-m+l} is the calculated value of $y(t_n - m + l)$ and

$$L_l(c_i) = \prod_{j_1=r1}^{s1} \frac{c_i - j_1}{1 - j_1}, \quad j_1 \neq 1 \text{ and } \ r_1, s_1 > 0.$$

At the s-stage for DDE, MERK1 method (3.1) is applied with delay $\tau = 1$. The below expression is the obtained results:

$$k_r = hf\left(t_n + c_r h, y_n + \sum_{u=1}^{r-1}\sum_{v=0}^{p-2} a_{r(u+v)} \frac{h^{v+1}}{v!}\left(f\frac{\partial}{\partial y}\right)^v f, \sum_{l=r_1}^{s_1} L_l(c_i) y_{n-m+l}\right) \tag{3.3}$$

$$y_{n+1} = y_n + \sum_{i=1}^{r} b_i k_i.$$

Define $u = (1, \cdots, 1)^T, k = (k^{(1)}, \cdots, k^{(q)})^T, b = (b_1, \cdots, b_q)^T$ and $L_l(c_i) = (L_l(c_1), \cdots, L_l(c_q))^T$. For $n \geq N$, considering f as in (3.1), (3.3) can be written as

$$k = \lambda(y_n u + hA_k) + \mu\left(\sum L_l(c_i) y_{n-m+l}\right) \tag{3.4}$$

$$y_{n+1} = y_n + hb^T k. \tag{3.5}$$

From equqtion (3.4),

$$k = \lambda y_n u[I - \lambda hA]^{-1} + \mu[I - \lambda hA]^{-1} \sum L_l(c_i) y_{n-m+l}$$

$$hk = \alpha y_n u \eta + \beta\eta \sum L_l(c) y_{n-m+l}, \tag{3.6}$$

where $\alpha = \lambda h, \beta = \mu h, \eta = [I - \lambda hA]^{-1}$ and I is the identity matrix. Substituting (3.6) in (3.5),

$$y_{n+1} = y_n + \alpha b^T \eta y_n u + \beta b^T \eta \sum L_l(c_i) y_{n-m+l}.$$

To maintain the stability polynomial in a standard form, the $n - m + l = 0$ is applied. If the stability polynomial is zero ζ_i then the recurrence is stable,

$$S(\alpha, \beta, \zeta) = \det[\zeta^{n+1} I - \zeta^n X - \sum_{l=r_1}^{s_1} \zeta^{1+l} Z_l]$$

satisfies the root condition $|\zeta_i| \leq 1$.

Next, the four point's interpolation is applied to evaluate $y(t_n + c_i h - 1)$ for obtaining the stability region of the method. The stability polynomial for the method is expressed as follows:

$$S(\alpha, \beta; \zeta) = \zeta^{n+1} - (1 + \alpha b^T \eta u)\zeta^n - \beta b^T \eta (L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2 + L_2(c)\zeta^3).$$

The stability polynomial of MERK1 method is given by

$$S_p(\alpha, \beta, \xi) = \xi^5 - \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{\alpha^4}{24}\right)\xi^4 + \left(\frac{2}{25}\beta\right)\xi^3 - \left(\frac{5}{9}\beta\right)\xi^2 - \left(\frac{11}{18}\beta + \frac{1}{2}\alpha\beta\right)\xi + \left(\frac{5}{54}\beta\right).$$

The stability region of the two-stage MERK1 fourth-order method using Lagrange interpolation is shown in Figure 1.

Similarly, the stability region can be obtained for the remaining members of the family of MERK methods.

4. Numerical Examples

Example 1. *Consider the linear DDE of the form*

$$y'(t) = 5y(t) + y(t-1), \quad y(t) = 5,$$

for $t \leq 0$ *with exact solution* $y(t) = 6e^{5t} - 1$.

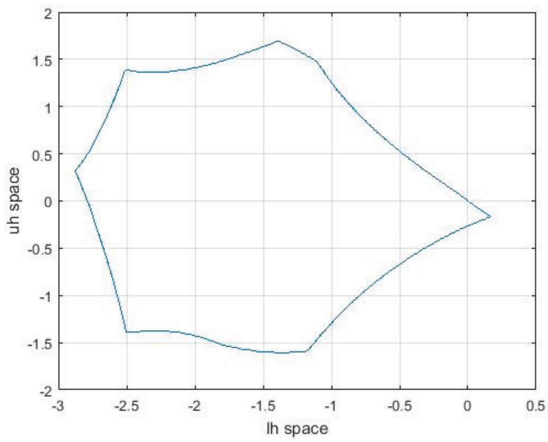

FIGURE 1. Example for Stability Region of MERK1

Example 2. *Consider the nonlinear DDE of the form*

$$y'(t) = y^2(t) + y(t-1) - t^4 + t + 1, \quad y(t) = t,$$

for $t \leq 0$ *with exact solution* $y(t) = t^2$.

The numerical results by two-stage MERK methods of order four have been compared with the results by the classical four-stage fourth-order clarifying Rung Kutta method (ERK4). The absolute error results of Examples (1) and (2) at $t = 1$ is given in Table 2.

TABLE 2. Absolute error results

Methods	Absolute error in example (1)	Absolute error in example (2)
ERK	2.523e-006	1.33e-006
MERK1	5.007e-006	4.600e-006
MERK2	2.829e-006	1.232e-006
MERK3	1.894e-006	2.166e-007
MERK3	1.265e-006	5.128e-006
MERK3	4.178e-006	3.319e-006

5. CONCLUSION

In this paper the two-stage MERK method of order 4 has been proposed to obtain the numerical solution of DDEs. For determining the approximate delay terms Lagrange interpolation is used. Stability polynomial of the method and the corresponding stability region is obtained. Our proposed method's efficiency is represented by examples of linear and nonlinear DDEs.

The numerical results by two-stage MERK method of order four have been compared with the results by the classical four-stage fourth-order clarifying Runge Kutta method deeply. From these results, it is evident that the accuracy of results by the four-stage method can be obtained in two stages by applying MERK method. Hence, it is concluded that the proposed family of MERK methods is suitable for solving DDEs.

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