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ON THE COMPARATIVE GROWTH OF GENERALIZE ITERATED ENTIRE FUNCTIONS

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ABSTRACT. In this article, we study the properties of generalize iterated entire functions and prove some results on the comparative growth properties of the maximum term of generalize iterated entire functions which improve and generalize some earlier results.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

$M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_n |a_n| r^n$ are respectively called the maximum modulus and maximum term of the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$, $\rho_f = \limsup_{r \to \infty} \dfrac{\log \log M(r, f)}{\log r}$ and $\lambda_f = \liminf_{r \to \infty} \dfrac{\log \log M(r, f)}{\log r}$ are respectively called the order and lower order of the entire function f.

Definition 1.1. *[8] We define $\log^{[0]}\theta = \theta$, $\exp^{[0]}\theta = \theta$ and for positive integer t, $\log^{[t]}\theta = \log(\log^{[t-1]}\theta)$, $\exp^{[t]}\theta = \exp(\exp^{[t-1]}\theta)$.*

In 1989 A. P. Singh [9] proved the following very important relation between the maximum modulus and maximum term of an entire function.

Theorem 1.1. *[9] Let f be an entire function defined in the open complex plane C. Then for $0 \leq \theta < \Theta$,*

$$\mu(\theta, f) \leq M(\theta, f) \leq \frac{\Theta}{\Theta - \theta} \mu(\Theta, f).$$

If $\Theta = 2\theta$ then,

(1.1) $$\mu(\theta, f) \leq M(\theta, f) \leq 2\mu(2\theta, f),$$

for all sufficiently large values of θ.

Using (1.1) we can easily prove that

$$\rho_f = \lim_{\theta \to \infty} \sup \frac{\log^{[2]} \mu(\theta, f)}{\log \theta},$$

and

$$\lambda_f = \lim_{\theta \to \infty} \inf \frac{\log^{[2]} \mu(\theta, f)}{\log \theta}.$$

Definition 1.2. *[1] Let $f(z)$ and $g(z)$ be two entire functions defined in the open complex plane C and $\alpha \in (0, 1]$. Then the generalize iterations of $f(z)$ with respect to $g(z)$ are defined as follows:*

$$
\begin{aligned}
f_{1,g}(z) &= (1 - \alpha)z + \alpha f(z) \\
f_{2,g}(z) &= (1 - \alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\
f_{3,g}(z) &= (1 - \alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\
&\quad \dots \qquad\qquad \dots \qquad\qquad \dots \\
&\quad \dots \qquad\qquad \dots \qquad\qquad \dots \\
f_{n,g}(z) &= (1 - \alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z))
\end{aligned}
$$

and so are

$$
\begin{aligned}
g_{1,f}(z) &= (1 - \alpha)z + \alpha g(z) \\
g_{2,f}(z) &= (1 - \alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)) \\
g_{3,f}(z) &= (1 - \alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z)) \\
&\quad \dots \qquad\qquad \dots \qquad\qquad \dots \\
&\quad \dots \qquad\qquad \dots \qquad\qquad \dots \\
g_{n,f}(z) &= (1 - \alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)).
\end{aligned}
$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

Many authors like Banerjee and Dutta [2], Dutta [3], [4], Dutta and Mandal [5], Mandal [7] proved some results on comparative growth properties of the maximum term of iterated entire function with that of the related function.

In this paper we not only study the growth properties of the maximum term of generalize iterated entire functions as compared to the growth of the maximum term of the related function but also study the comparative growth properties of the maximum terms of the generalize iterated entire functions to improve and generalize some earlier results.

Throughout the paper we denote by $f(z)$, $g(z)$ etc. non-constant entire functions of order (lower order) $\rho_f(\lambda_f)$, $\rho_g(\lambda_g)$ etc. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [6], [10] and [11].

2. PRELIMINARIES

The following lemmas will be needed in the sequel.

Lemma 2.1. *[4] If λ_g be finite, then*

$$\liminf_{r \to \infty} \frac{\log M(r,g)}{\log \mu(r,g)} \leq 2^{\lambda_g}.$$

Lemma 2.2. *[7] Let $f(z)$ and $g(z)$ be two non-constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for any $\varepsilon > 0$,*

$$\log^{[n]} \mu(r, f_{n,g}) \leq \begin{cases} (\rho_f + \varepsilon)(1 + 0(1)) \log M(r,g) + O(1) & \text{when n is even} \\ (\rho_g + \varepsilon)(1 + 0(1)) \log M(r,f) + O(1) & \text{when n is odd,} \end{cases}$$

for all sufficiently large values of r.

Lemma 2.3. *[5] Let $f(z)$ and $g(z)$ be two non-constant entire functions such that $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g \leq \rho_g < \infty$. Then for any ε $(0 < \varepsilon < \min\{\lambda_f, \lambda_g\})$,*

$$\log^{[n]} \mu(r, f_{n,g}) \geq \begin{cases} (\lambda_f - \varepsilon)(1 + 0(1)) \log M(\frac{r}{2^n}, g) + O(1) & \text{when n is even} \\ (\lambda_g - \varepsilon)(1 + 0(1)) \log M(\frac{r}{2^n}, f) + O(1) & \text{when n is odd,} \end{cases}$$

for all large values of r.

3. Main results

Theorem 3.1. *If ρ_f and ρ_g are finite, then*

$$\text{(i)} \quad \lim_{r\to\infty}\inf \frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,g)} \leq \rho_f 2^{\lambda_g}\,, \text{ when } n \text{ is even},$$

$$\text{(ii)} \quad \lim_{r\to\infty}\inf \frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,f)} \leq \rho_g 2^{\lambda_f}\,, \text{ when } n \text{ is odd}.$$

Proof. We have from Lemma 2.2 for all sufficiently large values of r,

$$\log^{[n]}\mu(r,f_{n,g}) \;\leq\; (1+0(1))(\rho_f+\varepsilon)\log M(r,g) + O(1).$$

Therefore $\quad \lim_{r\to\infty}\inf \dfrac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,g)} \;\leq\; (1+0(1))(\rho_f+\varepsilon)\lim_{r\to\infty}\inf \dfrac{\log M(r,g)}{\log\mu(r,g)}.$

Since $\varepsilon > 0$ is arbitrary, we get from Lemma 2.1,

$$\lim_{r\to\infty}\inf \frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,g)} \leq \rho_f 2^{\lambda_g}.$$

Similarly when n is odd then we get the second part of this theorem. This proves the theorem. $\qquad\square$

Theorem 3.2. *Let $f(z)$ and $g(z)$ be entire functions of finite order with $\rho_g < \lambda_f$ and n is even then*

$$\lim_{r\to\infty}\sup \frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,f)} = 0\,.$$

Proof. When n is even then we have from Lemma 2.2 for all sufficiently large values of r,

$$\log^{[n]}\mu(r,f_{n,g}) \;\leq\; (1+0(1))(\rho_f+\varepsilon)\log M(r,g) + O(1)$$

$$\text{(3.1)} \qquad\qquad\qquad \leq\; (1+0(1))(\rho_f+\varepsilon)r^{\rho_g+\varepsilon}+O(1).$$

Also from definition of lower order we have for $r \geq r_0$,

$$\text{(3.2)} \qquad\qquad\qquad \log\mu(r,f) \geq r^{\lambda_f-\varepsilon}.$$

So from (3.1) and (3.2) we get for $r \geq r_0$,

$$\frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,f)} \leq \frac{(1+0(1))(\rho_f+\varepsilon)r^{\rho_g+\varepsilon}+O(1)}{r^{\lambda_f-\varepsilon}}\,.$$

Since $\lambda_f > \rho_g$, we can choose $\varepsilon > 0$ such that $\lambda_f - \varepsilon > \rho_g + \varepsilon$ then

$$\lim_{r\to\infty}\sup \frac{\log^{[n]}\mu(r,f_{n,g})}{\log\mu(r,f)} = 0.$$

This proves the theorem. □

Remark 3.1. *If we take $\rho_g < \rho_f$ in Theorem 3.2, the result is still valid.*

Theorem 3.3. *Let $f(z)$ and $g(z)$ be entire functions of finite order with $\rho_f < \lambda_g$ and n is odd then*

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log \mu(r, g)} = 0.$$

Proof. Similar as Theorem 3.2. □

Theorem 3.4. *Let $f(z)$ and $g(z)$ be entire functions of finite order with $\lambda_g > \rho_f$ and n is even then*

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log \mu(r, f)} = \infty.$$

Proof. When n is even then from Lemma 2.3 we have for all sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_g\}$,

$$
\begin{aligned}
\log^{[n]} \mu(r, f_{n,g}) \quad & > \quad (1 + 0(1))(\lambda_f - \varepsilon) \log M(r, g) + O(1) \\
(3.3) \qquad & \geq \quad (1 + 0(1))(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1).
\end{aligned}
$$

Also for all sufficiently large values of r,

$$(3.4) \qquad\qquad \log \mu(r, f) \leq r^{\rho_f + \varepsilon}.$$

Therefore from (3.3) and (3.4) we get for $r \geq r_0$,

$$\frac{\log^{[n]} \mu(r, f_{n,g})}{\log \mu(r, f)} \geq \frac{(1 + 0(1))(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1)}{r^{\rho_f + \varepsilon}}.$$

Since $\lambda_g > \rho_f$, we can choose $\varepsilon > 0$ such that $\lambda_g - \varepsilon > \rho_f + \varepsilon$. Hence

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log \mu(r, f)} = \infty.$$

This proves the theorem. □

Theorem 3.5. *Let $f(z)$ and $g(z)$ be entire functions of finite order with $\lambda_f > \rho_g$ and n is odd then*

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log \mu(r, g)} = \infty.$$

Proof. Similar as Theorem 3.4. □

Theorem 3.6. *Let $f(z)$ and $g(z)$ be transcendental entire functions of non zero finite order then*

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} = \infty = \limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, g)}.$$

Proof. First we consider n is even then from (3.3) we have for sufficiently large values of r,

$$(3.5) \qquad \log^{[n]} \mu(r, f_{n,g}) > (1 + 0(1))(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1),$$

where $0 < \varepsilon < \min\{\lambda_f, \lambda_g\}$. From (3.4) we have

$$(3.6) \qquad \log^{[2]} \mu(r, f) \leq (\rho_f + \varepsilon) \log r.$$

So from (3.5) and (3.6) we get,

$$\frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} \geq \frac{(1 + 0(1))(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1)}{(\rho_f + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary therefore

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} = \infty.$$

Also when n is odd then from Lemma 2.3 we have for sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_g\}$,

$$\begin{aligned}
\log^{[n]} \mu(r, f_{n,g}) &> (1 + 0(1))(\lambda_g - \varepsilon) \log \mu(r, f) + O(1) \\
(3.7) \qquad &\geq (1 + 0(1))(\lambda_g - \varepsilon) r^{\lambda_f - \varepsilon} + O(1).
\end{aligned}$$

So from (3.6) and (3.7) we get,

$$\frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} \geq \frac{(1 + 0(1))(\lambda_g - \varepsilon) r^{\lambda_f - \varepsilon} + O(1)}{(\rho_f + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary therefore

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} = \infty.$$

Similarly we have

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, g)} = \infty.$$

This proves the theorem.   □

If we take one more logarithm of the numerator then the expression in Theorem 3.6 is finite. This motivates us to prove the following theorem.

Theorem 3.7. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order then:*

(i) $\displaystyle \limsup_{r \to \infty} \frac{\log^{[n+1]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, g)} \leq \frac{\rho_g}{\lambda_g}$ *, when n is even and $\lambda_g > 0$,*

(ii) $\displaystyle \limsup_{r \to \infty} \frac{\log^{[n+1]} \mu(r, f_{n,g})}{\log^{[2]} \mu(r, f)} \leq \frac{\rho_f}{\lambda_f}$ *, when n is odd and $\lambda_f > 0$.*

Proof. First we consider n is even then from Lemma 2.2 we have for sufficiently large values of r,

$$\log^{[n+1]} \mu(r, f_{n,g}) \leq \log^{[2]} M(r, g) + O(1)$$

(3.8)
$$\leq (\rho_g + \varepsilon) \log r + O(1).$$

Also we have for $r \geq r_0$ and $0 < \varepsilon < \lambda_g$,

(3.9)
$$\log^{[2]} \mu(r, f) \geq (\lambda_g - \varepsilon) \log r.$$

Therefore from (3.8) and (3.9) we get for $r \geq r_0$ and $0 < \varepsilon < \lambda_g$,

$$\frac{\log^{[n+1]} \mu(r, f_{n,g})}{\log \mu(r, f)} \leq \frac{(\rho_g + \varepsilon) \log r + O(1)}{(\lambda_g - \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary therefore

$$\limsup_{r \to \infty} \frac{\log^{[n+1]} \mu(r, f_{n,g})}{\log \mu(r, f)} \leq \frac{\rho_g}{\lambda_g}.$$

Similarly for odd n we get second part of the theorem.
This proves the theorem. $\qquad \square$

Theorem 3.8. *If f, g and h are three non constant entire functions of finite order and $\rho_f < \lambda_h$, then*

$$\lim_{r \to \infty} \frac{\log^{[n-2]} \mu(r, f_{n,g})}{\log^{[n-2]} \mu(r, h_{n,g})} = 0,$$

for n is even and $h_{n,g}(z) = (1 - \alpha)g_{n-1,h}(z) + \alpha h(g_{n-1,h}(z))$.

Proof. When n is even then from Lemma 2.2 we have for all large values of r,

$$\log^{[n]} \mu(r, f_{n,g}) \leq (1 + 0(1))(\rho_f + 2\varepsilon) \log M(r, g)$$

$$\leq \log[M(r, g)]^{\rho_f + 3\varepsilon}.$$

Therefore

(3.10) $$\log^{[n-2]} \mu(r, f_n) \leq e^{[M(r,g)]^{\rho_f + 3\varepsilon}}.$$

Also from Lemma 2.3 we have for sufficiently large values of r and $0 < \varepsilon < \frac{1}{6}(\lambda_h - \rho_f)$,

(3.11) $$\log^{[n-2]} \mu(r, h_{n,g}) > e^{[M(r,g)]^{\lambda_h - 3\varepsilon}}.$$

So from (3.10) and (3.11) we obtain,

$$\frac{\log^{[n-2]} \mu(r, f_{n,g})}{\log^{[n-2]} \mu(r, h_{n,g})} \leq \frac{e^{[M(r,g)]^{\rho_f + 3\varepsilon}}}{e^{[M(r,g)]^{\lambda_h - 3\varepsilon}}}$$

$$\leq \frac{1}{e^{[M(r,g)]^{\lambda_h - 3\varepsilon} - [M(r,g)]^{\rho_f + 3\varepsilon}}}.$$

Since $0 < \varepsilon < \frac{1}{6}(\lambda_h - \rho_f)$ is arbitrary and g is non constant therefore

$$\lim_{r \to \infty} \frac{\log^{[n-2]} \mu(r, f_{n,g})}{\log^{[n-2]} \mu(r, h_{n,g})} = 0.$$

This proves the theorem. □

Theorem 3.9. *If f, g and h are three entire functions with non zero lower order and finite order also $\rho_h < \lambda_g$ then*

$$\lim_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[n]} \mu(r, f_{n,h})} = \infty$$

for n is even and $f_{n,h}(z) = (1 - \alpha)h_{n-1,f}(z) + \alpha f(h_{n-1,f}(z))$.

Proof. When n is even then from Lemma 2.2 we have for $r > r_0$,

(3.12) $$\log^{[n]} \mu(r, f_{n,h}) \leq (1 + 0(1))(\rho_f + \varepsilon)r^{\rho_h + \varepsilon}.$$

Hence from (3.3) and (3.12) we have for sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_h\}$,

$$\frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[n]} \mu(r, f_{n,h})} \geq \frac{(1 + 0(1))(\lambda_f - \varepsilon)r^{\lambda_g - \varepsilon} + O(1)}{((1 + 0(1))\rho_f + \varepsilon)r^{\rho_h + \varepsilon} + O(1)}.$$

Since $\rho_h < \lambda_g$, we can choose $\varepsilon > 0$ such that $\rho_h + \varepsilon < \lambda_g - \varepsilon$. Hence

$$\lim_{r \to \infty} \frac{\log^{[n]} \mu(r, f_{n,g})}{\log^{[n]} \mu(r, f_{n,h})} = \infty.$$

This proves the theorem. □

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