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APPLICATIONS OF CUBIC LEVEL SET ON β-SUBALGEBRAS

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ABSTRACT. In this paper, the notion of cubic level set on fuzzy $\beta-$subalgebras has introduced and investigated few of its related outcomes.

1. INTRODUCTION

Neggers et al. [3] initiated the notion of $\beta-$algebra where two operations are coupled in such a way to reflect the natural coupling, which exists between the usual group operation and its associated B-algebra. The concept of fuzzy sets has been originated by Zadeh [7], which created a pathway for many researchers. Using a fuzzy set and interval valued fuzzy set, Jun et al. [4] introduced the concept of cubic sets in which the fascinating results have studied. In [6], Vijayabalaji et al. proposed the concept of cubic set theoretical approach to linear space. The thought of interval valued intuitionistic fuzzy $\beta-$subalgebras presented by Hemavathi et al. [1,2] and the level sets has extended in interval valued fuzzy $\beta-$subalgebra. Recently, Muralikrishna et al. [5] investigated the properties of cubic fuzzy β-subalgebras. This paper deals with the cubic level sets on $\beta-$subalgebra and its associated properties.

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2. PRELIMINARIES

This section provides the basic definitions required for this work.

Definition 2.1. *[3] A $\beta-$ algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:*

(i) $x - 0 = x$;
(ii) $(0 - x) + x = 0$;
(iii) $(x - y) - z = x - (z + y)$ *for all* $x, y, z \in X$.

Example 1. *[5] The following Cayley table shows $(X = \{0, 1, 2, 3\}, +, -, 0)$ is a $\beta-$algebra.*

$+$	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

$-$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Definition 2.2. *[3] A non empty subset A of a $\beta-$algebra $(X, +, -, 0)$ is called a β-subalgebra of X, if (i) $x + y \in A$; (ii) $x - y \in A$ $for\ all$ $x, y \in A$.*

Example 2. *In example 1, of $\beta-$algebra X,the subset $I = \{0, 2\}$ is a $\beta-$subalgebra of X.*

Definition 2.3. *[2] Let C be a fuzzy set of X and $\alpha \in [0, 1]$. Then $C_\alpha = \{x \in X : \zeta(x) \geq \alpha\}$ is known as a level set of C.*

Definition 2.4. *[4] Let X be a non empty set. By a cubic set in X we mean a structure $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ in which $\overline{\zeta}_C$ is an interval valued fuzzy set in X and η_C is a fuzzy set in X.*

Definition 2.5. *[5] Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ be a cubic set in X. Then the set C is a cubic fuzzy $\beta-$ subalgebra if it satisfies the following conditions*

(i) $\overline{\zeta}_C(x + y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ *and* $\overline{\zeta}_C(x - y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$
(ii) $\eta_C(x + y) \leq max\{\eta_C(x), \eta_C(y)\}$ *and* $\eta_C(x - y) \leq max\{\eta_C(x), \eta_C(y)\}$ $for\ all$ $x, y \in X$.

Example 3. *[5] For the $\beta-$algebra X given in the example 1, we define a cubic set $C = \{\langle x, \overline{\zeta}_C(x), \overline{\eta}_C(x)\rangle : x \in X\}$ on X as follows*

$$\overline{\zeta}_C = \begin{cases} [0.3, 0.6]: & x = 0 \\ [0.2, 0.5]: & x = 2 \\ [0.1, 0.4]: & x = 1, 3 \end{cases} \quad \textit{and} \quad \eta_C = \begin{cases} 0.7: & x = 0, 1 \\ 0.6: & x = 3 \\ 0.4: & x = 2 \end{cases}$$

Then C is a cubic fuzzy $\beta-$sub algebra of X.

3. Cubic Level set on $\beta-$subalgebra

This section presents the notions and related results of cubic level set on $\beta-$subalgebra.

Definition 3.1. *Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ be a cubic set of X. Define $C_{\overline{\alpha},\lambda} = \{x \in X : \overline{\zeta}_C \geq \overline{\alpha}, \eta_C \leq \lambda\}$, where $\overline{\alpha} \in D[0,1]$ and $\lambda \in [0,1]$, called a cubic level set of C.*

Theorem 3.1. *If $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ is a cubic fuzzy $\beta-$subalgebra in X, then $C_{\overline{\alpha},\lambda}$ is a $\beta-$subalgebra of X, for every $\overline{\alpha} \in D[0,1]$ and $\lambda \in [0,1]$.*

Proof. For $x, y \in C_{\overline{\alpha},\lambda}$ and $\overline{\zeta}_C(x) \geq \overline{\alpha}$ and $\overline{\zeta}_C(y) \geq \overline{\alpha}$, we can write

$$\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\} \geq rmin\{\overline{\alpha}, \overline{\alpha}\} \geq \overline{\alpha}.$$

This yields that $x + y \in C_{\overline{\alpha},\lambda}$. Similarly, we obtain $x - y \in C_{\overline{\alpha},\lambda}$.
For $x, y \in C_{\overline{\alpha},\lambda}$ and $\eta_C(x) \leq \lambda$ and $\eta_C(y) \leq \lambda$ we have

$$\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\} \leq \lambda.$$

This shows that $x + y \in C_{\overline{\alpha},\lambda}$. Similarly, we conclude that $x - y \in C_{\overline{\alpha},\lambda}$, and hence $C_{\overline{\alpha},\lambda}$ is $\beta-$subalgebra of X. □

Theorem 3.2. *Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic set in X such that $C_{\overline{\alpha},\lambda}$ is a $\beta-$subalgebra of X for every $\overline{\alpha} \in D[0,1]$ and $\lambda \in [0,1]$. Then C is a cubic fuzzy $\beta-$subalgebra of X.*

Proof. Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic set in X. Since $C_{\overline{\alpha},\lambda}$ is a $\beta-$subalgebra of X for $\overline{\alpha} \in D[0,1]$ and $\lambda \in [0,1]$, it follows that $x + y$ and $x - y \in C_{\overline{\alpha},\lambda}$. Now, take $\overline{\alpha} = rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ and $\lambda = max\{\eta_C(x), \eta_C(y)\}$ then we obtain $x + y \in C_{\overline{\alpha},\lambda}$ this implies that $\overline{\zeta}_C(x+y) \geq \overline{\alpha}$ and $\eta_C(x+y) \leq \lambda$. Also, $x - y \in C_{\overline{\alpha},\lambda}$ which yields that $\overline{\zeta}_C(x-y) \geq \overline{\alpha}$ and $\eta_c(x-y) \leq \lambda$. Therefore, we conclude that $\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$. Similarly, we have $\overline{\zeta}_C(x-y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$. Also, $\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\}$. Similarly, we have $\eta_C(x-y) \leq max\{\eta_C(x), \eta_C(y)\}$ hence C is a cubic fuzzy $\beta-$subalgebra of X. □

Theorem 3.3. *Any $\beta-$subalgebra of X can be realized as a level $\beta-$subalgebra of some cubic fuzzy $\beta-$subalgebra of X.*

Proof. Let C be a cubic fuzzy $\beta-$subalgebra of X. Let us define,

$$\overline{\zeta}_C(x) = \begin{cases} \overline{\alpha} & x \in X \\ [0,0], & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_C(x) = \begin{cases} \lambda & x \in X \\ 1, & \text{otherwise} \end{cases}$$

Then we discuss the following cases.

Case (i)

Both $x, y \in C$. Then we have

$$\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\} \geq rmin\{\overline{\alpha}, \overline{\alpha}\} = \overline{\alpha}.$$

Similarly, we have $\overline{\zeta}_C(x-y) \geq \overline{\alpha}$. Also,

$$\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\} \leq max\{\lambda, \lambda\} = \lambda.$$

In the same way, we have $\eta_C(x-y) \leq \lambda$.

Case (ii)

Both $x, y \notin A$. Now we consider,

$$\begin{aligned} \overline{\zeta}_C(x+y) &\geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\} \\ &\geq rmin\{[0,0], [0,0]\} \\ &= [0,0]. \end{aligned}$$

Similarly, we can write $\overline{\zeta}_C(x-y) \geq [0,0]$. Also, we get

$$\begin{aligned} \eta_C(x+y) &\leq max\{\lambda_C(x), \lambda_C(y)\} \\ &\leq max\{1,1\} \\ &= 1. \end{aligned}$$

Analogously, we have $\eta_C(x-y) \leq 1$.

Case (iii)

Let us take $x \in C \ and \ y \notin C$. Then we have

$$\begin{aligned}\overline{\zeta}_C(x+y) &\geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}\\ &\geq rmin\{\overline{\alpha}, [0,0]\}\\ &= [0,0].\end{aligned}$$

Moreover, we have $\overline{\zeta}_C(x-y) \geq [0,0]$.

$$\begin{aligned}\eta_C(x+y) &\leq max\{\eta_C(x), \eta_C(y)\}\\ &\leq max\{\lambda, 1\}\\ &= 1.\end{aligned}$$

Similarly, we have $\eta_C(x-y) \leq 1$.

Case (iv)

Let us consider $x \notin C \ and \ y \in C$. Then we obtain

$$\begin{aligned}\overline{\zeta}_C(x+y) &\geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}\\ &\geq rmin\{[0,0], \overline{\alpha}\}\\ &= [0,0].\end{aligned}$$

In the same manner, we have $\overline{\zeta}_C(x-y) \geq [0,0]$

$$\begin{aligned}\eta_C(x+y) &\leq max\{\eta_C(x), \eta_C(y)\}\\ &\leq max\{1, \lambda\}\\ &= 1.\end{aligned}$$

Likewise, we have $\eta_C(x-y) \leq 1$. Therefore, C is a cubic fuzzy $\beta-$subalgebra of X. □

Lemma 3.1. *If A and B be two level set of cubic fuzzy $\beta-$subalgebra of X, $\overline{\zeta}_A(x) \leq \overline{\zeta}_B(x)$ and $\eta_A(x) \leq \eta_B(x)$ then $A \subseteq B$.*

Proof. By definition 2.1, we have $A_{\overline{\alpha}_A, \lambda_A} = \{\langle x, \overline{\zeta}_A(x) \geq \overline{\alpha}_A, \eta_A(x) \leq \lambda_A \rangle\}$ and $B_{\overline{\alpha}_B, \lambda_B} = \{\langle x, \overline{\zeta}_B(x) \geq \overline{\alpha}_B, \eta_B(x) \leq \lambda_B \rangle\}$ where $\overline{\alpha}_A \leq \overline{\alpha}_B$ and $\lambda_A \geq \lambda_B$. If $x \in \overline{\zeta}_B(\overline{\alpha}_B)$ then $\overline{\zeta}_B(x) \geq \overline{\alpha}_B \geq \overline{\alpha}_A$ which implies that $x \in \overline{\zeta}_A(\overline{\alpha}_A)$. Therefore, $\overline{\zeta}_B(x) \geq \overline{\zeta}_A(x)$ and if $x \in \eta_B(\lambda_B)$ then $\eta_B(x) \leq \lambda_B \leq \lambda_A$ which implies that $x \in \eta_A(\lambda_A)$. Therefore, $\eta_B(x) \leq \eta_A(x)$ hence $A \subseteq B$. □

Theorem 3.4. *Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic fuzzy $\beta-$subalgebra of X. If $Im(C)$ is finite $\overline{\alpha}_0 < \overline{\alpha}_1 < ... < \overline{\alpha}_n$ and $\lambda_0 > \lambda_1 > \lambda_2 > ... > \lambda_n$ then any $\overline{\alpha}_i, \overline{\alpha}_j \in Im(\overline{\zeta}_C)$,$\overline{\zeta}_{\alpha_i} = \overline{\zeta}_{\alpha_j}$ implies $\alpha_i = \alpha_j$ and $\lambda_i, \lambda_j \in Im(\eta_C)$,$\eta_{\lambda_i} = \eta_{\lambda_j}$ implies $\lambda_i = \lambda_j$.*

Proof. Assume that $\overline{\alpha}_i \neq \overline{\alpha}_j$ and $\lambda_i = \lambda_j$. If $x \in \overline{\zeta}_{\alpha_j}$ then $\overline{\zeta}_C(x) \geq \overline{\alpha}_j > \overline{\alpha}_i$ this implies that $x \in \overline{\zeta}_{\alpha_i}$ there exists $x \in X$ such that $\overline{\alpha}_i \leq \overline{\zeta}(x) < \overline{\alpha}_j$ then $x \in \overline{\zeta}_{\overline{\alpha}_i}$ but $x \in \overline{\zeta}_{\overline{\alpha}_j}$. Moreover, if $x \in \eta_{\lambda_j}$ then $\eta_C(x) \leq \lambda_j \leq \lambda_i \Rightarrow x \in \eta_{\lambda_i}$ there exists $x \in X$ such that $\lambda_i \geq \eta(x) > \lambda_j$ then $x \in \eta_{\lambda_i}$ but $x \in \eta_{\lambda_j}$ therefore $\overline{\zeta}_{\alpha_j} \subset \overline{\zeta}_{\alpha_i}$ and $\overline{\zeta}_{\alpha_j} \neq \overline{\zeta}_{\alpha_i}$ and $\eta_{\lambda_j} \supset \eta_{\lambda_i}$ and $\eta_{\lambda_j} \neq \eta_{\lambda_i}$ which is a contradiction. □

Theorem 3.5. *Let $C = \{x, \overline{\zeta}_C(x), \eta_C(x) : x \in X\}$ be a cubic fuzzy $\beta-$subalgebra of X. Two level subalgebras $C_{\overline{\alpha}_1}$ and $C_{\overline{\alpha}_2}$(with $\overline{\alpha}_1 < \overline{\alpha}_2$) and C_{λ_1} and C_{λ_2}(with $\lambda_1 < \lambda_2$) of C are equal if and only if there is no $x \in X$ such that $\overline{\alpha}_1 \leq \overline{\zeta}_C(x) < \overline{\alpha}_2$ and $\lambda_1 \geq \eta_C(x) > \lambda_2$.*

Proof. Assume that $C_{\overline{\alpha}_1} = C_{\overline{\alpha}_2}$ for $\overline{\alpha}_1 < \overline{\alpha}_2$. Then there exists $x \in X$ such that the membership function $\overline{\alpha}_1 < \overline{\zeta}_C(x) < \overline{\alpha}_2$ and $\lambda_1 > \eta_C(x) > \lambda_2$. Hence $\overline{\zeta}_{\overline{\alpha}_2}$ is proper subset of $\overline{\zeta}_{\overline{\alpha}_1}$ and η_{λ_1} is proper subset of η_{λ_2} which is a contradiction.

Conversely, assume that there is no $x \in X$ such that the membership function $\overline{\alpha}_1 < \overline{\zeta}_C(x) < \overline{\alpha}_2$. Since $\overline{\alpha}_1 < \overline{\alpha}_2$ then $\overline{\zeta}_{\overline{\alpha}_2} \subseteq \overline{\zeta}_{\overline{\alpha}_1}$and $\lambda_1 > \lambda_2$ then $\eta_{\lambda_1} \supseteq \eta_{\lambda_2}$. If $x \in \overline{\zeta}_{\overline{\alpha}_1}$ then $\overline{\zeta}(x) \geq \overline{\alpha}_1$ and $\overline{\zeta}(x) \geq \overline{\alpha}_2$ because $\overline{\zeta}(x)$ does not lies between $\overline{\alpha}_1$ and $\overline{\alpha}_2$. If $x \in \eta_{\lambda_1}$ then $\eta(x) \leq \lambda_1$ and $\eta(x) \leq \lambda_2$ because $\eta(x)$ does not lies between λ_1 and λ_2. Hence $x \in \overline{\zeta}_{\overline{\alpha}_2}$ implies that $\overline{\zeta}_{\overline{\alpha}_1} \subseteq \overline{\zeta}_{\overline{\alpha}_2}$ and $x \in \eta_{\lambda_2}$ which yields that $\eta_{\lambda_1} \supseteq \eta_{\lambda_2}$. Therefore,$\overline{\zeta}_{\overline{\alpha}_1} = \overline{\zeta}_{\overline{\alpha}_2}$ and $\eta_{\lambda_1} = \eta_{\lambda_2}$. □

4. Conclusion

In the present work, the concept of levels set is applied into the structure of cubic fuzzy β-subalgebra and examined the related results. In future, this can be explored into several algebraic substructures.

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