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# RESULTS ON RELATIVELY PRIME DOMINATION NUMBER OF VERTEX SWITCHING OF COMPLEMENT GRAPHS

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ABSTRACT. Let $G$ be a non-trivial graph. A set $S \subseteq V$ is said to be a relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices $u$ and $v$ in $S$ such that $(d(u), d(v)) = 1$. The minimum cardinality of a relatively prime dominating set is called a relatively prime domination number and it is denoted by $\gamma_{\mathrm{rpd}}(G)$. For a finite undirected graph $G(V, E)$ and a subset $\sigma \subseteq V$, the switching of $G$ by $\sigma$ is defined as the graph $G^{\sigma}(V, E')$ which is obtained from $G$ by removing all edges between $\sigma$ and its complement $V - \sigma$ and adding as edges all non-edges between $\sigma$ and $V - \sigma$. In this paper we compute the relatively prime domination number of vertex switching of complement of path $P_n$, cycle $C_n$, star $K_{1,n}$ and complete bipartite graph $K_{m,n}$.

## 1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite undirected simple graph. The order and size of $G$ are denoted by $p$ and $q$ respectively. For graph theoretical terms, we refer to Harary [2] and for terms related to domination we refer to Haynes [3]. A subset $S$ of $V$ is said to be a dominating set in $G$ if every vertex not in $S$ is adjacent to at least one member of $S$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$.

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The originators of dominating sets are Berge and Ore [1, 8]. It was further extended to define many other domination related parameters in graphs. Let $G$ be a non-trivial graph. A set $S \subseteq V$ is said to be a relatively prime dominating set if it is a dominating set and for every pair of vertices $u$ and $v$ in $S$ such that $(d(u), d(v)) = 1$. The number of vertices in a smallest relatively prime dominating set is called the relatively prime domination number and it is denoted by $\gamma_{\mathrm{rpd}}(G)$ [5]. Switching in graphs was introduced by Lint and Seidel [7]. For a finite undirected graph $G(V, E)$ and $v \in V$, the vertex switching of $G$ by $v$ is the graph $G^v$ which is obtained from $G$ by removing all edges incident to $v$ and adding edges which are not adjacent to $v$ [4]. In this paper we determine the relatively prime domination number $\gamma_{\mathrm{rpd}}(\overline{G}^v)$ where $G$ is a path, cycle, star and complete bipartite graph.

## 2. Definition and Examples

**Definition 2.1.** *For a finite undirected graph $G(V, E)$ and $v \in V$, the vertex switching of $G$ by $v$ is the graph $G^v$ which is obtained from $G$ by removing all edges incident to $v$ and adding edges which are not adjacent to $v$.*

**Example 1.** *The graphs $C_5$ and $C_5^v$ are given in figures 2.1 and 2.2, respectively. Clearly $\{u, x\}$ is a minimal relatively prime dominating set of $C_5^v$ and hence $\gamma_{\mathrm{rpd}}(C_5^v) = 2$.*

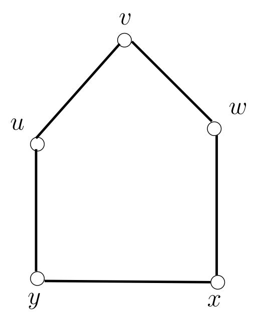


Fig 2.1. $C_5$

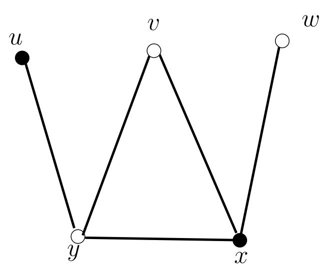


Fig 2.2. $C_5^v$

**Example 2.** Consider the graph $G$ given in figure 2.3. The graph $G^{v_1}$ is given in figure 2.4. Clearly $\{v_1, v_2, v_4\}$ is a dominating set of $G^{v_1}$. Also $(d(v_1), d(v_2)) =$

$(3,1)=1; (d(v_1),d(v_4))=(3,2)=1$ and $(d(v_2),d(v_4))=(1,2)=1$. By definition, $\{v_1,v_2,v_4\}$ is a relatively prime dominating set of $G^{v_1}$. Also $\{v_1,v_2,v_4\}$ is a minimal dominating set with this property and hence $\gamma_{\text{rpd}}(G^{v_1})=3$. But $\gamma(G^{v_1})=2$.

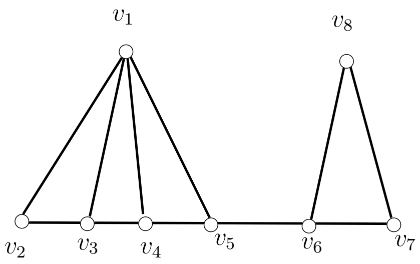


Fig 2.3 $G$

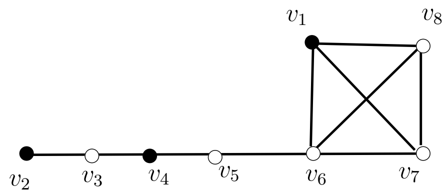


Fig 2.4 $G^{v_1}$

We recall the following results for future study.

**Theorem 2.1.** *[5] For a complete bipartite graph $K_{m,n}, \gamma_{\text{rpd}}(K_{m,n})=2$ if and only if $(m,n)=1$.*

**Result 1.** *[6]* If $G$ is a regular graph of degree $n \neq 1$, then $\gamma_{\text{rpd}}(G)=0$.

**Result 2.** *[6]* If $G=nK_2, n\geq 2$, then $\gamma(G)=\gamma_{\text{rpd}}(G)=n$. For $n=1, \gamma(G)=1$ and $\gamma_{\text{rpd}}(G)=2$.

**Theorem 2.2.** *[6]* $\gamma_{\text{rpd}}(P_n)=\begin{cases} 2 \text{ if } 2\leq \text{n} \leq 5 \\ 3 \text{ if n} = 6,7 \\ 0 \text{ otherwise} \end{cases}$.

**Notation 1.** *[4]* Consider a cycle $C_r=(v_1,v_2,\ldots,v_r)$ (clock-wise). For our convenience we denote it by $C_{r(v_1)}$. Identifying an end vertex of paths $P_m$ at $v_i$ and $P_s$ at $v_j$, then $C_{r(v_1)}$ is denoted by $C_{r(v_1)}(0,\ldots,P_m,0,\ldots,P_s,0,\ldots,0)$. Identifying an end vertex of paths $P_m$ and $P_s$ at the vertex $v_j$, then $C_{r(v_1)}$ is denoted by $C_{r(v_1)}(0,\ldots,P_m\cup P_s,0,\ldots,0)$. The graphs $C_{4(v)}(0,0,P_2,P_3)$ and $C_{4(v)}(0,2P_2\cup P_3,P_2,P_3)$ given in figure 2. 5 and 2. 6.

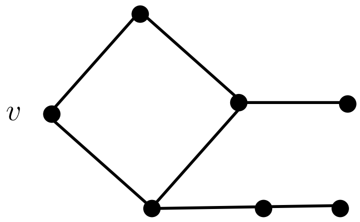

Fig. 2.5 $C_{4(v)}(0, 0, P_2, P_3)$

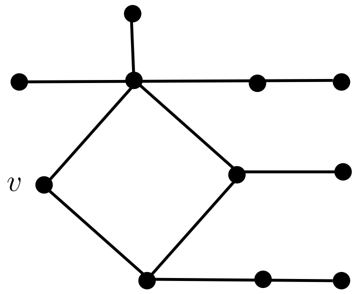

Fig. 2.6 $C_{4(v)}(0, 2P_2 \cup P_3, P_2, P_3)$

## 3. Main Results

In this section, we find $\gamma_{\text{rpd}}(\overline{G}^v)$ where $G$ is a cycle $C_n$, path $P_n$, star $K_{1,n}$ and complete bipartite graph $K_{m,n}$ and $v$ is any vertex of $G$.

**Theorem 3.1.** *For* $n \geq 3$, $\gamma_{\text{rpd}}(\overline{C}_n^v) = \begin{cases} 2 \text{ if n is odd} \\ 0 \text{ if n is even.} \end{cases}$

*Proof.* Let $v_1v_2v_3 \dots v_nv_1$ be the cycle $C_n$. It is clear that for $2 \leq i \leq n, \overline{C}_n^{v_i}$ are isomorphic to $\overline{C}_n^{v_1}$. We consider the following two cases. If $n = 3$, then $C_3 = K_3$ and hence $\overline{C}_3 = \overline{K}_3$. This implies that $\overline{C}_3^v = K_1 \cup K_2$, and hence $\gamma_{\text{rpd}}(\overline{C}_3^v) = 2$.

Consider $n \geq 4$. Without loss of generality, let $v = v_1$. In $C_n, d(v_i) = 2$, and hence in $\overline{C}_n, d(v_i) = n - 3, 1 \leq i \leq n$. Since $v_1$ is adjacent to $v_3, v_4, \dots, v_{n-1}$ in $\overline{C}_n, v_1$ is adjacent to $v_2$ and $v_n$ in $\overline{C}_n^{v_1}$. Hence in $\overline{C}_n^{v_1}, d(v_1) = 2, d(v_2) = d(v_n) = n - 2$ and $d(v_j) = n - 4, 3 \leq j \leq n - 1$. Since $v_n$ is adjacent to $v_1, v_2, \dots, v_{n-2}$ in $\overline{C}_n^{v_1}, \{v_{n-1}, v_n\}$ is a minimal dominating set of $\overline{C}_n^{v_1}$. Also in $\overline{C}_n^{v_1}, (d(v_{n-1}), d(v_n)) = (n - 4, n - 2)$. If $n$ is odd, then $(n - 2, n - 4) = 1$ and hence $\{v_{n-1}, v_n\}$ is a minimal relatively prime dominating set which implies that $\gamma_{\text{rpd}}(\overline{C}_n^v) = 2$. If $n$ is even, then $2, n - 2, n - 4$ are multiples of 2 and hence $\gamma_{\text{rpd}}(\overline{C}_n^v) = 0$. This completes the proof of the theorem. □

**Example 3.** The graphs $C_7, \overline{C}_7$ and $\overline{C}_7^{v_1}$ are given in figures 3.1, 3.2 and 3.3, respectively. Clearly, $\{v_6, v_7\}$ is a relatively prime dominating set of $\overline{C}_7^{v_1}$.

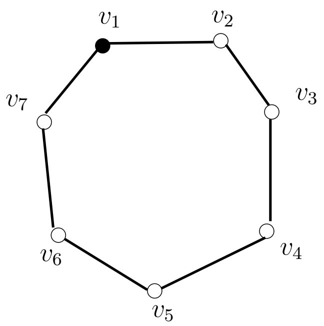


Fig 3.1 $C_7$

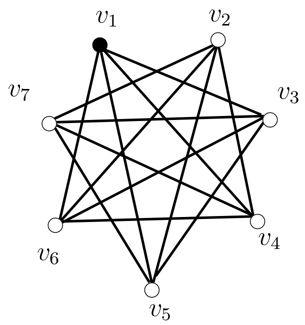


Fig 3.2 $\overline{C}_7$

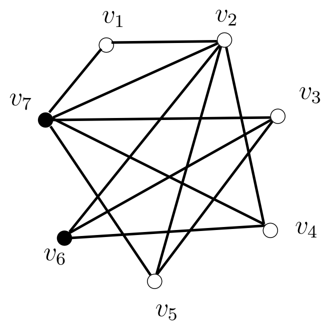


Fig 3.3 $\overline{C}_7^{v_1}$

**Theorem 3.2.**

(i) $\gamma_{\text{rpd}}(\overline{P}_2^v) = 2$

(ii) $\gamma_{\text{rpd}}(\overline{P}_3^v) = \begin{cases} 2 \text{ if v is an end vertex of P}_3 \\ 0 \text{ otherwise} \end{cases}$

(iii) $\gamma_{\text{rpd}}(\overline{P}_4^v) = \begin{cases} 2 \text{ if v is an support vertex of P}_4 \\ 3 \text{ if v is an end vertex} \end{cases}$

(iv) *For* $n \geq 5, \gamma_{\text{rpd}}(\overline{P}_n^v) = 2,$ *where* $v$ *is any vertex of* $P_n$.

*Proof.* Let $v_1 v_2 v_3 \dots v_n$ be the path $P_n$. Let $v$ be any vertex of $P_n$. We consider the following four cases.

**Case 1.** $n = 2$.

Then $P_2 = K_2$ and hence $\overline{P}_2 = \overline{K}_2$ This implies that $\overline{P}_2^v = K_2 = K_{1,1}$. By Theorem 2.1, $\gamma_{\text{rpd}}(\overline{P}_2^v) = 2$.

**Case 2.** $n = 3$.

Then $P_3 = K_{1,2}$ and hence $\overline{P}_3 = K_1 \cup K_2$. This implies that $\overline{P}_3^v$ is either $K_3$ or $K_1 \cup K_2$ according as $v$ is a support vertex or an end vertex of $P_3$. If $\overline{P}_3^v = K_3$, then by Result 1, $\gamma_{\text{rpd}}(\overline{P}_3^v) = 0$. If $\overline{P}_3^v = K_1 \cup K_2$, then clearly, $\gamma_{\text{rpd}}(\overline{P}_3^v) = 2$. Thus $\gamma_{\text{rpd}}(\overline{P}_3^v) = 0$ or 2 according as $v$ is the support vertex or an end vertex of $P_3$.

**Case 3.** $n = 4$.

In this case, $\overline{P}_4^v$ is either $K_1 \cup P_3$ with $v$ has degree one or $C_{3(v)}(0, 0, P_2)$ according as $v$ is an end vertex or a support vertex. If $\overline{P}_4^v = K_1 \cup P_3$, then the set containing the isolated vertex and the end vertices is the minimal relatively prime dominating set and hence $\gamma_{\text{rpd}}(\overline{P}_4^v) = 3$. If $\overline{P}_4^v = C_{3(v)}(0, 0, P_2)$, then the vertex $v$ and the end vertex is a minimal relatively prime dominating set and hence $\gamma_{\text{rpd}}(\overline{P}_4^v) = 2$. Thus $\gamma_{\text{rpd}}(\overline{P}_4^v) = 2$ or 3 according as $v$ is a support vertex or an end vertex of $P_4$.

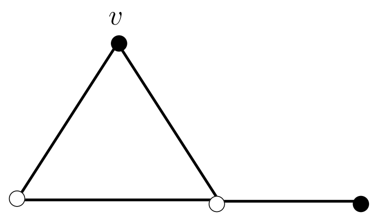


Fig 3.4 $C_{3(v)}(0, 0, P_2)$

**Case 4.** $n \geq 5$.

**Subcase 4.1.** $v$ is an end vertex.

In this case $v$ is either $v_1$ or $v_n$. Without loss of generality, let it be $v_1$. In $\overline{P}_n, d(v_1) = d(v_n) = n - 2$ and $d(v_i) = n - 3, 2 \leq i \leq n - 1$. Since $v_1$ is adjacent to $v_3, v_4, \ldots, v_n$ in $\overline{P}_n, v_1$ is adjacent to $v_2$ only in $\overline{P}_n^{v_1}$. Hence, in $\overline{P}_n^{v_1}, d(v_1) = 1, d(v_2) = n-3+1 = n-2, d(v_n) = n-2-1 = n-3, d(v_i) = n-3-1 = n-4, 3 \leq i \leq n - 1$. Now $\{v_2, v_n\}$ is a minimal dominating set and $(n - 2, n - 3) = 1$. This implies that $\{v_2, v_n\}$ is a minimal relatively prime dominating set of $\overline{P}_n^{v_1}$ and hence $\gamma_{\text{rpd}}(\overline{P}_n^{v_1}) = 2$.

**Subcase 4.2.** $v$ is an internal vertex but not a support vertex.

In this case $v$ is one of the $v_i, 3 \leq i \leq n - 2$. In $\overline{P}_n, d(v_1) = d(v_n) = n - 2$ and $d(v_j) = n - 3, 2 \leq j \leq n - 1$. Since $v_1$ is adjacent to $v_3, v_4, \ldots, v_n$ in $\overline{P}_n, v_1$ is adjacent to $v_3, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ in $\overline{P}_n^{v_i}$ and also $v_{i+1}$ is adjacent to $v_1, v_2, v_3, \ldots, v_{i-1}, v_i,\ v_{i+3}, \ldots, v_n$ in $\overline{P}_n^{v_i}$. Hence, in $\overline{P}_n^{v_i}, d(v_1) = d(v_n) = n - 3, d(v_i) = 2, d(v_{i-1}) = d(v_{i+1}) = n - 2$, and $d(v_j) = n - 4, 2 \leq j \neq i - 1, i, i + 1 \leq n - 1$. Now, $\{v_1, v_{i+1}\}$ is a dominating set of $\overline{P}_n^{v_i}$ and $(d(v_1), d(v_{i+1})) = (n - 3, n - 2) = 1$. This implies that $\{v_1, v_{i+1}\}$ is a minimal relatively prime dominating set of $\overline{P}_n^{v_i}$ and hence $\gamma_{\text{rpd}}(\overline{P}_n^{v_i}) = 2$.

**Subcase 4.3.** $v$ is a support vertex.

Then $v$ is either $v_2$ or $v_{n-1}$. Clearly $P_n^{v_2} \cong P_n^{v_{n-1}}$. Without loss of generality, let $v$ be $v_2$. Since $v_1$ is adjacent to $v_3, v_4, \ldots, v_n$ in $\overline{P}_n, v_1$ is adjacent to $v_2, v_3, v_4, \ldots, v_n$

in $\overline{P}_n^{v_2}$.$v_3$ is adjacent to $v_1, v_4, v_5, \ldots, v_n$ in $\overline{P}_n$ implies that $v_3$ is adjacent to $v_1, v_2, v_4, \ldots, v_n$ in $\overline{P}_n^{v_2}$. Hence in $\overline{P}_n^{v_2}$, $d(v_1) = n-1$ and $d(v_3) = n-2$. Now,$\{v_1, v_3\}$ is a dominating set of $\overline{P}_n^{v_2}$ and $(d(v_1), d(v_3)) = (n-1, n-2) = 1$. This implies that $\{v_1, v_3\}$ is a minimal relatively prime dominating set of $\overline{P}_n^{v_2}$ and hence $\gamma_{\text{rpd}}(\overline{P}_n^{v_2}) = 2$. Thus, $\gamma_{\text{rpd}}(\overline{P}_n^{v}) = 2$. The theorem follows from cases 1, 2, 3 and 4. □

**Example 4.** The graphs $P_5, \overline{P}_5, \overline{P}_5^{v_1}, \overline{P}_5^{v_2}$ and $\overline{P}_5^{v_3}$ are given in figures 3.5, 3.6, 3.7, 3.8 and 3.9 respectively. Clearly, $\{v_2, v_3\}$ is a relatively prime dominating set of $\overline{P}_5^{v_1}$.

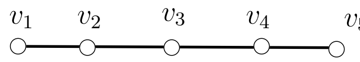

Fig 3.5 $P_5$

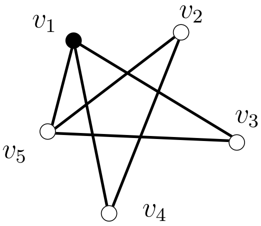

Fig 3.6 $\overline{P}_5$

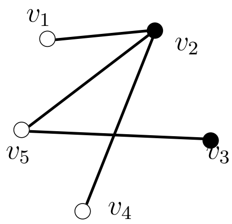

Fig 3.7 $\overline{P}_5^{v_1}$

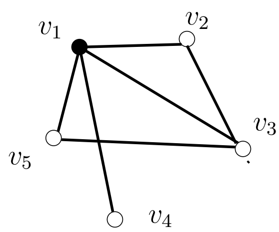

Fig 3.8 $\overline{P}_5^{v_2}$

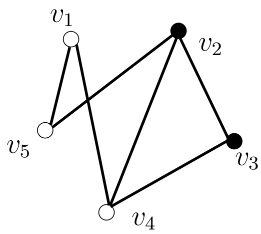

Fig 3.9 $\overline{P}_5^{v_3}$

**Theorem 3.3.** *For* $n \geq 2, \gamma_{\text{rpd}}(\overline{K}_{1,n}^{v}) = \begin{cases} 2 \text{ if v is an end vertex of } K_{1,n} \\ 0 \text{ otherwise} \end{cases}$.

*Proof.* Let $u$ be the center and $w$ be any end vertex of $K_{1,n}, n \geq 2$.

**Case 1.** $n = 2$.

Then $K_{1,2} = P_3$. By Theorem 2.2, $\gamma_{\text{rpd}}(\overline{K}_{1,2}^{v}) = \begin{cases} 2 \text{ if v is an end vertex of } K_{1,2} \\ 0 \text{ otherwise} \end{cases}$

**Case 2.** $n \geq 3$.

Then $\overline{K}_{1,n} = K_1 \cup K_n$ where $K_1$ is the vertex $u$. Clearly, $\overline{K}_{1,n}^{v}$ is $K_{n+1}$ if $v = u$

and $K_2 \cup K_{n-1}$ if $v = w$. If $v = u$, then by Result 1, $\gamma_{\text{rpd}}(\overline{K}^v_{1,n}) = 0$. If $v = w$, then $\gamma_{\text{rpd}}(\overline{K}^v_{1,n}) = K_2 \cup K_{n-1}$. In this case $\{u, x\}$ is a minimal dominating set of $\overline{K}^v_{1,n}$ where $u$ is in $K_2$ and $x$ is a vertex of $K_{n-1}$ and $(d(u), d(x)) = (1, n-2) = 1$. This implies that $\{u, x\}$ is a minimal relatively prime dominating set of $\overline{K}^v_{1,n}$ and hence $\gamma_{\text{rpd}}(\overline{K}^v_{1,n}) = 2$. Thus $\gamma_{\text{rpd}}(\overline{K}^v_{1,n}) = \begin{cases} 2 \text{ if v is an end vertex of } \mathrm{K}_{1,\mathrm{n}} \\ 0 \text{ otherwise} \end{cases}$ □

**Theorem 3.4.** *For $n \geq 2, \gamma_{\text{rpd}}(\overline{K}^v_{m,n}) = 2$, where $m \neq n$ and $m + n$ is odd.*

*Proof.* Let $(V_1, V_2)$ be the bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Clearly $\overline{K}_{m,n} = K_m \cup K_n$. Now, $\overline{K}^v_{m,n} = K_{m-1} \cup K_{n+1}$ or $K_{m+1} \cup K_{n-1}$ according as $v \in V_1$ or $v \in V_2$. Let $x$ and $y$ be two vertices which are in different components of $\overline{K}^v_{m,n}$. Then $\{x, y\}$ is a minimal dominating set of $\overline{K}^v_{m,n}$. In $\overline{K}^v_{m,n}, d(x) = m - 2, d(y) = n$ if $v \in V_1$ and $d(x) = m, d(y) = n - 2$ if $v \in V_2$. If $m + n$ is odd, then either $m$ or $n$ is odd but not both. This implies that $(m - 2, n) = 1$ and $(m, n - 2) = 1$. Hence $\{x, y\}$ is a minimal relatively prime dominating set of $\overline{K}^v_{m,n}$ and hence $\gamma_{\text{rpd}}(\overline{K}^v_{m,n}) = 2$. □

**Observation 1.** $\gamma_{\text{rpd}}(\overline{K}^v_{n,n}) = 2$, if $n$ is odd.

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