ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.4, 1487–1493
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.4.2 Spec. Issue on NCFCTA-2020

RECONSTRUCTION OF FINITE TOPOLOGICAL SPACES WITH MORE THAN ONE ISOLATED POINT

A. ANAT JASLIN JINI[1] AND S. MONIKANDAN

ABSTRACT. The deck of a topological space X is the set $\mathscr{D}(X) = \{[X - \{x\}] : x \in X\}$, where $[Z]$ denotes the homeomorphism class of Z. A space X is topologically reconstructible if whenever $\mathscr{D}(X) = \mathscr{D}(Y)$ then X is homeomorphic to Y. For $|\mathscr{D}(X)| \geq 3$, it is shown that all finite topological spaces with more than one isolated point are reconstructible.

1. FIRST SECTION: IMPORTANT

A vertex-deleted subgraph or *card* $G-v$ of a graph G is obtained by deleting the vertex v and all edges incident with v. The collection of all cards of G is called the *deck* of G. A graph H is a *reconstruction* of G if H has the same deck as G. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. The graph reconstruction conjecture, posed by Kelly and Ulam [7] in 1941, asserts that every graph G on n (≥ 3) vertices is reconstructible. More precisely, if G and H are finite graphs with at least three vertices such that $\mathscr{D}(H) = \mathscr{D}(G)$, then G and H are isomorphic.

In 2016, Pitz and Suabedissen [6] have introduced the concept of reconstruction in topological spaces as follows. For a topological space X, the subspace $X - \{x\}$ is called a card of X. The set $\mathscr{D}(X) = \{[X - \{x\}] : x \in X\}$ of

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 05C60, 54A05.
Key words and phrases. Reconstruction, Finite topological space, Homeomorphism.

subspaces of X is called the deck of X, where $[X - \{x\}]$ denotes the homeomorphism class of the card $X - \{x\}$. Given topological spaces X and Z, we say that Z is a reconstruction of X if their decks agree. A topological space X is said to be reconstructible if the only reconstructions of it are the spaces homeomorphic to X. Formally, a space X is reconstructible if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies $X \cong Z$ and a property $\mathscr{P}$ of topological spaces is reconstructible if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies "X has $\mathscr{P}$ if and only if Z has $\mathscr{P}$".

The number of elements in a topological space X is called the *size* of X. Terms not defined here are taken as in [2]. Gartside et al [3, 4, 6] have proved that the space of real numbers, the space of rational numbers, the space of irrational numbers, every compact Hausdorff space that has a card with a maximal finite compactification, and every Hausdorff continuum X with weight $\omega(X) < |X|$ are reconstructible. In their above paper, they also proved certain properties of a space, namely all hereditary separation axioms and all cardinal invariants are reconstructible. All finite sequences are reconstructed by Manvel et al [5].

In this paper, it is shown that every finite topological space with at least n (≥ 4) elements and more than one isolated point with $|\mathscr{D}(X)| \geq 3$ is reconstructible. Also, for $|\mathscr{D}(X)| = 2$, we prove that the finite topological spaces with more than one isolated point and with one discrete card is reconstructible. The condition that $n \geq 4$ is needed because there are nonreconstructible topological spaces of size 2 or 3. For $n = 2$, the set $X = \{a, b\}$ endowed with any of the three topologies $\tau_1 = \{\phi, \{a\}, \{b\}, X\}, \tau_2 = \{\phi, \{a\}, X\}$ or $\tau_3 = \{\phi, X\}$ is not reconstructible, since all these topological spaces have the same deck. For $n = 3$, the set $X = \{a, b, c\}$ endowed with any of the two topologies $\tau_1 = \{\phi, \{c\}, X\}, \tau_2 = \{\phi, \{a, b\}, X\}$ is not reconstructible.

2. Finite Topological Spaces

Since every discrete topological space is reconstructible [1], we assume that X is a finite topological space of size n, which is not discrete, where $n \geq 4$ and $X = \{x_1, x_2, \ldots, x_n\}$. Let $m = |\mathscr{D}(X)|$. The next lemma is proved in [1].

Lemma 2.1. *[1] Let X be a topological space with isolated points and for $m = 2$, all but one isolated point in a card must belong to at least one open set in the other cards. Then the property that whether X has one isolated point or at least two isolated points is reconstructible.*

For a collection of open subsets Y of a topological space X, $\vee(Y)$ denotes the set consisting of elements of Y together with all possible union of elements of Y.

Theorem 2.1. *Every space with at least three mutually non-homeomorphic cards and more than one isolated point is reconstructible.*

Proof. Let X be a space with more than one isolated point. Then every card of X has at least one isolated point. If an isolated point, say x_1 of a card X_x is not an isolated point of any other card of X, then x_1 is not an isolated point of X (as otherwise, x_1 would be an isolated point in all but one card of X) and hence the set $\{x_1, x\}$ is open in X. Thus, an isolated point, say x_1 in a card X_x is an isolated point of X if it is isolated in all but one card of X; $\{x_1, x\}$ is open in X otherwise. Repeat these arguments for each of the remaining isolated points in every card in the deck of X to identify the isolated points of X. Finally, we arrive at two disjoint new sets, say O_1 and O_2, where O_1 consists of all the isolated points of X and O_2 consists of all such open sets $\{x_1, x\}$ of X. Let $O_1 = \{y_1, y_2, ..., y_k\}$, where $k \geq 2$ and let $\mathscr{C}_1 = \vee(O_1 \cup O_2)$.

Now consider a card X_x and an open set $U_2 = \{a, b\}$ of X_x such that $U \notin \mathscr{C}_1$. Then $|U_2 \cap O_1| = 1$ or 0. If the former holds, without loss of generality, let $a = y_i$, for some $i, 1 \leq i \leq k$. If $\{b\}$ is not open in none of the cards, then $\{a, b\}$ is not open in X (as otherwise U_2 would be an open set in $n-2$ cards, $\{b\}$ would be an open set in $X - \{a\}$ and $\{a\}$ would be an open set in $X - \{b\}$, giving a contradiction) and hence the set $U_2 \cup \{x\}$ is open in X. If $\{b\}$ is open in a card, which is not open in X, then $\{b\}$ along with the deleted point for the card, in which $\{b\}$ is open, is open in X. Note that this open set already in the collection O_2. So, assume that the latter holds. If one of $\{a\}$ and $\{b\}$ are not open in none of the cards, then U_2 is not open in X and hence the set $U_2 \cup \{x\}$ is open in X. So assume $\{a\}$ and $\{b\}$ are open in at least one of the cards. If one of $\{a\}$ and $\{b\}$ are not open in X, then the 1- subset which is not open in X along with the deleted point for the card, in which the 1-subset is present, is open in X. Therefore, U_2 is not open in X and hence $U_2 \cup \{x\}$ is open in X. Repeat these steps for each $2-$open set W in every card in the deck of X. Finally, we will get collections of open sets, O_3 consists of some 3-open sets of X that is not in $\mathscr{C}_1$. Let $\mathscr{C}_2 = \vee(\mathscr{C}_1 \cup O_3)$.

Again we proceed with the similar arguments to 3-open sets. Consider any card X_x and a 3-open set, say U_3 in X_x such that $U_3 \notin \mathscr{C}_2$. If one of the 2-subsets of U_3 is not open in none of the cards, then U_3 is not open in X (as otherwise V would be open in $n-3$ cards and 2-subsets of U_3 are open in the three cards X_z, where $z \in U_3$, giving a contradiction) and hence $U_3 \cup \{x\}$ is open in X. So assume that, all the 2-subsets of U_3 is open in at least one the cards. If one of the 2- subset is not open in X, then the 2-subset which is not open in X along with deleted point for the corresponding card, in which the 2-subset is present, is open in X. Therefore, U_3 is not open in X and hence $U_3 \cup \{x\}$ is open in X. Repeat these steps for each $3-$open set U_3 in every card in the deck of X. Finally, we shall arrive at collections of open sets, say O_4 consists of some 4-open sets of X that is not in $\mathscr{C}_2$. Let $\mathscr{C}_3 = \vee(\mathscr{C}_2 \cup O_4)$.

In general, consider a card X_x and a k-open set, say $U_k, k \leq n-2$, in X_x such that $U_k \notin \mathscr{C}_{k-1}$. If one of the $(k-1)$-subsets of U_k is not open in the none of the cards, then U_k is not open in X and hence $U_k \cup \{x\}$ is open in X. So assume that, all the $(k-1)$-subsets of U_k is open in at least one the cards. If one of the $(k-1)$- subset is not open in X, then the $(k-1)$-subset along with deleted point for the corresponding card, in which the $(k-1)$-subset is present, is open in X. Therefore, U_k is not open in X and hence $U_k \cup \{x\}$ is open in X. Repeat these steps for each $k-$open set U_k in every card in the deck of X. Finally, we shall arrive at collections of open sets, say O_{k+1} consists of some $(k+1)$-open sets of X that is not in $\mathscr{C}_{k-1}$. Let $\mathscr{C}_k = \vee(\mathscr{C}_{k-1} \cup O_{k+1})$.

The proof completes once we identified the remaining $(n-1)-$open sets, if any, in X. For this, we consider a card X_x such that the unique $(n-1)$-open set $X - \{x\}$ in it is not in the collection $\mathscr{C}_{n-2}$ so formed. Since each card has at least one isolated point of X, it follows that each $(n-1)$-open set in a card contains at least one isolated point of X. Now, let $\mathscr{U}(X_x) = \{X_x - y_i \ : \ y_i$ is an isolated point of $X\}$. If an element of $\mathscr{U}(X_x)$ does not belong to any card, then X_x does not open in X, since the element itself is not in the space X. So, assume that each element of $\mathscr{U}(X_x)$ is an open set of at least one card of X. If at least one of the elements of the set $\mathscr{U}(X_x)$ is open in X, then the set X_x is open in X. So, assume that none of the elements of the set $\mathscr{U}(X_x)$ is open in X. Then each element in $\mathscr{U}(X_x)$ together with the deleted point of the card, in which the element is open, is open in X and hence X_x is not open in X. Repeat these steps for the $(n-1)-$open set in every card in the deck of X. Let O'_{n-1} be the set of these new $(n-1)-$open sets. Then $\vee(\mathscr{C}_{n-2} \cup O'_{n-1})$ is the desired topology on X. $\square$

Lemma 2.2. *Let X be a space with only two non-homeomorphic cards and more than one isolated point. If the subspace topology on one card, say X_x is the discrete topology, then τ_X must be equal to one of the following three collections:*

(i) $\tau_{X_x} \cup X$;

(ii) $\tau_{X_x} \cup \{x, y\} \cup \{\{x, y\} \cup U \mid y \in X_x$ *and* $U \in \tau_{X_x}\}$;

(iii) $\tau_{(X_x - y)} \cup \{x, y\} \cup \{\{x, y\} \cup U \mid y \in X_x$ *and* $U \in \tau_{(X_x - y)}\}$.

Proof. Assume to the contrary, that τ_X was not equal to the collection given in (i), (ii) and (iii). We proceed by three cases depending on the number of isolated points of X.

Case 1. The space X has $n-1$ isolated points.

Let $\{y_1, y_2, ..., y_{n-1}\}$ be the set of all isolated points of X. By our contrary assumption, there exists a smallest $i-$open set, say W in X containing the point, say $x \in X - \{y_1, ..., y_{n-1}\}$ for some $i,\ 3 \leq i \leq n-1$. Then the only card having discrete topology is X_x. Consider now the two cards X_{y_k} and X_{y_s}, where $y_k \in W$ and $y_s \notin W$. We claim that the two cards X_{y_k} and X_{y_s} are non-homeomorphic. Suppose, to the contrary, that there is a homeomorphism $f : X_{y_k} \longrightarrow X_{y_s}$. Then x must be mapped to x under f. It is clear that the smallest open set containing the point x in X_{y_k} is $W - \{y_k\}$ while the smallest open set containing the point x in X_{y_s} is W, giving a contradiction to f. This completes the claim and hence the space X has at least three mutually non-homeomorphic cards, giving a contradiction.

Case 2. The space X has $n-2$ isolated points.

Let $\{y_1, y_2, ..., y_{n-2}\}$ be the set of all isolated points of X. By our contrary assumption, in X, there exists an open set $U \cup \{x, y\}$ or $U \cup \{x\}$ or $U \cup \{y\}$ where $\phi \neq U \in \vee(\{y_1, y_2, ..., y_{n-2}\})$ and $x, y \in X - \{y_1, y_2, ..., y_{n-2}\}$. If the former holds, then no card has the discrete topology, a contradiction. So, assume that the latter holds. If $|U| > 1$, then no card will have the discrete topology, a contradiction. So, let us assume that $|U| = 1$. If X has only one open set $\{y_j\} \cup \{x\}$, where $j = 1, 2, ..., n-2$, then no cards will have the discrete topology, again a contradiction. So, assume that X has two open sets $\{y_i\} \cup \{x\}$ and $\{y_j\} \cup \{y\}$, where $i, j = 1, 2, ..., n-2$. If $i \neq j$, then no card will have the discrete topology. Otherwise, the only card having the discrete topology is X_{y_i}. Since the card X_{y_e}, where $e \neq i$, has $n-3$ isolated points while X_x has $n-2$ isolated points, the two cards are non-homeomorphic, giving a contradiction.

Case 3. The space X has at most $n-3$ isolated points, where $n \geq 5$.

The isolated points of X are denoted by $y_1,\ y_2, ..., y_i$. Then $2 \leq i \leq n-3$, since the space X under consideration has at least two isolated points. If X has no 2-open sets of the type $\{y_i, x_j\}$, where $x_j \in X - \{y_1, y_2, ..., y_i\}, 1 \leq j \leq n-i$, then no card has the discrete topology. So, assume that X has $2-$open sets $\{y_i, x_j\}$. If X has at most k, where $k < n-i$, 2-open sets of the above type, then no card has the discrete topology. So, we assume that X has $n-i$ 2-open sets of the form $\{y_i\} \cup \{x_j\}$. If any two of the isolated points $y_i's$ are distinct, then clearly no card has the discrete topology. Finally, we consider the case that all the $y_i's$ are equal and they are y_a. Now, the only card having the discrete topology is X_{y_a}. Since the card X_{y_i} has $i-1$ isolated points while the card X_{x_j} has i isolated points, it follows that they are non-homeomorphic, giving a contradiction and completes the proof. □

Theorem 2.2. *Let X be a space with only two non-homeomorphic cards and more than one isolated point. If one card has discrete topology, then X is reconstructible.*

Proof. Let the two cards be X_x, X_y, where X_x is endowed with discrete topology. By Lemma 2.2, τ_X must be equal to one of the collections given in (i), (ii) or (iii). Therefore X_y has $n-2$ or $n-3$ isolated points. We proceed by two cases depending on the number of isolated points in the card X_y.

Case 1. The card X_y has $n-2$ isolated points.

Suppose X has $n-2$ isolated points. Then X must be of the form (iii) of Lemma 2.2 and hence the card X_y has $n-3$ isolated points, a contradiction. Hence X must contain exactly $n-1$ isolated points and consequently τ_X must be equal to the form (i) or (ii) of Lemma 2.2. Let the set of all isolated points in X be $\{y_1, y_2, ..., y_{n-1}\}$. If the open sets of X_y are in $\vee(\{y_1, y_2 ..., y_i\}) \cup X_y$, where $2 \leq i \leq n-2$, then X has no open set of the form $\{y_i, x\}$, where $i = 1, 2, ..., n-1$ (as otherwise X_y would contain the $2-$open set $\{y_j, x\}$, where $j = 1, 2, ..., n-1$). Then, by Lemma 2.2, X must be of the form (i) of Lemma 2.2. Now the collection $\{U \mid U \in X_y \text{ and } U \in \vee(\{y_1, y_2, ..., y_{n-2}\})\} \cup \{U \cup \{y\} \mid U \in X_y\}$ is the desired topology on X. Suppose that X_y contains the open set of the form $\{y_i, x\}$. Then, by Lemma 2.2, X must be of the form (ii) of Lemma 2.2 and hence the collection $\{U \mid U \in X_y\} \cup \{U \cup \{y\} \mid U \in X_y\}$ is the desired topology on X.

Case 2. The card X_y has $n-3$ isolated points.

Now X has $n-2$ isolated points. By Lemma 2.2, τ_X is of the form (iii) of Lemma 2.2. Since X has $n-2$ isolated points, one isolated point in the card X_x is not open in X; let it be y. Then the collection $\{U \cup \{x\} \mid U \in X_x$ and $y \in U\} \cup \vee(\{y_1, y_2, ..., y_{n-2}\})$ is the desired topology on X. $\square$

REFERENCES

[1] A. ANAT JASLIN JINI, S. MONIKANDAN: *Reconstruction of finite topological spaces with at most one isolated point*, *Cubo A Math.J.*, (submitted).

[2] R. ENGELKING: *General Topology*, Heldermann Verlag, Berlin, 1989.

[3] P. GARTSIDE, M. F. PITZ, R. SUABEDISSEN: *Reconstructing topological graphs and continua*, Colloq. Math., **148** (2017), 107-122.

[4] P. GARTSIDE, M. F. PITZ, R. SUABEDISSEN: *Reconstructing compact metrizable spaces*, Proc. Amer. Math. Soc., **145** (2017), 429-443.

[5] B. MANVEL, A. MEYEROWITZ, A. SCHWENK, K. SMITH, P. STOCKMEYER: *Reconstruction of sequences*, Discrete Math., **94**(1991), 209-219.

[6] M. F. PITZ, R. SUABEDISSEN: *A topological variation of the reconstruction conjecture*, Glasg. Math. J., **59**(1) (2016), 1-15.

[7] S. M. ULAM: *A Collection of Mathematical Problems, in: Interscience Tracts in Pure and Applied Mathematics*, Interscience Publishers, **8** 1960.

DEPARTMENT OF MATHEMATICS
MANONMANIAM SUNDARANAR UNIVERSITY
TIRUNELVELI-627012, TAMIL NADU, INDIA.
Email address: jaslina.142@gmail.com

DEPARTMENT OF MATHEMATICS
MANONMANIAM SUNDARANAR UNIVERSITY
TIRUNELVELI-627012, TAMIL NADU, INDIA.
Email address: monikandans@gmail.com