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A STUDY ON COMPLEMENTEDNESS IN THE SUBGROUP LATTICES OF 3×3 MATRICES OVER Z_3

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ABSTRACT. In this paper, we verify the complementedness in the subgroup lattice of the group of 3×3 matrices over Z_3.

1. INTRODUCTION

Let $L(G)$ be the Lattice of Subgroups of G, where G is a group of 3×3 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number.

Let

$$\mathcal{G} = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a, b, c, d, e, f, g, h, i \in Z_p, \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \right\} \neq 0.$$

Then $\mathcal{G}$ is a group under matrix multiplication modulo p. Let

$$\mathcal{G} = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathcal{G} : \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1 \right\}$$

Then G is a subgroup of $\mathcal{G}$. we have, $o(\mathcal{G}) = (p^n - 1)(p^n - p)(p^n - p^2)...(p^n - p^{n-1})$ [2] and $o(G) = \dfrac{(p^n - 1)(p^n - p)(p^n - p^2)...(p^n - p^{n-1})}{p - 1}$, see [2]. In this paper,

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we are going to study about the complementedness in the subgroup lattice of the group of 3×3 matrices over Z_3.

For ready reference we give the diagram of the lattice of subgroups of G when $p = 2$ and for $p = 3$ we give a split up of the diagram which are found in the thesis of V. Durai Murugan [1]. For additions details, see for instance [3, 4].

2. Preliminaries

Definition 2.1. ***(Poset)*** *A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair $(P, \leq)$ is called a partially ordered set or poset. A poset. $(P, \leq)$ is totally ordered if every $x, y \in P$ are comparable, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.*

Definition 2.2. *Let $(P, \leq)$ be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the* ***supremum or join*** *of S.A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the* ***infimum or meet*** *of S.*

Definition 2.3. ***(Lattice)*** *Poset $(P, \leq)$ is called a lattice if every pair x, y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.*

Definition 2.4. ***(Covering Relation)*** *In the poset $(P, \leq)$, a covers b or b is covered by a (in notation, $a > b$ or $b < a$) if and only if $b < a$ and for no $x, b < x < a$.*

Definition 2.5. ***(Atom)*** *An element a is an* ***atom****, if $a > 0$ and a dual atom, if $a < 1$.*

Definition 2.6. ***(Complete Lattice)*** *A poset is said to be* ***complete*** *lattice if all its subsets have both join and meet. In particular, every complete lattice is a* ***bounded*** *lattice.*

Definition 2.7. ***(Complemented Lattice)*** *Let L be a bounded lattice with greatest element 1 and least element 0. Two elements x and y of L are said to be complements of each other if $x \vee y = 1$ and $x \wedge y = 0$. If every element of L has a complement, then L is called a complemented Lattice.*

3. Subgroups of G of different orders in $L(G)$ over Z_3

Let H be an arbitrary subgroup of G of order 2. Then the number of subgroups of order 2 is 117. Let K be an arbitrary subgroup of G of order 3. Then the number of subgroups of order 3 is 364. Let L be an arbitrary subgroup of G of order 4. Then the number of subgroups of order 4 is 351.

Let M be an arbitrary subgroup of G of order 6. Then the number of subgroups of order 6 is 468. Let N be an arbitrary subgroup of G of order 8. Then the number of subgroups of order 8 is 468. Let O be an arbitrary subgroup of G of order 9. Then the number of subgroups of order 9 is 117.

Let Q be an arbitrary subgroup of G of order 13. Then the number of subgroups of order 13 is 144. Let R be an arbitrary subgroup of G of order 16. Then the number of subgroups of order 16 is 351. Let V be an arbitrary subgroup of G of order 27. Then the number of subgroups of order 27 is 52.

We observe that, for each subgroups of order 4 contain only one subgroup of order 2. Each subgroups of order 6 contains one subgroup of order 3 and one subgroup of order 2. The first 351 subgroups of order 8 contains only one subgroups of order 4 and next 117 subgroups of order 8 contains exactly three subgroups of order 4.

Each subgroups of order 9 contains exactly three subgroups of order 3. Each subgroups of order 16 contains one subgroup of order 8, two subgroups of order 4 and four subgroups of order 2. Also, each subgroups of order 27 contains three subgroups of order 9 and three subgroups of order 3.

4. Complementedness in the lattice of subgroups of the group of 3×3 matrices over Z_3

Lemma 4.1. *$L(G)$ is complemented lattice if $p = 3$.*

Proof. For $L(G)$ when $p = 3$, the elements of $L(G)$ when $p = 3$ and their respective complements are given in the following table.

Subgroup	**Complement**	**Subgroup**	**Complement**	**Subgroup**	**Complement**
H_1	K_1	H_{31}	K_{31}	H_{61}	K_{61}
H_2	K_2	H_{32}	K_{32}	H_{62}	K_{62}
H_3	K_3	H_{33}	K_{33}	H_{63}	K_{63}
H_4	K_4	H_{34}	K_{34}	H_{64}	K_{64}
H_5	K_5	H_{35}	K_{35}	H_{65}	K_{65}
H_6	K_6	H_{36}	K_{36}	H_{66}	K_{66}
H_7	K_7	H_{37}	K_{37}	H_{67}	K_{67}
H_8	K_8	H_{38}	K_{38}	H_{68}	K_{68}
H_9	K_9	H_{39}	K_{39}	H_{69}	K_{69}
H_{10}	K_{10}	H_{40}	K_{40}	H_{70}	K_{70}
H_{11}	K_{11}	H_{41}	K_{41}	H_{71}	K_{71}
H_{12}	K_{12}	H_{42}	K_{42}	H_{72}	K_{72}
H_{13}	K_{13}	H_{43}	K_{43}	H_{73}	K_{73}
H_{14}	K_{14}	H_{44}	K_{44}	H_{74}	K_{74}
H_{15}	K_{15}	H_{45}	K_{45}	H_{75}	K_{75}
H_{16}	K_{16}	H_{46}	K_{46}	H_{76}	K_{76}
H_{17}	K_{17}	H_{47}	K_{47}	H_{77}	K_{77}
H_{18}	K_{18}	H_{48}	K_{48}	H_{78}	K_{78}
H_{19}	K_{19}	H_{49}	K_{49}	H_{79}	K_{79}
H_{20}	K_{20}	H_{50}	K_{50}	H_{80}	K_{80}
H_{21}	K_{21}	H_{51}	K_{51}	H_{81}	K_{81}
H_{22}	K_{22}	H_{52}	K_{52}	H_{82}	K_{82}
H_{23}	K_{23}	H_{53}	K_{53}	H_{83}	K_{83}
H_{24}	K_{24}	H_{54}	K_{54}	H_{84}	K_{84}
H_{25}	K_{25}	H_{55}	K_{55}	H_{85}	K_{85}
H_{26}	K_{26}	H_{56}	K_{56}	H_{86}	K_{86}
H_{27}	K_{27}	H_{57}	K_{57}	H_{87}	K_{87}
H_{28}	K_{28}	H_{58}	K_{58}	H_{88}	K_{88}
H_{29}	K_{29}	H_{59}	K_{59}	H_{89}	K_{89}
H_{30}	K_{30}	H_{60}	K_{60}	H_{90}	K_{90}

Subgroup	**Complement**	**Subgroup**	**Complement**	**Subgroup**	**Complement**
H_{91}	K_{91}	K_5	L_5	K_{36}	L_{36}
H_{92}	K_{92}	K_6	L_6	K_{37}	L_{37}
H_{93}	K_{93}	K_7	L_7	K_{38}	L_{38}
H_{94}	K_{94}	K_8	L_8	K_{39}	L_{39}
H_{95}	K_{95}	K_9	L_9	K_{40}	L_{40}
H_{96}	K_{96}	K_{10}	L_{10}	K_{41}	L_{41}
H_{97}	K_{97}	K_{11}	L_{11}	K_{42}	L_{42}
H_{98}	K_{98}	K_{12}	L_{12}	K_{43}	L_{43}
H_{99}	K_{99}	K_{13}	L_{13}	K_{44}	L_{44}
H_{100}	K_{100}	K_{14}	L_{14}	K_{45}	L_{45}
H_{101}	K_{101}	K_{15}	L_{15}	K_{46}	L_{46}
H_{102}	K_{102}	K_{16}	L_{16}	K_{47}	L_{47}
H_{103}	K_{103}	K_{17}	L_{17}	K_{48}	L_{48}
H_{104}	K_{104}	K_{18}	L_{18}	K_{49}	L_{49}
H_{105}	K_{105}	K_{19}	L_{19}	K_{50}	L_{50}
H_{106}	K_{106}	K_{20}	L_{20}	K_{51}	L_{51}
H_{107}	K_{107}	K_{21}	L_{21}	K_{52}	L_{52}
H_{108}	K_{108}	K_{22}	L_{22}	K_{53}	L_{53}
H_{109}	K_{109}	K_{23}	L_{23}	K_{54}	L_{54}
H_{110}	K_{110}	K_{24}	L_{24}	K_{55}	L_{55}
H_{111}	K_{111}	K_{25}	L_{25}	K_{56}	L_{56}
H_{112}	K_{112}	K_{26}	L_{26}	K_{57}	L_{57}
H_{113}	K_{113}	K_{27}	L_{27}	K_{58}	L_{58}
H_{114}	K_{114}	K_{28}	L_{28}	K_{59}	L_{59}
H_{115}	K_{115}	K_{29}	L_{29}	K_{60}	L_{60}
H_{116}	K_{116}	K_{30}	L_{30}	K_{61}	L_{61}
H_{117}	K_{117}	K_{31}	L_{31}	K_{62}	L_{62}
K_1	L_1	K_{32}	L_{32}	K_{63}	L_{63}
K_2	L_2	K_{33}	L_{33}	K_{64}	L_{64}
K_3	L_3	K_{34}	L_{34}	K_{65}	L_{65}
K_4	L_4	K_{35}	L_{35}	K_{66}	L_{66}

Subgroup	**Complement**	**Subgroup**	**Complement**	**Subgroup**	**Complement**
K_{67}	L_{67}	K_{98}	L_{98}	K_{129}	L_{129}
K_{68}	L_{68}	K_{99}	L_{99}	K_{130}	L_{130}
K_{69}	L_{69}	K_{100}	L_{100}	K_{131}	L_{131}
K_{70}	L_{70}	K_{101}	L_{101}	K_{132}	L_{132}
K_{71}	L_{71}	K_{102}	L_{102}	K_{133}	L_{133}
K_{72}	L_{72}	K_{103}	L_{103}	K_{134}	L_{134}
K_{73}	L_{73}	K_{104}	L_{104}	K_{135}	L_{135}
K_{74}	L_{74}	K_{105}	L_{105}	K_{136}	L_{136}
K_{75}	L_{75}	K_{106}	L_{106}	K_{137}	L_{137}
K_{76}	L_{76}	K_{107}	L_{107}	K_{137}	L_{137}
K_{77}	L_{77}	K_{108}	L_{108}	K_{138}	L_{138}
K_{78}	L_{78}	K_{109}	L_{109}	K_{139}	L_{139}
K_{79}	L_{79}	K_{110}	L_{110}	K_{140}	L_{140}
K_{80}	L_{80}	K_{111}	L_{111}	K_{141}	L_{141}
K_{81}	L_{81}	K_{112}	L_{112}	K_{142}	L_{142}
K_{82}	L_{82}	K_{113}	L_{113}	K_{143}	L_{143}
K_{83}	L_{83}	K_{114}	L_{114}	K_{144}	L_{144}
K_{84}	L_{84}	K_{115}	L_{115}	K_{145}	L_{145}
K_{85}	L_{85}	K_{116}	L_{116}	K_{146}	L_{146}
K_{86}	L_{86}	K_{117}	L_{117}	K_{147}	L_{147}
K_{87}	L_{87}	K_{118}	L_{118}	K_{148}	L_{148}
K_{88}	L_{88}	K_{119}	L_{119}	K_{149}	L_{149}
K_{89}	L_{89}	K_{120}	L_{120}	K_{150}	L_{150}
K_{90}	L_{90}	K_{121}	L_{121}	K_{151}	L_{151}
K_{91}	L_{91}	K_{122}	L_{122}	K_{152}	L_{152}
K_{92}	L_{92}	K_{123}	L_{123}	K_{153}	L_{153}
K_{93}	L_{93}	K_{124}	L_{124}	K_{154}	L_{154}
K_{94}	L_{94}	K_{125}	L_{125}	K_{155}	L_{155}
K_{95}	L_{95}	K_{126}	L_{126}	K_{156}	L_{156}
K_{96}	L_{96}	K_{127}	L_{127}	K_{157}	L_{157}
K_{97}	L_{97}	K_{128}	L_{128}	K_{158}	L_{158}

Subgroup	**Complement**	**Subgroup**	**Complement**	**Subgroup**	**Complement**
K_{159}	L_{159}	K_{190}	L_{190}	K_{221}	L_{221}
K_{160}	L_{160}	K_{191}	L_{191}	K_{222}	L_{222}
K_{161}	L_{161}	K_{192}	L_{192}	K_{223}	L_{223}
K_{162}	L_{162}	K_{193}	L_{193}	K_{224}	L_{224}
K_{163}	L_{163}	K_{194}	L_{194}	K_{225}	L_{225}
K_{164}	L_{164}	K_{195}	L_{195}	K_{226}	L_{226}
K_{165}	L_{165}	K_{196}	L_{196}	K_{227}	L_{227}
K_{166}	L_{166}	K_{197}	L_{197}	K_{228}	L_{228}
K_{167}	L_{167}	K_{198}	L_{198}	K_{229}	L_{229}
K_{168}	L_{168}	K_{199}	L_{199}	K_{230}	L_{230}
K_{169}	L_{169}	K_{200}	L_{200}	K_{231}	L_{231}
K_{170}	L_{170}	K_{201}	L_{201}	K_{232}	L_{232}
K_{171}	L_{171}	K_{202}	L_{202}	K_{233}	L_{233}
K_{172}	L_{172}	K_{203}	L_{203}	K_{235}	L_{235}
K_{173}	L_{173}	K_{204}	L_{204}	K_{236}	L_{236}
K_{174}	L_{174}	K_{205}	L_{205}	K_{237}	L_{237}
K_{175}	L_{175}	K_{206}	L_{206}	K_{238}	L_{238}
K_{176}	L_{176}	K_{207}	L_{207}	K_{239}	L_{239}
K_{177}	L_{177}	K_{208}	L_{208}	K_{240}	L_{240}
K_{178}	L_{178}	K_{209}	L_{209}	K_{241}	L_{241}
K_{179}	L_{179}	K_{210}	L_{210}	K_{242}	L_{242}
K_{180}	L_{180}	K_{211}	L_{211}	K_{243}	L_{243}
K_{181}	L_{181}	K_{212}	L_{212}	K_{244}	L_{244}
K_{182}	L_{182}	K_{213}	L_{213}	K_{245}	L_{245}
K_{183}	L_{183}	K_{214}	L_{214}	K_{246}	L_{246}
K_{184}	L_{184}	K_{215}	L_{215}	K_{247}	L_{247}
K_{185}	L_{185}	K_{216}	L_{216}	K_{248}	L_{248}
K_{186}	L_{186}	K_{217}	L_{217}	K_{249}	L_{249}
K_{187}	L_{187}	K_{218}	L_{218}	K_{250}	L_{250}
K_{188}	L_{188}	K_{219}	L_{219}	K_{251}	L_{251}
K_{189}	L_{189}	K_{220}	L_{220}	K_{252}	L_{252}

Subgroup	Complement	Subgroup	Complement	Subgroup	Complement
K_{253}	L_{253}	K_{284}	L_{284}	K_{315}	L_{315}
K_{254}	L_{254}	K_{285}	L_{285}	K_{316}	L_{316}
K_{255}	L_{255}	K_{286}	L_{286}	K_{317}	L_{317}
K_{256}	L_{256}	K_{287}	L_{287}	K_{318}	L_{318}
K_{257}	L_{257}	K_{288}	L_{288}	K_{319}	L_{319}
K_{258}	L_{258}	K_{289}	L_{289}	K_{320}	L_{320}
K_{259}	L_{259}	K_{290}	L_{290}	K_{321}	L_{321}
K_{260}	L_{260}	K_{291}	L_{291}	K_{322}	L_{322}
K_{261}	L_{261}	K_{292}	L_{292}	K_{323}	L_{323}
K_{262}	L_{262}	K_{293}	L_{293}	K_{324}	L_{324}
K_{263}	L_{263}	K_{294}	L_{294}	K_{325}	L_{325}
K_{264}	L_{264}	K_{295}	L_{295}	K_{326}	L_{326}
K_{265}	L_{265}	K_{296}	L_{296}	K_{327}	L_{327}
K_{266}	L_{266}	K_{297}	L_{297}	K_{328}	L_{328}
K_{267}	L_{267}	K_{298}	L_{298}	K_{329}	L_{329}
K_{268}	L_{268}	K_{299}	L_{299}	K_{330}	L_{330}
K_{269}	L_{269}	K_{300}	L_{300}	K_{331}	L_{331}
K_{270}	L_{270}	K_{301}	L_{301}	K_{332}	L_{332}
K_{271}	L_{271}	K_{302}	L_{302}	K_{333}	L_{333}
K_{272}	L_{272}	K_{303}	L_{303}	K_{334}	L_{334}
K_{273}	L_{273}	K_{304}	L_{304}	K_{335}	L_{335}
K_{274}	L_{274}	K_{305}	L_{305}	K_{336}	L_{336}
K_{275}	L_{275}	K_{306}	L_{306}	K_{337}	L_{337}
K_{276}	L_{276}	K_{307}	L_{307}	K_{338}	L_{338}
K_{277}	L_{277}	K_{308}	L_{308}	K_{339}	L_{339}
K_{278}	L_{278}	K_{309}	L_{309}	K_{340}	L_{340}
K_{279}	L_{279}	K_{310}	L_{310}	K_{341}	L_{341}
K_{280}	L_{280}	K_{311}	L_{311}	K_{342}	L_{342}
K_{281}	L_{281}	K_{312}	L_{312}	K_{343}	L_{343}
K_{282}	L_{282}	K_{313}	L_{313}	K_{344}	L_{344}
K_{283}	L_{283}	K_{314}	L_{314}	K_{345}	L_{345}

Subgroup	**Complement**	**Subgroup**	**Complement**
K_{346}	L_{346}	M_{201}- M_{250}	Q_5
K_{347}	L_{347}	M_{251}- M_{300}	Q_6
K_{348}	L_{348}	M_{301}- M_{350}	Q_7
K_{349}	L_{349}	M_{351}-M_{400}	Q_8
K_{350}	L_{350}	M_{401}-M_{450}	Q_9
K_{351}	L_{351}	M_{451}-M_{468}	Q_{10}
K_{352}	Q_1	N_1- N_{50}	V_1
K_{353}	Q_2	N_{51}- N_{100}	V_2
K_{354}	Q_3	N_{101}- N_{150}	V_3
K_{355}	Q_4	N_{151}- N_{200}	V_4
K_{356}	Q_5	N_{201}-N_{250}	V_5
K_{357}	Q_6	N_{251}-N_{300}	V_6
K_{358}	Q_7	N_{301}-N_{350}	V_7
K_{359}	Q_8	N_{351}- N_{400}	V_8
K_{360}	Q_9	N_{401}- N_{450}	V_9
K_{361}	Q_{10}	N_{451}- N_{468}	V_{10}
K_{362}	Q_{11}	O_1- O_{50}	H_1
K_{363}	Q_{12}	O_{51}- O_{100}	H_2
K_{364}	Q_{13}	O_{101}- O_{117}	H_3
K_{234}	L_{234}	Q_1- Q_{50}	R_1
L_1-L_{50}	O_1	Q_{51}- Q_{100}	R_2
L_{51}-L_{100}	O_2	Q_{101}- Q_{117}	R_3
L_{101}-L_{150}	O_3	R_1- R_{50}	K_1
L_{151}-L_{200}	O_4	R_{51}- R_{100}	K_2
L_{201}-L_{250}	O_5	R_{101}- R_{150}	K_3
L_{251}-L_{300}	O_6	R_{151}- R_{200}	K_4
L_{301}-L_{351}	O_7	R_{201}- R_{250}	K_5
M_{1-} M_{50}	Q_1	R_{251}- R_{300}	K_6
M_{51}- M_{100}	Q_2	R_{301}- R_{351}	K_7
M_{101}- M_{150}	Q_3	V_1- V_{50}	N_1
M_{151}- M_{200}	Q_4	V_{51}- V_{52}	N_2

□

5. Conclusion

In this paper, we proved that the complementedness in the subgroup lattice of the group of 3×3 matrices over Z_3.

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