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A STUDY ON SUBDIRECT IRREDUCIBILITY OF THE SUBGROUP LATTICES OF THE GROUP OF 2×2 MATRICES OVER Z_7

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ABSTRACT. In this paper, we determine subdirect irreducibility of the subgroup lattice of the group of 2×2 matrices over Z_7.

1. INTRODUCTION

Let $L(G)$ denotes the Lattice of Subgroups of G, where G is a group of 2×2 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number.

Let

$$\mathcal{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z_p, ad - bc \neq 0.$$

Then $\mathcal{G}$ is a group under matrix multiplication modulo p.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G} : ad - bc = 1. \right\}$$

Then $\mathcal{G}$ is a subgroup of G.

We have, $o(\mathcal{G}) = p(p^2-1)(p-1)$ and $o(G) = p(p^2-1)$ [6]. For more details on this theory and on its applications, we suggest the reader to refer [1–5, 7–9].

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2. Preliminaries

Definition 2.1. ***(Poset)*** *A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair $(P, \leq)$ is called a partially ordered set or poset. A poset. $(P, \leq)$ is totally ordered if every $x, y \in P$ are comparable, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.*

Definition 2.2. *Let $(P, \leq)$ be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the* ***supremum or join*** *of S.A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the* ***infimum or meet*** *of S.*

Definition 2.3. ***(Lattice)*** *Poset $(P, \leq)$ is called a lattice if every pair x, y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.*

Definition 2.4. ***(Atom)*** *An element a is an atom, if $a > 0$ and a dual atom, if $a < 1$.*

Definition 2.5. *An equivalence relation θ on a lattice L is called a congruence relation on L iff $(a_0, b_0) \in \theta$ and $(a_1, b_1) \in \theta$ imply that $(a_0 \wedge a_1, b_0 \wedge b_1) \in \theta$ and $(a_0 \vee a_1, b_0 \vee b_1) \in \theta$.*

Definition 2.6. *The collection of all congruence relations on L, is denoted by Con L.*

Note: Con L with respect to the set inclusion relation becomes an algebraic lattice [1].

Definition 2.7. *If a lattice L has only two trivial congruence relations, namely ω, the diagonal and $\tau = L \times L$, then L is said to be simple. (e.g. M_3 is simple)*

Definition 2.8. *If Con L contains a unique atom, then we say that L is subdirectly irreducible. (e.g N_5 is subdirectly irreducible)*

We tabulate the subgroups of G, when $p = 7$ in the order in which they lie in different maximal subgroups (co-atoms). This will make our work easy.

TABLE 1. Intervals $[\{e\}, U_i]$ in $L(G), i = 1, 2, ...8$

Order	**Subgroups**	**Order**	**Subgroups**
42	U_1	42	U_2
21	T_1	21	T_2
14	R_1	14	R_2
7	N_1	7	N_2
6	M_1, M_{12}, M_{14}, M_{16}, M_{17}, M_{21}, M_{23}	6	M_1, M_{11}, M_{13}, M_{15}, M_{18}, M_{22}, M_{24}
3	K_1, K_{12}, K_{14}, K_{16}, K_{17}, K_{21}, K_{23}	3	K_1, K_{11}, K_{13}, K_{15}, K_{18}, K_{22}, K_{24}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_3	42	U_4
21	T_3	21	T_4
14	R_3	14	R_4
7	N_3	7	N_4
6	M_4, M_6, M_9, M_{17}, M_{20}, M_{24}, M_{26}	6	M_4, M_5, M_{10}, M_{18}, M_{19}, M_{23}, M_{25}
3	K_4, K_6, K_9, K_{17}, K_{20}, K_{24}, K_{26}	3	K_4, K_5, K_{10}, K_{18}, K_{19}, K_{23}, K_{25}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_5	42	U_6
21	T_5	21	T_6
14	R_5	14	R_6
7	N_5	7	N_6
6	M_3, M_6, M_8, M_{16}, M_{19}, M_{22}, M_{28}	6	M_3, M_5, M_7, M_{15}, M_{20}, M_{21}, M_{27}
3	K_3, K_6, K_8, K_{16}, K_{19}, K_{22}, K_{28}	3	K_3, K_5, K_7, K_{15}, K_{20}, K_{21}, K_{27}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_7	42	U_8
21	T_7	21	T_8
14	R_7	14	R_8
7	N_7	7	N_8
6	M_2, M_8, M_{10}, M_{12}, M_{13}, M_{26}, M_{27}	6	M_2, M_7, M_9, M_{11}, M_{14}, M_{25}, M_{28}
3	K_2, K_8, K_{10}, K_{12}, K_{13}, K_{26}, K_{27}	3	K_2, K_7, K_9, K_{11}, K_{14}, K_{25}, K_{28}

TABLE 2. Intervals $[\{e\}, V_i]$ in $L(G), i = 1, 2, ...14$

Order	**Subgroups**	**Order**	**Subgroups**
48	V_1	48	V_2
16	S_{12}, S_{16}, S_{17}	16	S_{13}, S_{18}, S_{19}
12	Q_1, Q_4, Q_7, Q_8	12	Q_1, Q_3, Q_9, Q_{10}
8	P_{12}, P_{16}, P_{17}	8	P_{13}, P_{18}, P_{19}
6	M_1, M_4, M_7, M_8	6	M_1, M_3, M_9, M_{10}
4	$L_1, L_2, L_3, L_{10}, L_{11}, L_{12}, L_{14}, L_{16}, L_{17}$	4	$L_1, L_2, L_3, L_8, L_9, L_{13}, L_{15}, L_{18}, L_{19}$
3	K_1, K_4, K_7, K_8	3	K_1, K_3, K_9, K_{10}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_3	48	V_4
16	S_4, S_5, S_{15}	16	S_6, S_7, S_{14}
12	Q_2, Q_4, Q_{15}, Q_{16}	12	Q_2, Q_3, Q_{17}, Q_{18}
8	P_4, P_5, P_{15}	8	P_6, P_7, P_{14}
6	M_2, M_4, M_{15}, M_{16}	6	M_2, M_3, M_{17}, M_{18}
4	$L_1, L_4, L_5, L_{10}, L_{11}, L_{13}, L_{15}, L_{20}, L_{21}$	4	$L_1, L_6, L_7, L_8, L_9, L_{12}, L_{14}, L_{20}, L_{21}$
3	K_2, K_4, K_{15}, K_{16}	3	K_2, K_3, K_{17}, K_{18}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_5	48	V_6
16	S_5, S_{10}, S_{21}	16	S_4, S_{11}, S_{20}
12	Q_5, Q_9, Q_{12}, Q_{22}	12	$Q_6, Q_{10}, Q_{11}, Q_{21}$
8	P_5, P_{10}, P_{21}	8	P_4, P_{11}, P_{20}
6	M_5, M_9, M_{12}, M_{22}	6	$M_6, M_{10}, M_{11}, M_{21}$
4	$L_3, L_4, L_5, L_7, L_8, L_{10}, L_{15}, L_{16}, L_{21}$	4	$L_2, L_4, L_5, L_6, L_9, L_{11}, L_{15}, L_{17}, L_{20}$
3	K_5, K_9, K_{12}, K_{22}	3	$K_6, K_{10}, K_{11}, K_{21}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_7	48	V_8
16	S_6, S_9, S_{20}	16	S_7, S_8, S_{21}
12	Q_5, Q_8, Q_{14}, Q_{24}	12	Q_6, Q_7, Q_{13}, Q_{23}
8	P_6, P_9, P_{20}	8	P_7, P_8, P_{21}
6	M_5, M_8, M_{14}, M_{24}	6	M_6, M_7, M_{13}, M_{23}
4	$L_3, L_4, L_6, L_7, L_9, L_{11}, L_{14}, L_{19}, L_{20}$	4	$L_2, L_5, L_6, L_7, L_8, L_{10}, L_{14}, L_{18}, L_{21}$
3	K_5, K_8, K_{14}, K_{24}	3	K_6, K_7, K_{13}, K_{23}

Order	Subgroups
48	V_9
16	S_3, S_{10}, S_{16}
12	$Q_{11}, Q_{17}, Q_{19}, Q_{27}$
8	P_3, P_{10}, P_{16}
6	$M_{11}, M_{17}, M_{19}, M_{27}$
4	$L_3, L_5, L_9, L_{10}, L_{12}, L_{16}, L_{17}, L_{19}, L_{21}$
3	$K_{11}, K_{17}, K_{19}, K_{27}$

Order	Subgroups
48	V_{10}
16	S_2, S_{11}, S_{17}
12	$Q_{12}, Q_{18}, Q_{20}, Q_{28}$
8	P_2, P_{11}, P_{17}
6	$M_{12}, M_{18}, M_{20}, M_{28}$
4	$L_2, L_4, L_8, L_{11}, L_{12}, L_{16}, L_{17}, L_{18}, L_{20}$
3	$K_{12}, K_{18}, K_{20}, K_{28}$

Order	Subgroups
48	V_{11}
16	S_2, S_8, S_{18}
12	$Q_{14}, Q_{15}, Q_{19}, Q_{26}$
8	P_2, P_8, P_{18}
6	$M_{14}, M_{15}, M_{19}, M_{26}$
4	$L_2, L_7, L_8, L_{11}, L_{13}, L_{17}, L_{18}, L_{19}, L_{21}$
3	$K_{14}, K_{15}, K_{19}, K_{26}$

Order	Subgroups
48	V_{12}
16	S_3, S_9, S_{19}
12	$Q_{13}, Q_{16}, Q_{20}, Q_{25}$
8	P_3, P_9, P_{19}
6	$M_{13}, M_{16}, M_{20}, M_{25}$
4	$L_3, L_6, L_9, L_{10}, L_{13}, L_{16}, L_{18}, L_{19}, L_{20}$
3	$K_{13}, K_{16}, K_{20}, K_{25}$

Order	Subgroups
48	V_{13}
16	S_1, S_{12}, S_{14}
12	$Q_{21}, Q_{22}, Q_{25}, Q_{26}$
8	P_1, P_{12}, P_{14}
6	$M_{21}, M_{22}, M_{25}, M_{26}$
4	$L_1, L_6, L_7, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}$
3	$K_{21}, K_{22}, K_{25}, K_{26}$

Order	Subgroups
48	V_{14}
16	S_1, S_{13}, S_{15}
12	$Q_{23}, Q_{24}, Q_{27}, Q_{28}$
8	P_1, P_{13}, P_{15}
6	$M_{23}, M_{24}, M_{27}, M_{28}$
4	$L_1, L_4, L_5, L_{12}, L_{13}, L_{14}, L_{15}, L_{18}, L_{19}$
3	$K_{23}, K_{24}, K_{27}, K_{28}$

When $p = 7$, we display two typical intervals $[\{e\}, U_1]$ and $[\{e\}, V_1]$ of $L(G)$ in the following figures.

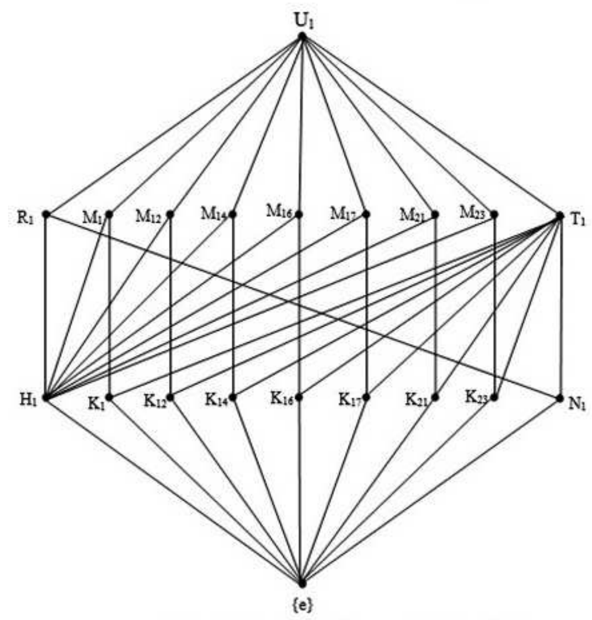

FIGURE 1. The Interval $[\{e\}, U_1]$

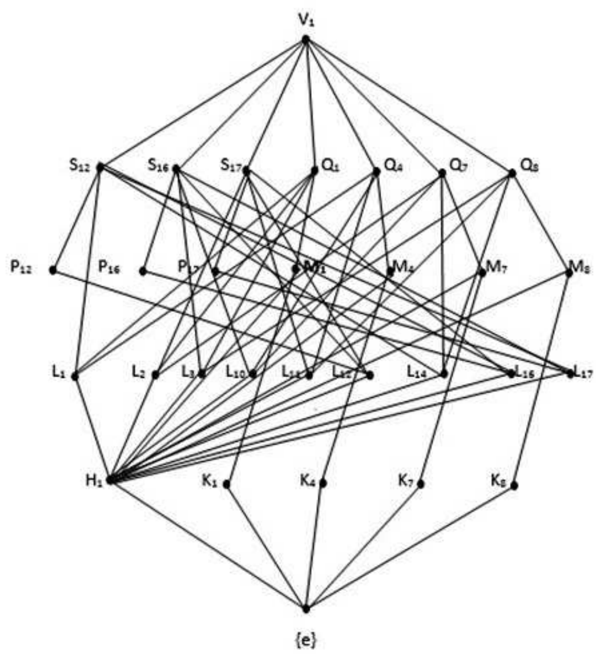

FIGURE 2. The Interval $[\{e\}, V_1]$

3. SUBDIRECT IRREDUCIBILITY OF $L(G)$ WHEN $p = 7$

In this section, while computing the congruences we refer to tables 1 and 2.

Lemma 3.1. *$\theta(\{e\}, H_1)$ is a proper congruence relation on $L(G)$.*

Proof. Let $\theta(\{e\}, H_1) = \theta_1$. Let K be a subgroup of odd order then

$$(\{e\}, H_1) \vee (K, K) = (K, S) \tag{3.1}$$

where K covers S and S is of even order. Therefore $(K, S) \in \theta_1$, for every subgroup K of odd order.

θ_1 contains no other pair (X, Y), where $X \neq Y$. Since, if X is any subgroup of even order in $L(G)$, then $(\{e\}, H_1) \wedge (X, X) = (\{e\}, H_1)$ and $(\{e\}, H_1) \vee (X, X) = (X, X)$.

If X is any subgroup of odd order other than K and S then $(K, S) \wedge (X, X) = (\{e\}, \{e\})$ and $(K, S) \vee (X, X) = (K \vee X, S \vee X)$ which is the same type of the pair (K, S) as in (3.1).

Therefore, we do not get a new element other than that found in (3.1). So we conclude that θ_1 is a proper congruence relation on $L(G)$. □

Lemma 3.2. *The principal congruence generated by any pair of the form (K, S), where K is of odd order and S is of even order immediately above K, is equal to θ_1.*

Proof. Let $\theta_2 = (K, S)$, where K is of odd order and S is of even order immediately above K. We have to prove that $\theta_1 = \theta_2$. Now, $(K, S) \in \theta_2$. Therefore,

$$(K, S) \wedge (H_1, H_1) = (K \wedge H_1, S \wedge H_1) = (\{e\}, H_1).$$

Therefore, $(\{e\}, H_1) \in \theta_2$.

Therefore, $\theta_1 \subseteq \theta_2$. Also, as seen in the previous lemma, $(K, S) \in \theta_1$ which implies that $\theta_2 \subseteq \theta_1$.

Therefore, $\theta_1 = \theta_2$ and hence the proof. □

Lemma 3.3. $\theta(H_1, G) = L(G) \times L(G)$.

Proof. Now, $(H_1, G) \wedge (X_i, X_i) = (\{e\}, X_i)$, for any odd order subgroup X_i of G. Then, the supremum of all $(\{e\}, X_i)$ is $(\{e\}, G)$ which belongs to $\theta(H_1, G)$. Therefore, $\theta(H_1, G) = L(G) \times L(G)$, is an improper congruence. □

Lemma 3.4. *$\theta(H_1, X) = L(G) \times L(G)$, where X is a subgroup of any order.*

Proof. Let X be either an odd order or even order subgroup of G. Now, $(H_1, X) \vee (U_i, U_i) = (U_i, G)$, where U_i's are co-atoms and non-comparable with X and we observe that there are atleast 3 such U_i's and if we take take the infimum of all (U_i, G), we get $(H_1, G) \in \theta(H_1, X)$ which generates $L(G) \times L(G)$ [by Lemma 3.3]. Therefore, $\theta(H_1, X) = L(G) \times L(G)$, where X is a subgroup of any order. □

Lemma 3.5. *$\theta(\{e\}, X) = L(G) \times L(G)$, where X is a subgroup of any order.*

Proof. Now, $(\{e\}, X) \vee (U_i, U_i) = (U_i, G)$, where U_i's are co-atoms and non-comparable with X.

We observe that there are atleast 3 such U_i's and if we take infimum of all (U_i, G), we get $(H_1, G) \in \theta(\{e\}, X)$ which genetates $L(G) \times L(G)$. [by Lemma 3.3].

Therefore, $\theta(\{e\}, X) = L(G) \times L(G)$, where X is a subgroup of any order. □

Lemma 3.6. *$\theta(X, Y) = L(G) \times L(G)$, where both X and Y are of even order subgroups.*

Proof. **Case (i):** Suppose that $o(X) \neq 42$ and $o(Y) = 42$.

(i(a)): If X is not a subset of Y then, $\theta(X, Y)$ contains $(X \wedge Y, X \vee Y) = (H_1, G)$. By lemma 3.3, $\theta(X, Y) = L(G) \times L(G)$.

(i(b)): Suppose X is a subset of Y, that is, X is of order 2 or 6. Take a subgroup S of odd order such that S is not a subset of X and S is a subset of Y. Such an S exists. Then $(S \wedge X, S \wedge Y) = (\{e\}, S) \in \theta(X, Y)$ which generates $L(G) \times L(G)$ [by lemma 3.5].

Case (ii): Symmetry of $\theta(X, Y)$ implies the roles of X and Y can be interchanged. □

Lemma 3.7. *$\theta(X, Y) = L(G) \times L(G)$, where both X and Y are subgroups of odd order.*

Proof. **Case (i):** Let X and Y be non-comparable subgroups. Now, $\theta(X, Y)$ contains $(X \wedge Y, X \vee Y) = (\{e\}, Z)$, Z is of any order.

By lemma 3.5, $\theta(X, Y) = L(G) \times L(G)$.

Case (ii): Let X and Y be comparable. Now, $(X \wedge K_i, Y \wedge K_i) = (\{e\}, K_i)$, K_i is a subgroup of order 3 and $K_i \subset Y$ but $K_i \not\subset X$. Such a K_i exists. Then,

$$(\{e\}, K_i) \in \theta(X, Y).$$

By lemma 3.5, $(\{e\}, K_i)$ generates $L(G) \times L(G)$. Therefore,

$$\theta(X, Y) = L(G) \times L(G)$$

when both X and Y are of odd order subgroups. $\square$

Lemma 3.8. *$\theta(X, Y) = L(G) \times L(G)$, where X is an odd order subgroup and Y is an even order subgroup.*

Proof. Let X be not a subset of Y. Now, $\theta(X, Y)$ contains $(X \wedge Y, X \vee Y) = (\{e\}, Z)$, Z is of any order and $Z \neq \{e\}$.

That is,

$$(\{e\}, Z) \in \theta(X, Y).$$

By lemma 3.5, $(\{e\}, Z)$ generates $L(G) \times L(G)$. Therefore, $\theta(X, Y)$ generates $L(G) \times L(G)$.

If X is a subset of Y and X does not cover Y, then there exists a subgroup T of odd order such that T is not a subset of X but T is a subset of Y. Then

$$(X \wedge T, Y \wedge T) = (\{e\}, T) \in \theta(X, Y).$$

But, $(\{e\}, T)$ generates $L(G) \times L(G)$.

Therefore, $\theta(X, Y)$ is improper.

Hence $\theta(X, Y) = L(G) \times L(G)$ when X is an odd order subgroup and Y is an even order subgroup. $\square$

Theorem 3.1. *$L(G)$ is subdirectly irreducible when $p = 7$.*

Proof. From lemma 3.1 to 3.8, we conclude that the only proper congruence relation on $L(G)$ is $\theta(\{e\}, H_1)$. Thus $Con(L(G))$ contains a unique atom $\theta(\{e\}, H_1)$.

Therefore, $Con(L(G))$ is isomorphic to the three element chain. Hence, $L(G)$ is subdirectly irreducible. $\square$

4. Conclusion

In this paper we proved that the subgroup lattices of the group of 2×2 matrices over Z_7 is subdirect irreducibility.

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